Final Exam: IERG 6300

Total points: 50

The exam is due by 12:00 noon on Monday December 23rd, 2019

Even if you opt for this course to be Pass/Fail you need to answer this exam

This exam contains 2 pages (including this cover page) and 6 questions.

Important Notes

• Use of internet is not allowed.

• You may use the lecture notes and quote relevant results. You can also use standard theorems from analysis.

1. (a) (2 points) Let $X$ be a non-negative random variable such that $E(X^2) < \infty$. Let $0 \leq a < E(X)$. Show that

$$P(X > a) \geq \frac{(E(X) - a)^2}{E(X^2)}.$$

Hint: Apply Cauchy-Schwartz inequality to $X^{1\cdot X>a}$.

(b) (3 points) Let $\{A_n\}_{n \geq 1} \in \mathcal{F}$ such that $\sum_{n=1}^{\infty} P(A_n) = \infty$. Further let

$$\limsup_{n \to \infty} \frac{\left(\sum_{k=1}^{n} P(A_k)\right)^2}{\sum_{1 \leq j \leq n} P(A_j \cap A_k)} = \alpha > 0.$$

Show that $P(A_n \ i.o.) \geq \alpha$.

Hint: Use previous part with $X_n = \sum_{k=1}^{n} 1_{A_k}$ and $a_n = \lambda E(X_n)$.

2. (6 points) Suppose $X$ and $Y$ are $[0, 1]$-valued random variables such that $E(X^n) = E(Y^n)$ for $n = 1, 2, ...$. Show that $X$ has the same distribution as $Y$.

Hint: Recall (Weierstrass approximation theorem) that if $h(x)$ is continuous in $[0, 1]$ then there exists a sequence of polynomials $p_n(x)$ such that $\sup_{x \in [0,1]} |h(x) - p_n(x)| \to 0$ as $n \to \infty$.

3. Let $X$ be an $\mathcal{F}$-measurable random variable satisfying $E(X^2) < \infty$. For any $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$ define

$$\text{Var}(X|\mathcal{G}) = E((X - E(X|\mathcal{G}))^2|\mathcal{G}).$$

(a) (3 points) Consider $\sigma$-algebras satisfying $\mathcal{G}_2 \subseteq \mathcal{G}_1 \subseteq \mathcal{F}$. Show that $E(\text{Var}(X|\mathcal{G}_2)) \leq E(\text{Var}(X|\mathcal{G}_1))$

(b) (3 points) Show that for any $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$

$$E(X^2) = E(\text{Var}(X)|\mathcal{G}) + \text{Var}(E(X|\mathcal{G})).$$

4. Let $s > 0$ and let $\{Z_n\}_{n \geq 1}$ be a sequence of independent random variables such that $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2n}$, $P(Z_n = 0) = 1 - \frac{1}{2n}$, $n \geq 1$. Let $Y_0 = 0$ and for $n \geq 1$, define

$$Y_n = n^s Y_{n-1}|Z_n| + Z_n 1_{\{Y_{n-1} = 0\}}.$$
(a) (2 points) Show that \( \{Y_n\} \) is a martingale w.r.t. to the canonical filtration \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) and \( \mathcal{F}_i = \sigma(Z_1, ..., Z_i), i \geq 1 \).

(b) (4 points) Show that

\[
P(\max_{1 \leq k \leq n} Y_k \geq x) \leq \frac{1}{2x} \left( 1 + \sum_{k=2}^{n} \frac{1}{k^s} \left( 1 - \frac{1}{(k-1)^s} \right) \right)
\]

(Hint: Use Doob’s inequality. Theorem 1 in lecture notes 6)

(c) Show that

i. (2 points) \( Y_n \to 0 \) in probability.

ii. (3 points) \( Y_n \to 0 \) almost surely if and only if \( s > 1 \)

iii. (1 point) For no value of \( s > 0 \) does \( \mathbb{E}(|Y_n|) \to 0 \).

5. (6 points) For \( n \geq 1 \), let \( X_n, Y_n \) be non-negative integrable random variables adapted to the filtration \( \mathcal{F}_n \). Assume that \( \mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq Y_n + X_n(1 + Y_n) \). Further let \( \sum_n Y_n < \infty \) almost surely. Show that \( X_n \) converges almost surely.

Hint: Construct a suitable non-negative supermartingale, i.e. one whose convergence implies the convergence of \( X_n \), and use the Martingale Convergence Theorem for non-negative supermartingales in the lecture notes.

6. Let \( \{X_i\}_{i \geq 1} \) denote a sequence of independent and identically distributed Bernoulli random variables with

\[
P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.
\]

Let \( S_0 = 0 \) and let \( S_n = \sum_{i=1}^{n} X_i \). Further let \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) and \( \mathcal{F}_i = \sigma(X_1, ..., X_i), i \geq 1 \) denote the canonical filtration.

(a) (2 points) Show that \( P(S_n - S_k \geq 0) \geq \frac{1}{2} \) for \( k = 1, ..., n \).

(b) (3 points) Fix \( a > 0 \). Let \( \tau_a = \inf\{k \geq 1 : S_k > a\} \). Show that

\[
P(S_n > a) \geq \sum_{k=1}^{n} P(\tau_a = k, S_n - S_k \geq 0) \geq \frac{1}{2} P(\tau_a = k).
\]

(c) (1 point) Deduce that for any \( n \geq 1 \) and \( a > 0 \)

\[
P(\max_{1 \leq k \leq n} S_k > a) \leq 2P(S_n > a).
\]

(d) (3 points) For any \( p, n \in \mathbb{N} \), show that

\[
P(\max_{1 \leq k \leq n} S_k \geq p) = 2P(S_n \geq p) - P(S_n = p).
\]

(e) (3 points) Let \( Z_n \) denote the number of strict sign changes within \( \{S_0, S_1, ..., S_n\} \). Show that

\[
P(Z_{2n+1} \geq p|S_1 = -1) = P(\max_{1 \leq k \leq 2n+1} S_k \geq 2p - 1|S_1 = -1).
\]

Hint: flip the signs of \( \{X_i\} \)'s between the odd and even strict sign changes of \( S_k \).

(f) (3 points) Use the above parts to show that \( Z_{2n+1} \) has the same distribution as \( \frac{|S_{2n+1}| - 1}{2} \).