Question 1 [10 points]: Convergence
Show that $X_n \overset{a.s.}{\to} 0$ if and only if for each $\epsilon > 0$, there is an $n$ such that for every random integer $M$ with $M(\omega) \geq n, \forall \omega \in \Omega$ we have that
$$P(\{\omega : |X_M(\omega)| > \epsilon\}) < \epsilon.$$ 

Question 2 [5 points]: Almost sure convergence
Let $X_1, X_2, \ldots$ be a sequence of random variables such that
$$X_k = \begin{cases} 
3 & \text{with probability } 1 - \frac{1}{k^2} \\
k^2 & \text{with probability } \frac{1}{k^2}
\end{cases}.$$ 
Let $A_n = \frac{X_1 + \cdots + X_n}{n}$. Define
$$C = \{\omega : \lim_n A_n(\omega) \text{ exists}\}.$$ 
Further, let
$$A_\infty = \begin{cases} 
\lim_n A_n & \omega \in C \\
0 & \omega \notin C
\end{cases}.$$ 
(a) [3 points] Determine $P(C)$
(b) [2 points] Determine $E(A_\infty)$. 

NOTE: Describing your ideas can help even if execution is not perfect.
Question 3 [16 points]: Independence

Let $s > 1$ and define $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$. Let $X$ and $Y$ be independent random variables taking values in $\mathbb{N}$ according to

$$P(X = n) = P(Y = n) = \frac{n^{-s}}{\zeta(s)}.$$ 

For $p > 1$, $p$ being a prime number, define the events

$$E_p = \{\omega : X(\omega) \equiv 0 \pmod{p}\},$$

corresponding to $X$ being a multiple of $p$. Let $C = \cup_p E_p$.

(a) [5 points] Prove that the events $\{E_p\}$ are mutually independent.

(b) [2 points] Using $C^c = \cap_p E_p^c$ prove that

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

(c) [4 points] Prove that

$$P(\text{no square other than 1 is a factor of } X) = \frac{1}{\zeta(2s)}.$$ 

(d) [5 points] Prove that

$$P(g.c.d.(X, Y) = n) = \frac{n^{-2s}}{\zeta(2s)}.$$

Question 4 [18 points]: Strong Law

Let $X_1, \ldots, X_n$ be pairwise independent non-negative random variables satisfying $\sup_i E(\phi(X_i)) \leq B < \infty$ for some convex non-negative non-decreasing function $\phi(x)$ satisfying $\lim_{x \to \infty} \frac{\phi(x)}{x} = \infty$.

Further assume that we have $\lim_n \frac{\sum_{k=1}^n E(X_k)}{n} = \mu$ and let $\sum_{k=1}^\infty \frac{1}{\phi(k)} < \infty$. Define $S_n = \sum_{i=1}^n (X_i - E(X_i))$; then $\frac{S_n}{n} \to 0$ a.s.

(Hint: mimic the steps of Etemadi’s proof of SLLN, which is given in the notes, with estimates for the i.i.d. case replaced by the given conditions).

Question 5 [13 points]: Metrizing Weak Convergence

For two distribution functions $F, G$ on $\mathbb{R}$ define

$$d(F, G) = \inf \{\varepsilon > 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon, \forall x\}.$$ 

Show that

(a) $d(F, G)$ is a metric in the space of distribution functions, i.e.

i. [3 points] $d(F, G) \geq 0$ with equality iff $F = G$.

ii. [3 points] $d(F, G) = d(G, F)$.

iii. [3 points] $d(F, G) + d(G, H) \geq d(F, H)$.

(b) [4 points] $F_n \Rightarrow F$ iff $d(F_n, F) \to 0$. 
Question 6 [18 points]: Intersection of \( \Sigma \)-algebras

Let \( X_0, X_1, X_2, \ldots \) be (mutually) independent random variables with \( P(X_k = 1) = P(X_k = -1) = \frac{1}{2} \). For \( n \geq 1 \), define

\[ Y_n = X_0X_1 \ldots X_n. \]

Let \( F = \sigma(X_1, X_2, \ldots) \) and \( G_n = \sigma(\{Y_r\}_{r>n}) \). Thus \( G_\infty = \cap_{n \geq 1} G_n \) is the tail \( \sigma \)-algebra.

(a) [3 points] Show that \( Y_1, Y_2, \ldots \) are mutually independent.

(b) [15 points] Show that

\[ F_1 := \cap_n \sigma(F, G_n) \neq \sigma(F, G_\infty) =: F_2. \]

(Hint: Show that for every \( n \), \( X_0 \) is \( \sigma(F, G_n) \)-measurable, while \( X_0 \) is not \( \sigma(F, G_\infty) \)-measurable. One way is to characterize \( F_2 \). Kolmogorov’s 0-1 law regarding tail \( \sigma \)-algebras may be useful.)