Homework 3: IERG 6300

Due date: 17 February, 2022.

Exercises

1. Let $X$ be integrable. Define variance of $X$, $\text{var}(X)$, as

$$\int X^2dP - \left(\int XdP\right)^2.$$

Show that $\text{var}(X) \geq 0$. Further if $E(X) = \int XdP = \mu_X$, then $\text{Var}(X) = 0$ if and only if $P(X = \mu_X) = 1$.

2. Show that for any $X \geq 0$ (no other assumptions on existence of moments) both

(a) $\lim_{y \to \infty} y E(X^{-1}1_{X>y}) = 0$,
(b) $\lim_{y \downarrow 0} y E(X^{-1}1_{X>y}) = 0$

3. Consider the following:

Definition 1. A collection of random variables $\{X_\alpha\}$ is called uniformly integrable (U-I) if

$$\lim_{M \to \infty} \sup_\alpha E(|X_\alpha|1_{|X_\alpha|>M}) = 0.$$

(a) Construct a sequence of random variables such that $\sup_n E(|X_n|) < \infty$ but $\{X_n\}$ is not U-I.
(b) If $\{X_n\}$ is a U-I sequence and $\{Y_n\}$ is another U-I sequence, then $\{X_n + Y_n\}$ is a U-I sequence.

4. Show that if for some $r \in \mathbb{N}$, $E(|X|^r) < \infty$, then the characteristic function $\phi(t)$ is $r$-times continuously differentiable. If $r$ is even, show that the converse is true (Hint: use Fatou).