IERG 6300 Theory of Probability

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Homework 5 Student's Id: ____________

Due Date: 31 March, 2022

The notation P-*a.e.* stands for almost everywhere with respect to the probability distribution P.

Question 1 [5 points]: Equal on a field

Let X, Y be two integrable random variables measurable with respect to $\mathcal{F}$ and let $\mathcal{A}$ be an algebra such that $\mathcal{F} = \sigma(\mathcal{A})$. Suppose we are given that

$$\int_A X dP = \int_A Y dP$$

for all $A \in \mathcal{A}$, the show that $X = Y$, P-*a.e.*

Question 2 [10 points]: Lipschitz continuity

Let $F(x)$ be a distribution function such that $F(0) = 0$ and $F(1) = 1$, let α be the probability measure corresponding to the distribution function. If $F(x)$ satisfies the Lipschitz condition

$$|F(x) - F(y)| \leq A|x - y|,$$

then show that $\alpha \ll m$ where m is the Lebesgue measure on $[0, 1]$. Show also that the Radon-Nikodym derivative $\frac{d\alpha}{dm}$ satisfies

$$0 \leq \frac{d\alpha}{dm} \leq A, \ m\text{-a.e.}$$

Question 3 [10 points]: Chain Rule

Let ν, λ, μ be three non-negative finite measures on the same measurable space $(\Omega, \mathcal{F})$ such that $\nu \ll \lambda$ and $\lambda \ll \mu$, then show that $\nu \ll \mu$ and

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu}, \ \mu\text{-a.e.}$$

Question 4 [13 points]: Uniform absolute continuity

Let $\mathcal{A}$ be a collection of subsets of Ω. Given two non-negative finite measures λ and μ, we say that λ *is uniformly absolutely continuous with respect to a non-negative finite measure* μ *on* $\mathcal{A}$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $A \in \mathcal{A}$ with $\mu(A) < \delta$, we have $\lambda(A) < \epsilon$. On the other hand, given two non-negative finite measures λ and μ, we say that λ *is absolutely continuous with respect to a non-negative finite measure* μ *on* $\mathcal{A}$ if for any $A \in \mathcal{A}$ such that $\mu(A) = 0$, we also have $\lambda(A) = 0$.

(a) [5 points] Show that absolute continuity on σ-algebra $\mathcal{F}$, implies uniform absolute continuity on the σ-algebra $\mathcal{F}$.

(b) [3 points] Show by example that absolute continuity on an algebra $\mathcal{A}$ does not imply absolute continuity on $\sigma(\mathcal{A})$.

(c) [5 points] Show that uniform absolute continuity on an algebra $\mathcal{A}$ implies absolute continuity on $\sigma(\mathcal{A})$.

Question 5 [10 points]: Absolute continuity revisited

Let F_X be a distribution function on the line. Show that its induced measure is absolutely continuous with respect to the Lebesgue measure on the line if and only if for any $\epsilon > 0$ there exists an $\delta > 0$ such that for an arbitrary finite collection of disjoint intervals $I_j = [a_j, b_j]$ with $\sum_j |b_j - a_j| < \delta$, it follows that $\sum_j [F(b_j) - F(a_j)] \leq \epsilon$.

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Question 6 [10 points]: Conditional expectation as a projection

Let $\mathbb{H} = L_2[\Omega, \mathcal{F}, P]$ be the Hilbert space of all $\mathcal{F}$-measurable square integrable functions. Equip $\mathbb{H}$ with an inner product defined according to

$$\langle f, g \rangle_P = \int_\Omega f g dP.$$

Given another σ-algebra $\Sigma \subseteq \mathcal{F}$, let $\mathbb{H}_1 = L_2[\Omega, \Sigma, P]$. Show that $f \mapsto \mathrm{E}(f|\Sigma)$ is an orthogonal projection from $\mathbb{H}$ to $\mathbb{H}_1$, i.e. show that if $g = \mathrm{E}(f|\Sigma)$, then for any other $h \in L_2[\Omega, \Sigma, P]$, we have $\|f - h\| \geq \|f - g\|$, where $\|g\| = \langle g, g \rangle^{\frac{1}{2}}$. Additionally, show that equality holds only if $h = g$ P-$a.e.$.