IERG 6300 Theory of Probability

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Homework 6 Student's Id: ____________

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The notation P-*a.e.* stands for almost everywhere with respect to the probability distribution P.

Question 1 [10 points]: Martingale as a difference of two non-negative martingales

Let X_n be a martingale such that $\sup_n \mathrm{E}(|X_n|) < \infty$. Define, for $n \geq j$,

$$Y_{j,n} = \mathrm{E}(|X_n||\mathcal{F}_j).$$

Show the following:

(a) [3 points] $Y_{j,n}$ is non-decreasing almost surely, i.e. $Y_{j,n+1} \geq Y_{j,n}$ almost surely.

(b) [5 points] Show that there exists a Y_j such that $Y_{j,n} \to Y_j$ almost surely and $\mathrm{E}(|Y_{j,n} - Y_j|) \to 0$. Further, show that Y_j is a martingale.

(c) [2 points] Show that $Y_j + X_j \geq 0$, and hence $X_j = (X_j + Y_j) - Y_j$ is a decomposition of a martingale as a difference of two non-negative martingales.

Question 2 [11 points]: Stopped σ-algebra

Let $\mathcal{F}_n \subset \mathcal{F}$ and $\{\mathcal{F}_n\}$ be a filtration. Let τ be a stopping time adapted to the filtration. Define

$$\mathcal{F}_\tau := \{A : A \in \mathcal{F}, \text{ and } A \cap \{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n, \forall n\}.$$

(a) [3 points] Show that $\mathcal{F}_\tau$ is a σ-algebra.

(b) [2 points] Show that τ is $\mathcal{F}_\tau$ measurable.

(c) [3 points] If $\tau_1 \leq \tau_2$ then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.

(d) [3 points] If stopping times $\tau_n \uparrow \tau$ then

$$\sigma\left(\cup_n \mathcal{F}_{\tau_n}\right) = \mathcal{F}_\tau.$$

Question 3 [13 points]: Asymmetric random walk

Let $X_1, X_2, .., $ be i.i.d., with $\mathrm{P}(X_i = 1) = p$ and $\mathrm{P}(X_i = -1) = 1 - p$, and $p > \frac{1}{2}$. Define $S_n = X_1 + \cdots X_n$, and $\mathcal{F}_n = \sigma(X_1, .., X_n)$.

(a) [2 points] Let $\phi(S_n) := \left(\frac{1-p}{p}\right)^{S_n}$. Show that $\phi(S_n)$ is a Martingale.

(b) [2 points] Let $T_k = \inf\{n : S_n = k\}$. Then for $l < 0 < k$, show that

$$\mathrm{P}(T_k < T_l) = \frac{\phi(0) - \phi(l)}{\phi(k) - \phi(l)}.$$

(c) [3 points] If $l < 0$ then show that $P(T_l < \infty) = \left(\frac{1-p}{p}\right)^{-l}$. If $k > 0$, then show that $P(T_k < \infty) = 1$.

(d) [3 points] If $k > 0$ then $E(T_k) = \frac{k}{2p-1}$. (Hint: Consider $Z_n = S_n - (2p-1)n$)

(e) [3 points] Show that $\mathrm{var}(T_1) = \frac{1-(2p-1)^2}{(2p-1)^3}$. (Hint: Consider $Z_n = (S_n - (2p-1)n)^2 - (1 - (2p-1)^2)n$.)

Question 4 [5 points]: Doob's inequality revisited

Let X_n be a martingale with $X_0 = 0$ and $\mathrm{E}(X_n^2) < \infty$. Show that, for any $\lambda \geq 0$,

$$\mathrm{P}\left(\max_{1 \leq m \leq n} X_m \geq \lambda\right) \leq \frac{\mathrm{E}(X_n^2)}{\mathrm{E}(X_n^2) + \lambda^2}.$$