Homework 3: IERG 6300

Due date: 17 February, 2022.

Exercises

1. Let X be integrable. Define *variance* of X, $var(X)$, as
$$\int X^2 dP - \left(\int X dP\right)^2.$$
Show that $var(X) \geq 0$. Further if $E(X) = \int X dP = \mu_X$, then $\mathrm{Var}(X) = 0$ *if and only if* $P(X = \mu_X) = 1$.

2. Show that for any $X \geq 0$ (no other assumptions on existence of moments) both
   (a) $\lim_{y\to\infty} \quad y\,\mathrm{E}(X^{-1} 1_{X>y}) = 0$,
   (b) $\lim_{y\downarrow 0} \quad y\,\mathrm{E}(X^{-1} 1_{X>y}) = 0$

3. Consider the following:

   Definition 1. A collection of random variables $\{X_\alpha\}$ is called uniformly integrable (U-I) if
$$\lim_{M\to\infty} \sup_\alpha \mathrm{E}(|X_\alpha| 1_{|X_\alpha|>M}) = 0.$$
   (a) Construct a sequence of random variables such that $\sup_n \mathrm{E}(|X_n|) < \infty$ but $\{X_n\}$ is not U-I.
   (b) If $\{X_n\}$ is a U-I sequence and $\{Y_n\}$ is another U-I sequence, then $\{X_n + Y_n\}$ is a U-I sequence.

4. Show that if for some $r \in \mathbb{N}$, $\mathrm{E}(|X|^r) < \infty$, then the characteristic function $\phi(t)$ is r-times continuously differentiable. If r is even, show that the converse is true (Hint: use Fatou).