 Student's Id: ____________

Midterm Examination I

Due: March 7, 2022

Question	Points	Score
Convergence	10	
Almost sure convergence	5	
Independence	16	
Strong Law	18	
Metrizing Weak Convergence	13	
Intersection of Σ-algebras	18	
Total:	80	

You are not allowed to use resources and forums outside those given in Piazza for this class or your own notes from this class
Always justify your answers.
NOTE: Describing your ideas can help even if execution is not perfect.

Question 1 [10 points]: Convergence

Show that $X_n \overset{a.s.}{\to} 0$ if and only if for each $\epsilon > 0$, there is an n such that for every random integer M with $M(\omega) \geq n, \forall \omega \in \Omega$ we have that

$$\mathrm{P}(\{\omega : |X_{M(\omega)}(\omega)| > \epsilon\}) < \epsilon.$$

Question 2 [5 points]: Almost sure convergence

Let $X_1, X_2, ..$ be a sequence of random variables such that

$$X_k = \begin{cases} 3 & \text{with probability } 1 - \frac{1}{k^2} \\ k^2 & \text{with probabilty } \frac{1}{k^2} \end{cases}.$$

Let $A_n = \frac{X_1 + \cdots + X_n}{n}$. Define

$$C = \{\omega : \lim_n A_n(\omega) \text{ exists}\}.$$

Further, let

$$A_\infty = \begin{cases} \lim_n A_n & \omega \in C \\ 0 & \omega \notin C \end{cases}.$$

(a) [3 points] Determine $P(C)$

(b) [2 points] Determine $E(A_\infty)$.

Question 3 [16 points]: Independence

Let $s > 1$ and define $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$. Let X and Y be independent random variables taking values in $\mathbb{N}$ according to

$$P(X = n) = P(Y = n) = \frac{n^{-s}}{\zeta(s)}.$$

For $p > 1$, p being a prime number, define the events

$$E_p = \{\omega : X(\omega) \equiv 0 \pmod{p}\},$$

correspoding to X being a multiple of p. Let $C = \cup_p E_p$.

(a) [5 points] Prove that the events $\{E_p\}$ are mutually independent.

(b) [2 points] Using $C^c = \cap_p E_p^c$ prove that

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right).$$

(c) [4 points] Prove that

$$P(\text{no square other than 1 is a factor of} X) = \frac{1}{\zeta(2s)}.$$

(d) [5 points] Prove that

$$P(g.c.d.(X, Y) = n) = \frac{n^{-2s}}{\zeta(2s)}.$$

Question 4 [18 points]: Strong Law

Let $X_1, \ldots, X_n$ be pairwise independent non-negative random variables satisfying $\sup_i E(\phi(X_i)) \leq B < \infty$ for some convex non-negative non-decreasing function $\phi(x)$ satisfying $\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty$. Further assume that we have $\lim_n \frac{\sum_{k=1}^n E(X_k)}{n} = \mu$ and let $\sum_{k=1}^\infty \frac{1}{\phi(k)} < \infty$. Define $S_n = \sum_{i=1}^n (X_i - E(X_i))$; then $\frac{S_n}{n} \to 0$ *a.s.*
(Hint: mimic the steps of Etemadi's proof of SLLN, which is given in the notes, with estimates for the i.i.d. case replaced by the given conditions).

Question 5 [13 points]: Metrizing Weak Convergence

For two distribution functions F, G on $\mathbb{R}$ define

$$d(F, G) = \inf\{\varepsilon > 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon, \forall x\}.$$

Show that

(a) $d(F, G)$ is a metric in the space of distribution functions, i.e.
   i. [3 points] $d(F, G) \geq 0$ with equality *iff* $F = G$.
   ii. [3 points] $d(F, G) = d(G, F)$.
   iii. [3 points] $d(F, G) + d(G, H) \geq d(F, H)$.

(b) [4 points] $F_n \overset{w}{\Rightarrow} F$ *iff* $d(F_n, F) \to 0$.

Question 6 [18 points]: Intersection of Σ-algebras

Let $X_0, X_1, X_2, ...$ be (mutually) independent random variables with $\mathrm{P}(X_k = 1) = \mathrm{P}(X_k = -1) = \frac{1}{2}$. For $n \geq 1$, define

$$Y_n = X_0 X_1 \dots X_n.$$

Let $\mathcal{F} = \sigma(X_1, .X_2, ..)$ and $\mathcal{G}_n = \sigma(\{Y_r\}_{r>n})$. Thus $\mathcal{G}_\infty = \cap_{n \geq 1} \mathcal{G}_n$ is the tail σ-algebra.

(a) [3 points] Show that $Y_1, Y_2, \dots$ are mutually independent.

(b) [15 points] Show that

$$\mathcal{F}_1 := \cap_n \sigma(\mathcal{F}, \mathcal{G}_n) \neq \sigma(\mathcal{F}, \mathcal{G}_\infty) =: \mathcal{F}_2.$$

(Hint: Show that for every n, X_0 is $\sigma(\mathcal{F}, \mathcal{G}_n)$-measurable, while X_0 is not $\sigma(\mathcal{F}, \mathcal{G}_\infty)$ measurable. One way is to characterize $\mathcal{F}_2$. Kolmogorov's 0-1 law regarding tail σ-algebras may be useful.)