Optimization of Some Non-convex Functionals Arising in Information Theory

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The thesis concerns optimization of some non-convex functionals arising in Information Theory. Computation of achievable regions or outer bounds to capacity regions in Information Theory can be formulated as optimization of certain non-convex functional family.

The first part is about evaluating forward hypercontractivity and reverse hypercontractivity region for the pair of variables (X, Y) where X is a uniformly distributed binary random variable and Y (a ternary random variable) is obtained by passing X through a binary erasure channel (BEC), for a non-trivial range of parameters. Our technique uses an equivalent characterization of forward hypercontractivity and reverse hypercontractivity using Kullback-Leibler Divergence, which is in general a non-convex functional optimization problem. A similar analysis also recovers the celebrated results for the pair of variables (X, Y) where X is a uniformly distributed binary random variable and Y (a binary random variable) is obtained by passing X through a binary symmetric channel (BSC), also called the Bonami-Beckner inequality. This optimization problem is also equivalent to the computation of the capacity region for the Gray-Wyner source coding problem of network information theory.

The second part starts from a new non-convex weighted sum rate outer bound for the Körner and Marton's sum modulo two problem. In a seminal work Körner and Marton showed that linear codes achieved the optimal rates and outperformed random coding and binning based arguments. Körner also showed the optimality of Slepian-Wolf based random coding for the same problem for a different class of pairwise distributions. By optimizing over this outer bound, we could show

that the optimal sum rate is given by random linear codes for a larger class of binary distributions, thus extending the known optimality results for this problem. Via using similar ideas, we could derive outer bounds for Quadratic Gaussian Distributed Source Coding Problem and Quadratic Gaussian CEO Problem, and present alternative proofs for the optimality of Berger-Tung inner bound in these two settings.

The third part is related to the non-convex functional $H(Y_t) - \gamma H(X_t)$, where $X_t := X + \sqrt{t}Z$ is in the set of distributions along the heat flow and Y_t is obtained by passing X_t through an additive Gaussian noise channel. We show that if t is re-scaled so that $H(X_t)$ is linear in t, then $H(Y_t)$ is convex in t. This problem is equivalent to showing the log-convexity of Fisher Information, resolving a conjecture in [15] and implicitly in the 1966 paper [41] by McKean.

摘要： 本畢業論文主要考慮的是信息論中出現的一些非凸函數的優化問題。計算信息論中的信道容量的可達到的編碼速率區域或者其外界的問題，能夠轉化為某種特定類型的非凸函數族的優化問題。

第一部分是關於在某些非平凡的參數範圍内，計算forward hypercontractivity還有reverse hypercontractivity的區域，考察對象是一組XY隨機變量：X是二元平均分佈的隨機變量，Y是把X通過一個二元擦除信道（BEC）獲得的三元隨機變量。我們使用的方法是借助於forward hypercontractivity還有reverse hypercontractivity的用Kullback-Leibler Divergence表達的等價描述，將問題轉換為一個非凸函數優化問題。類似的分析也可以得到Bonami-Beckner不等式。這個著名結果考察的對象是一組XY隨機變量：X是二元平均分佈的隨機變量，Y是把X通過一個二元對稱信道（BSC）獲得的二元隨機變量。這個優化問題同時也是等價於網絡信息論中Gray-Wyner源碼壓縮問題的信道容量的計算。

第二部分開始於我們證明出來的一個新的關於Körner還有Marton的模二和的源碼壓縮問題的非凸的加權源碼壓縮率和的外界。Körner和Marton在一片開創性的論文裡面證明了隨機線型碼可以打敗隨機碼和隨機哈希函數的壓縮方法，並且在輸入源分佈是某些分佈的情況下達到了最優的源碼壓縮率。Körner也證明了Slepian-Wolf創造的隨機碼和隨機哈希函數的壓縮方法在輸入源分佈是其他特定分佈的情況下可以達到最優的源碼壓縮率。通過優化我們的新的外界，我們可以證明對於更多的輸入源分佈隨機線型碼可以達到最優的加權源碼壓縮率和，因此擴展了這個問題已知的最優源碼壓縮率的結果。通過使用類似的思路，我們可以推導出適用於二次高斯分布式源碼壓縮問題還有二次高斯CEO源碼壓縮問題的壓縮速率的外界，並且給出這兩種設定下Berger-Tung發明的可達到的源碼壓縮率區域的最優性的另一種證明方法。

第三個部分是關於非凸函數$H(Y_t)-\gamma H(X_t)$，這裏信道輸入$X_t := X+\sqrt{t}Z$的分佈服從滿足熱流方程式的解，信道輸出Y_t則是服從把X_t通過一個加性高斯白噪聲信道獲得的輸出信號的分佈。我們證明了如果把t重新縮放使得信道輸入的熵是t的線型函數，那麼信道輸出的熵就是t的凸函數。這個問題等價於證明Fisher信息的log凸性。我們的這個工作解決了耿艷林和程帆2015發表的論文裡面的一個猜想。這個猜想也隱含地出現在1966年Mckean的論文裡面。

Acknowledgement

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I wish to thank Prof. Shun Watanabe who kindly told us about exercise 16.23 page 390 in Csiszár and Körner's book [20] and hence prompted us to investigate the optimality for other parameter regimes for the Körner and Marton's modulo two sum problem in Chapter 3.

I would like to thank the 2020 ISIT reviewers for asking whether our weighted sum rate lower bound for Körner and Marton's modulo two sum problem could be applied to Gaussian CEO distributed source coding and quadratic Gaussian distributed source coding. This motivated us to study these two problems in Chapter 3.

Lastly, I would like to thank my family for their encouragement and support.

Dedicated to My Family.

Contents

List of Figures

Notations

Mathematics

iff	if and only if
$q \ll p$	absolute continuity of measure q with respect to measure p
i.i.d.	identically and independently distributed
$\oplus$	Minkowski sum
$\mathbb{R}$	real line
$\mathbb{R}^d$	the d-dimensional Euclidean space
$\mathbb{R}^d_+$	the nonnegative orthant of the d-dimensional Euclidean space
$\mathbb{R}^d_{++}$	the strictly positive orthant of the d-dimensional Euclidean space
$\mathbb{N}$	natural number
$\mathbb{N}_+$	positive natural number
$\bar{x}$	$1-x$
$\mathcal{D}$	domain of function
$\mathfrak{C}_x[f]$	the upper concave envelope of the function $f(x)$ over domain $\mathcal{D}$, i.e., $\mathfrak{C}_x[f](x_0) = \inf\{g(x_0) : g(x) \text{ is concave in } x \in \mathcal{D}, g(x) \geq f(x) \forall x \in \mathcal{D}\}$
$\mathfrak{K}_x[f]$	the lower convex envelope of the function $f(x)$ over domain $\mathcal{D}$, i.e., $\mathfrak{K}_x[f](x_0) = \inf\{g(x_0) : g(x) \text{ is convex in } x \in \mathcal{D}, g(x) \leq f(x) \forall x \in \mathcal{D}\}$
$[i : 2^{nR}]$	the set $\{i, i+1, \cdots, 2^{\lceil nR \rceil}\}$, where $\lceil nR \rceil$ is the smallest integer $\geq nR$

$[i : 2^{nR})$	the set $\{i, i+1, \cdots, 2^{\lfloor nR \rfloor}\}$, where $\lfloor nR \rfloor$ is the integer part of nR
$\log_+ x$	$\max\{\log x, 0\}$
$\ln_+ x$	$\max\{\ln x, 0\}$

Probability Theory

$X, Y, \cdots$	scalar random variables
$\mathcal{X}, \mathcal{Y}, \cdots$	the finite sets where the discrete random variables $X, Y, \cdots$ take values from
$x, y, \cdots$	constants or values of scalar random variable
$\lvert\mathcal{X}\rvert, \lvert\mathcal{Y}\rvert, \cdots$	size of the finite sets $\mathcal{X}, \mathcal{Y}, \cdots$
X_i^j	sequence of random variables $(X_i, X_{i+1}, \cdots, X_j)$ with length $j - i + 1$ for $1 \leq i \leq j$
X^j	sequence of random variables $(X_1, X_2, \cdots, X_j)$ with length j for $j \geq 1$
$\mathcal{X} \times \mathcal{Y}$	the Cartesian product of two finite sets $\mathcal{X}$ and $\mathcal{Y}$
$\mathcal{X}^n$	the n-th Cartesian product of the finite set $\mathcal{X}$
$\prod_{i=1}^n \mathcal{X}_i$	$\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n$
p_X	the probability vector $[p_X(x)]_{x \in \mathcal{X}}$ of discrete random variable X indexed by $x \in \mathcal{X}$ with each entry denoted as $p_X(x)$
p_{XY}	the joint probability vector $[p_{XY}(x, y)]_{x \in \mathcal{X}, y \in \mathcal{Y}}$ of discrete random variables (X, Y) indexed by $(x, y) \in \mathcal{X} \times \mathcal{Y}$, with each entry denoted as $p_{XY}(x, y)$
$p_{Y\|X}$	the conditional probability vector $[p_{Y\|X}(y\|x)]_{x \in \mathcal{X}, y \in \mathcal{Y}}$ of discrete random variable X given Y indexed by $(x, y) \in \mathcal{X} \times \mathcal{Y}$, with each entry denoted as $p_{Y\|X}(y\|x)$
$p_{Y\|X=x}$	the conditional probability vector $[p_{Y\|X}(y\|x)]_{y \in \mathcal{Y}}$ of Y given $X = x$, indexed by $y \in \mathcal{Y}$ and each entry denoted as $p_{Y\|X}(y\|x)$
$p_X p_{Y\|X}$	the probability vector $[p_X(x) p_{Y\|X}(y\|x)]_{x \in \mathcal{X}, y \in \mathcal{Y}}$
$p_X p_Y$	the probability vector $[p_X(x) p_Y(y)]_{x \in \mathcal{X}, y \in \mathcal{Y}}$
$p_X^{\otimes n}$	the nth Kronecker product of the probability vector p_X

Information Theory

W	channel
$W_{Y\|X}$	stochastic matrix $[W(y\|x)]_{x\in\mathcal{X},y\in\mathcal{Y}}$ where rows are indexed by $x \in \mathcal{X}$, columns are indexed by $y \in \mathcal{Y}$ and each entry is denoted as $W(y\|x)$
$W_{Y\|X}^{\otimes n}$	the nth Kronecker product of the stochastic matrix $W_{Y\|X}$
$\mathcal{C}$	code
$\mathscr{C}$	capacity for channel coding probelm
$\mathscr{R}$	optimal rate region for source coding problem
$\mathscr{A}$	achievable rate region
BEC(ε)	the binary erasure channel $W_{Y\|X}$ with input $X \in \{0,1\}$ and output $Y \in \{0,E,1\}$, whose conditional probability law of Y given X is given by $W_{Y\|X}(E\|0) = W_{Y\|X}(E\|1) = \varepsilon, W_{Y\|X}(0\|0) = W_{Y\|X}(1\|1) = 1-\varepsilon$, $\varepsilon \in [0,1]$
BSC(ρ)	the binary symmetric channel $W_{Y\|X}$ with input $X \in \{0,1\}$ and output $Y \in \{0,1\}$, whose conditional probability law of Y given X is given by $W_{Y\|X}(0\|0) = W_{Y\|X}(1\|1) = \frac{1+\rho}{2}, W_{Y\|X}(1\|0) = W_{Y\|X}(0\|1) = \frac{1-\rho}{2}$, $\rho \in [-1,1]$
$p_{XY}^{BEC(\varepsilon)}$	X is binary and uniformly distributed, and Y is obtained by passing X through a binary erasure channel $BEC(\varepsilon)$ to produce Y. The joint distribution of (X,Y) will be denoted as $p_{XY}^{BEC(\varepsilon)}$
$p_{XY}^{BSC(\rho)}$	X is binary and uniformly distributed, and Y is obtained by passing X through a binary symmetric channel $BSC(\rho)$ to produce Y. The joint distribution of (X,Y) will be denoted as $p_{XY}^{BSC(\rho)}$
DSBS	Doubly Symmetric Binary Source. The joint distribution of (X,Y) follows $p_{XY}^{BSC(\rho)}$

BISO($\vec{p}$)	X is binary and uniformly distributed, and Y is obtained via a channel $W_{Y\|X}$ that satisfies a symmetry property, $W_{Y\|X}(i\|1) = W_{Y\|X}(-i\|0) = p_i$, for integer $i \in [-K : K]$. The joint distribution of (X, Y) will be denoted as BISO($\vec{p}$), where $\vec{p} = [p_i], -K \leq i \leq K$
$H_2(x)$	the binary entropy function $H_2(x) := -x \log x - \bar{x} \log \bar{x}$
H_2^{-1}	the inverse of binary entropy function $H_2^{-1} : [0, 1] \mapsto [0, \frac{1}{2}]$
$H(X)$	the entropy of a discrete random variable X taking values from a finite set $\mathcal{X}$, $H(X) := -\sum_{x \in \mathcal{X}} p_X(x) \log p_X(x)$
$h(X)$	the differential entropy of a random variable X taking values from $\mathbb{R}$, $h(X) = -\int_{\mathbb{R}} f(x) \ln f(x) dx$

Chapter 1

Introduction

Shannon's seminal work [57] laid down the theoretical foundations of information theory. The idea of the point-to-point communications problem and his source and channel coding theorems have many profound implications in areas like wireless communications and data compression. Network information theory, on the other hand, focuses on the limits of reliable communication over a network with multiple senders and receivers, and some channel transition matrix that models the effects of the interference and noise in the network. One fundamental problem is to determine whether certain communication strategies could achieve the limit of reliable communication, which could be reduced to testing certain properties of some non-convex information-theoretic functionals. This thesis will focus on such non-convex functionals.

Let X be a *discrete random variable* that takes values from some finite set $\mathcal{X}$, with the probability mass function denoted by p_X. The *entropy* of a discrete random variable X, $H(X)$, is defined as

$$H(X) := \sum_{x \in \mathcal{X}} p_X(x) \log p_X(x)$$

where the logarithm is base 2.

Given a vector of discrete random variables $X^n := (X_1, \cdots, X_n)$ taking values from some finite set $\otimes_{i=1}^n \mathcal{X}_i$, and for any $\vec{d} := (d_{x^n} : x^n \in \otimes_{i=1}^n \mathcal{X}_i)$, which is an arbitrary real-valued vector, we are interested in computing the following function $G(\vec{d})$:

$$G(\vec{d}) := \max_{p_{X^n}} \left(\sum_{S \subset [1:n]} \alpha_S H(X_S) - E_{p_{X^n}}(\vec{d}) \right) \tag{1.1}$$

where S is a subset of $[1:n]$, X_S denotes the set $\{X_i : i \in S\}$, and $\alpha_S \in \mathbb{R}$ depends on S.

Here $E_{p_{X^n}}(\vec{d}) = \sum_{x^n \in \otimes_{i=1}^n \mathcal{X}_i} p_{X^n}(x^n) d_{x^n}$. Computing $G(\vec{d})$ requires evaluating the global maximizer over p_{X^n} of the functional $\sum_{S \subset [1:n]} \alpha_S H(X_S) - E_{p_{X^n}}(\vec{d})$, which can, in general, be non-convex.

The evaluation of certain achievable rate regions or bounds to the capacity region canonically involves functionals of the above form (as will be made clear in the rest of this chapter). Further, optimality of certain achievable regions can also be cast in the language of properties of the maximizers of the above functionals. For instance, if the global optimizers of a natural extension of a functional (corresponding to an achievable rate region) defined on so-called product-spaces are product distributions, then the achievable rate regions can be shown to be optimal in many settings.

The rest of this chapter will try to illustrate how the above family of functionals arise in communication settings, as well as elucidate some of the key questions related to the functionals that are of interest.

1.1 Some Communication Models

1.1.1 Point-to-point channel coding

In the celebrated work [57], the point-to-point communication model was first proposed by Shannon. Figure 1.1 depicts this model, where a sender wishes to communicate reliably with a receiver through certain channel. We are interested in maximizing the amount of the information that can be reliably transmitted from the sender to the receiver.

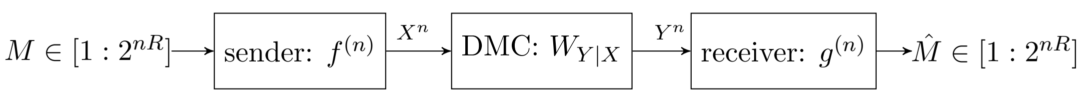

Figure 1.1: Point-to-point communication channel model

More specifically, a *channel*, denoted by W, is a stochastic mapping from $\mathcal{X}$ to $\mathcal{Y}$ that will output symbol $y \in \mathcal{Y}$ given some input symbol $x \in \mathcal{X}$ with certain probability. When both $\mathcal{X}$ and $\mathcal{Y}$ are finite, the channel is called a *discrete channel*.

For $n \in \mathbb{N}_+$, the n *uses of a discrete channel* is defined as the stochastic mapping from $\mathcal{X}^n$ to $\mathcal{Y}^n$ specified by a stochastic matrix $W_{Y^n|X^n}$, where $\mathcal{X}^n$ and $\mathcal{Y}^n$ are the n-th Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$ respectively, and $W_{Y^n|X^n}$ is the stochastic matrix where rows are indexed by elements in $\mathcal{X}^n$, columns are indexed by elements in $\mathcal{Y}^n$ and each entry $W_{Y^n|X^n}(y^n|x^n)$ is the conditional probability that the channel outputs y^n given certain input x^n. When $n = 1$, we will simply write the stochastic matrix $W_{Y^1|X^1}$ as $W_{Y|X}$.

A discrete channel is called a *discrete memoryless channel* (DMC), if for any $n \in \mathbb{N}_+$ the stochastic matrix $W_{Y^n|X^n}$ of the n uses of the channel is the nth tensor product of the stochastic matrix $W_{Y|X}$ of the channel. We will use $W_{Y|X}$ to denote a DMC omitting the input set $\mathcal{X}$ and the output set $\mathcal{Y}$ if there is no danger of confusion, and the n uses of a DMC will be denoted as $W_{Y|X}^{\otimes n}$.

Let $R \in \mathbb{R}_+$and $n \in \mathbb{N}_+$, a (n, R) *code* for a DMC $W_{Y|X}$ is defined as the function pair $(f^{(n)}, g^{(n)})$, where $f^{(n)}$ is some mapping from $[1 : 2^{nR}]$ to $\mathcal{X}^n$ called encoding function, and $g^{(n)}$ is some mapping from $\mathcal{Y}^n$ to $[1 : 2^{nR}]$ called decoding function. Here R is called the rate of the code, and n is called the block-length of the code.

In the model shown in Figure 1.1, the channel is a DMC $W_{Y|X}$ and a (n, R) code $\mathcal{C}$ is applied in the communication: the message M is distributed uniformly in the set $[1 : 2^{nR}]$. The sender will map the generated message M to a sequence X^n by the encoding function $f^{(n)}$, and pass the X^n to the receiver through the n uses of a DMC $W_{Y|X}$. The receiver maps the output sequence Y^n back to some estimation of message M, $\hat{M} \in [1 : 2^{nR}]$, by the decoding function $g^{(n)}$.

One way to measure the performance of this (n, R) code for the DMC $W(Y|X)$ is to compute the average error probability that $M \neq \hat{M}$, defined as $P_e(\mathcal{C}, W_{Y|X}) := P(M \neq \hat{M})$.

A rate R is *achievable* for a DMC $W_{Y|X}$ if there exists a sequence of (n, R) codes $\mathcal{C}_n$ such that $\lim_{n\to\infty} P_e(\mathcal{C}_n, W_{Y|X}) = 0$. The capacity of a DMC $W_{Y|X}$, denoted as $\mathscr{C}(W_{Y|X})$, is defined as the closure of the set of all achievable rates. Intuitively speaking, $\mathscr{C}(W_{Y|X})$ measures how much information can be transmitted reliably from the sender to the receiver.

In [57], Shannon used the mutual information $I(X; Y)$ between two random variables X, Y to express the capacity for a DMC:

Theorem 1.1. *The capacity of a DMC $W_{Y|X}$ is given by*

$$\mathscr{C}(W_{Y|X}) = \{R \geq 0 : R \leq \max_{p_X} I(X;Y)\}. \tag{1.2}$$

where $I(X;Y) := \sum_{x\in\mathcal{X},y\in\mathcal{Y}} p_{XY}(x,y) \log \frac{p_{XY}(x,y)}{p_X(x)p_Y(y)}$.

Remark 1.1. Notice that $I(X;Y)$ is a concave function in p_X, so $\mathscr{C}(W_{Y|X})$ can be computed directly from the stochastic matrix $W_{Y|X}$ of the DMC. Such a characterization of the capacity region, without involving multiple uses of the channel W, eliminates the computation difficulty in finding the limit when $n \to \infty$ and is informally called the *single-letter characterization* of the capacity region.

One can use random coding and joint typicality decoding to prove that when $R < \max_{p_X} I(X;Y)$, there exists a sequence of (n,R) codes $\mathcal{C}_n$ such that $\lim_{n\to\infty} \mathrm{P}_e(\mathcal{C}_n, W_{Y|X}) = 0$. In this case, we say that the set $\{R \geq 0 : R < \max_{p_X} I(X;Y)\}$ is an *achievable rate region* for the DMC $W_{Y|X}$, denoted as $\mathscr{A}(W_{Y|X})$.

When the capacity $\mathscr{C}(W_{Y|X})$ matches the closure of the achievable rate region $\mathscr{A}(W_{Y|X})$, we will say that the achievable rate region $\mathscr{A}(W_{Y|X})$ *is optimal.*

1.1.2 Multiple Access Channel Coding

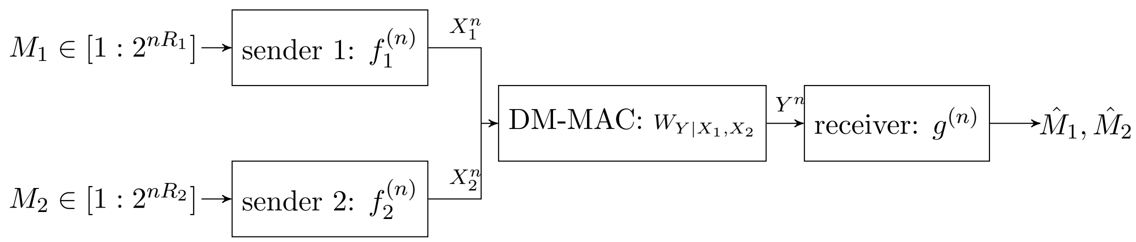

Figure 1.2: Multiple access channel coding

A natural extension for the point-to-point communication model is a multiple access communication model shown in Figure 1.2, where each sender wishes to transmit an independent messages reliably to the receiver. This is first alluded to in Shannon's paper [56].

Similarly to the point-to-point channel coding, one could define a *discrete memoryless multiple access channel* (DM-MAC) $W_{Y|X_1,X_2}$ and a (n,R_1,R_2) code $\mathcal{C} := (f_1^{(n)}, f_2^{(n)}, g^{(n)})$ for this multiple access communication model. Figure 1.2 shows how a (n,R_1,R_2) code $\mathcal{C}$ is applied in the communication over a DM-MAC $W_{Y|X_1,X_2}$.

To measure the performance of the code $\mathcal{C}$ for this DM-MAC $W_{Y|X_1,X_2}$, the average probability of error $P_e(\mathcal{C}, W_{Y|X_1,X_2}) := P((M_1, M_2) \neq (\hat{M}_1, \hat{M}_2))$ is employed. A rate pair (R_1, R_2) is achievable for a DM-MAC $W_{Y|X_1,X_2}$ if there exists a sequence of (n, R_1, R_2) code $\mathcal{C}_n$ such that $\lim_{n\to\infty} P_e(\mathcal{C}_n, W_{Y|X_1,X_2}) = 0$. The capacity of a DM-MAC $W_{Y|X_1,X_2}$ is defined as the closure of the set of all achievable rate pairs (R_1, R_2) for this DM-MAC, denoted as $\mathscr{C}(W_{Y|X_1,X_2})$.

Ahlswede [3], [1] and Liao [39] established a single-letter characterization for $\mathscr{C}(W_{Y|X_1,X_2})$.

Theorem 1.2. *The capacity region of the DM-MAC $W_{Y|X_1,X_2}$ is the set of rate pairs (R_1, R_2) satisfying*

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2, Q) \\ R_2 &\leq I(X_2; Y|X_1, Q) \\ R_1 + R_2 &\leq I(X_1, X_2; Y|Q) \end{aligned} \tag{1.3}$$

for some probability mass function (pmf) $p_Q p_{X_1|Q} p_{X_2|Q}$, where $|\mathcal{Q}| \leq 2$.

Remark 1.2. Notice that the Q in this theorem 1.2 doesn't appear in the original communication setting, but is needed in making the rate region convex. Such random variable is called *auxiliary random variable.*

Denote the achievable rate region for a DM-MAC $W_{Y|X_1,X_2}$, i.e., the interior of the set of rate pairs (R_1, R_2) satisfying inequalities (1.3), as $\mathscr{A}(W_{Y|X_1,X_2})$.

1.1.3 Broadcast Channel Coding

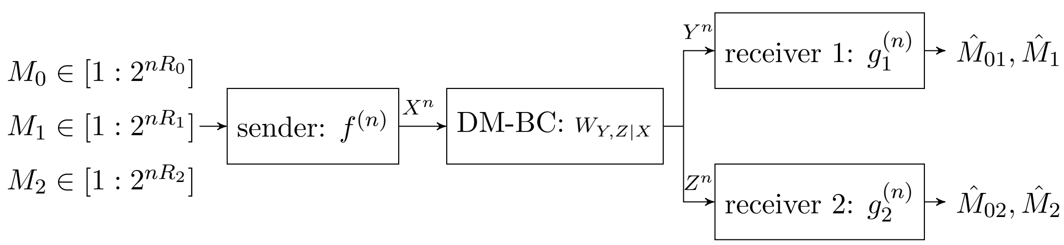

Figure 1.3: Broadcast channel coding

As another natural extension for the point-to-point channel coding, a broadcast channel coding is shown in figure 1.3, where one sender wishes to transmit two private messages to each receiver and a common message to both receivers. This communication setting was first introduced by Cover in [18].

Similarly to the point-to-point channel coding, one could define a *discrete memoryless broadcast channel* (DM-BC) $W_{Y,Z|X}$ and a (n, R_0, R_1, R_2) code $\mathcal{C} := (f^{(n)}, g_1^{(n)}, g_2^{(n)})$ for broadcast channel coding. Figure 1.2 shows how a (n, R_0, R_1, R_2) code $\mathcal{C}$ is applied in the communication over a DM-BC $W_{Y,Z|X}$.

We employ the average probability of error criterion $P_e(\mathcal{C}, W_{Y,Z|X}) := P((M_0, M_1) \neq (\hat{M}_{01}, \hat{M}_1) \text{ or } (M_0, M_2) \neq (\hat{M}_{02}, \hat{M}_2))$ to measure the "reliability" of a code. A rate tuple (R_0, R_1, R_2) is achievable for a DM-BC $W_{Y,Z|X}$ if there exists a sequence of (n, R_0, R_1, R_2) code $\mathcal{C}_n$ such that $\lim_{n\to\infty} P_e(\mathcal{C}_n, W_{Y,Z|X}) = 0$. The capacity of a DM-BC $W_{Y,Z|X}$ is defined as the closure of the set of all achievable rate pairs (R_0, R_1, R_2) for this DM-BC, denoted as $\mathscr{C}(W_{Y,Z|X})$.

In 1979 [40], Marton used the idea of multicoding and joint typicality encoding to give an achievable rate region for a DM-BC $W_{Y,Z|X}$:

Theorem 1.3 (Marton's inner bound)**.** *A rate tuple* (R_0, R_1, R_2) *is achievable for a DM-BC* $W_{Y,Z|X}$ *if*

$$\begin{aligned} R_0 &< \min\{I(W;Y), I(W;Z)\}, \\ R_0 + R_1 &< I(W,U;Y), \\ R_0 + R_2 &< I(W,V;Z), \\ R_0 + R_1 + R_2 &< \min\{I(W;Y), I(W;Z)\} + I(U;Y|W) + I(V;Z|W) - I(U;V|W) \end{aligned} \tag{1.4}$$

for some pmf p_{UVW} *and function* $x(u,v,w)$*, where* $|\mathcal{W}| \leq |\mathcal{X}|+4$*,* $|\mathcal{U}| \leq |\mathcal{X}|$*,* $|\mathcal{V}| \leq |\mathcal{X}|$*.*

Remark 1.3. The cardinality bounds on the region was determined in [29].

Denote this achievable rate region by $\mathscr{A}(W_{Y,Z|X})$.

Open Question: Is $\mathscr{A}(W_{Y,Z|X}) = \mathscr{C}(W_{Y,Z|X})$ for all $W_{Y,Z|X}$?

1.1.4 Gray-Wyner Source Coding Setting

Another fundamental setting in information theory is communication of uncompressed sources over multiple noiseless channels, and we are interested in how much can be compressed by encoding the sources separately.

Consider a *2-component discrete memoryless source* (2-DMS), (X, Y), which is defined as the source pair that generates an i.i.d. sequence of random vari-

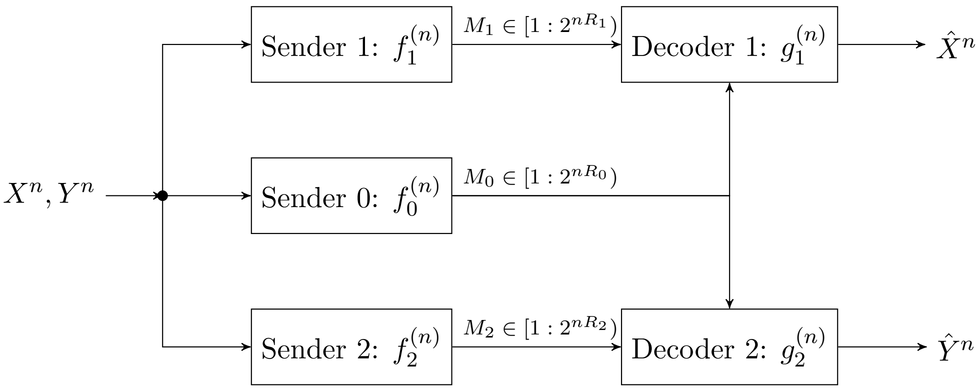

Figure 1.4: Gray-Wyner Source Coding Setting

able pairs (X_i, Y_i) from some finite set $\mathcal{X} \times \mathcal{Y}$ according to some joint distribution p_{XY}. One distributed lossless source coding problem on a 2-DMS (X, Y) following distribution p_{XY}, called Gray-Wyner source coding, is shown in Figure 1.4. Similar to previous cases, one could define a (n, R_0, R_1, R_2) code $\mathcal{C} := (f_0^{(n)}, f_1^{(n)}, f_2^{(n)}, g_1^{(n)}, g_2^{(n)})$ for this setup. The average probability of error $P_e(\mathcal{C}, p_{XY}) := P(X^n \neq \hat{X}^n \text{ or } Y^n \neq \hat{Y}^n)$ is used to measure the code performance for this 2-DMS. A rate tuple (R_0, R_1, R_2) is achievable for a 2-DMS if there exists a sequence of (n, R_0, R_1, R_2) code $\mathcal{C}_n$ such that $\lim_{n\to\infty} P_e(\mathcal{C}_n, p_{XY}) = 0$. The optimal rate region of Gray-Wyner source coding on a 2-DMS (X, Y) is defined as the closure of the set of all achievable rate tuples (R_0, R_1, R_2), denoted as $\mathscr{R}(p_{XY})$.

Gray and Wyner in [30] gave a single-letter characterization for $\mathscr{R}(p_{XY})$:

Theorem 1.4. *The optimal rate region $\mathscr{R}(p_{XY})$ for the Gray-Wyner source coding with 2-DMS (X, Y) is the set of rate triplets (R_0, R_1, R_2) such that*

$$\begin{aligned} R_0 &\geq I(X, Y; V), \\ R_1 &\geq H(X|V), \\ R_2 &\geq H(Y|V) \end{aligned} \tag{1.5}$$

for some conditional pmf $p_{V|XY}$ with $|\mathcal{V}| \leq |\mathcal{X}||\mathcal{Y}| + 2$.

Denote the interior of the set of (R_0, R_1, R_2) satisfying equations 1.5 as $\mathscr{A}_{GW}(p_{XY})$.

1.1.5 Lossless Source Coding with One Helper

Consider the distributed source coding problem depicted in Figure 1.5, where two senders separately encode two correlated sources into two indexes, and transmit the indexes to the receiver so that one of the sources can be reconstructed losslessly at the receiver.

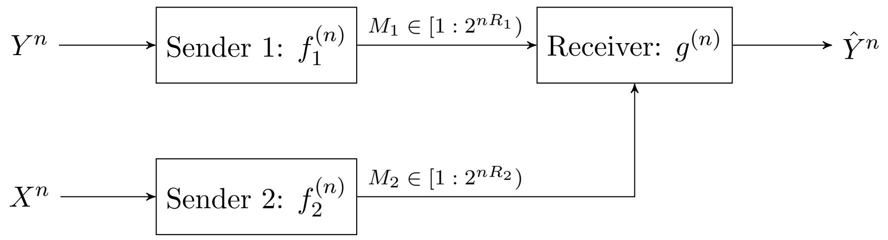

Figure 1.5: Lossless source coding with one helper

Similar to Gray-Wyner source coding, the source is a 2-DMS (X, Y) following distribution p_{XY}. And one could define a (n, R_1, R_2) *distributed source code* $\mathcal{C} := (f_1^{(n)}, f_2^{(n)}, g^{(n)})$ for this setup.

Since the receiver aims to reconstruct sequence Y^n losslessly, the average probability of error $P_e(\mathcal{C}, p_{XY}) := P\left(\hat{Y}^n \neq Y^n\right)$ is used to measure the performance of the distributed source code $\mathcal{C}$. A rate pair (R_1, R_2) is achievable for a 2-DMS (X, Y) if there exists a sequence of (n, R_1, R_2) distributed source codes $\mathcal{C}_n$ such that $\lim_{n\to\infty} P_e(\mathcal{C}, p_{XY}) = 0$. The optimal rate region for loseless source coding of X with one helper observing Y is defined as the closure of the set of all achievable rate pairs (R_1, R_2), denoted as $\mathscr{R}(p_{XY})$.

Ahlswede and Körner [4] and Wyner [69] independently established the following singlet characterization:

Theorem 1.5. *Let (X, Y) be a 2-DMS following distribution p_{XY}. The optimal rate region $\mathscr{R}(p_{XY})$ for loseless source coding of Y with a helper observing X is the set of rate pairs (R_1, R_2) such that*

$$\begin{aligned} R_1 \geq & H(Y|U), \\ R_2 \geq & I(U; X) \end{aligned} \tag{1.6}$$

for some conditional pmf $p_{U|X}$, where $|\mathcal{U}| \leq |\mathcal{X}| + 1$.

Denote the interior of the set of rate pairs (R_1, R_2) satisfying inequalities (1.6) as $\mathscr{A}(p_{XY})$.

1.1.6 Lossless Source Coding with Two Helpers

A natural question after lossless source coding with one helper is to consider a distributed source coding network with two helpers, see [35], where three senders separately encode three correlated sources into three indexes, and transmit the indexes to the receiver so that one of the sources can be reconstructed losslessly at the receiver.

In Figure 1.6, the source is a 3-DMS (X, Y, Z) following distribution p_{XYZ}. And one could define a (n, R_0, R_1, R_2) distributed source code $\mathcal{C} := (f_0^{(n)}, f_1^{(n)}, f_2^{(n)}, g^{(n)})$ for this setup.

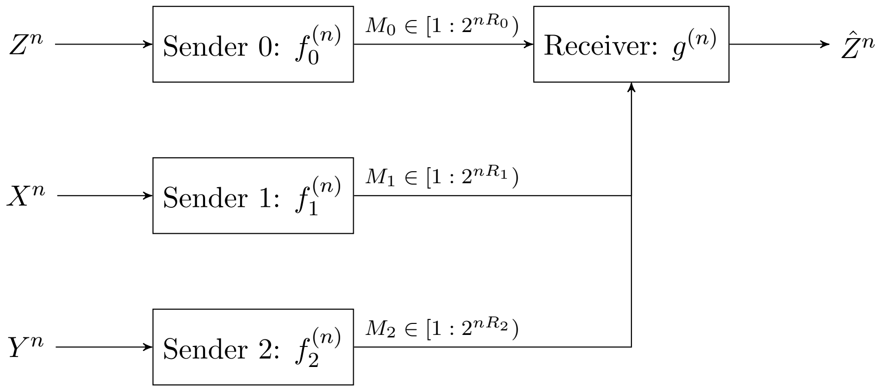

Figure 1.6: Lossless source coding with two helpers

The average probability of error $P_e(\mathcal{C}, p_{XYZ}) := P\left(\hat{Z}^n \neq Z^n\right)$ is used to measure the performance of the distributed source code $\mathcal{C}$. A rate pair (R_0, R_1, R_2) is achievable for a 3-DMS (X, Y, Z) if there exists a sequence of (n, R_0, R_1, R_2) distributed source codes $\mathcal{C}_n$ such that $\lim_{n\to\infty} P_e(\mathcal{C}, p_{XYZ}) = 0$. The optimal rate region for loseless source coding of Z with two helpers observing Y and X is defined as the closure of the set of all achievable rate pairs (R_0, R_1, R_2), denoted as $\mathscr{R}(p_{XYZ})$.

The single-letter characterization of $\mathscr{R}(p_{XYZ})$ is unknown in general. In this thesis, we will consider the projection of $\mathscr{R}(p_{XYZ})$ onto the subspace where $R_0 = 0$, that is, sender 0 is not allowed to send information on Z^n to receiver.

In [58], a remarkable result by Slepian and Wolf showed that when $Z = (X, Y)$ random binning ideas can be used to achieve the following rate region:

$$\begin{aligned} R_1 &\geq H(X|Y) \\ R_2 &\geq H(Y|X) \end{aligned} \tag{1.7}$$

$$R_1 + R_2 \geq H(XY)$$

and hence this becomes an achievable region for any function $f(X,Y)$. We shall call this region the *Slepian-Wolf region.* Random coding and random binning ideas were used subsequently for many network information theory problems to yield the capacity results and still drives most of the achievable regions studied in the community.

Körner and Marton considered the case when (X,Y) follows from the DSBS distribution, and investigated the capacity region when $Z = X \oplus Y$, i.e. the receiver wishes to compute the bit-wise modulo-two sum of the sequences X^n, Y^n, which we will refer to as *the Körner and Marton's modulo two sum problem.* And we will use $\mathscr{R}_{KM}(p_{XY})$ to denote the optimal rate region for this problem. In particular they showed that linear codes can be used to achieve the rate region:

$$\begin{aligned} R_1 &\geq H(Z) \\ R_2 &\geq H(Z) \end{aligned} \tag{1.8}$$

and further that this matches the capacity region when $p(x,y)$ is DSBS distribution. We shall call this region the *Körner-Marton region.* For any $\rho \neq 0$ it is immediate that the above region is strictly larger than the region given by (1.7). Thus it became apparent that random coding ideas had its limitations and structured codes were needed for multiuser information theory problems. This has then led to development of lattice codes, coset codes, and other ideas that have spurred a sub-field of algebraic network information theory.

In 1982, Ahlswede and Han [2] combined both the coding schemes above and obtained the following achievable rate region:

Theorem 1.6 (Ahlswede and Han [1]). *A rate pair (R_1, R_2) is achievable if*

$$\begin{aligned} R_1 &\geq I(U;X|V) + H(Z|UV) \\ R_2 &\geq I(V;Y|U) + H(Z|UV) \\ R_1 + R_2 &\geq I(UV;XY) + 2H(Z|UV) \end{aligned} \tag{1.9}$$

for some U and V that satisfy the Markov chain $U \to X \to Y \to V$.

Remark 1.4. The following remarks are worth noting.

1. Observe that when $U = X, V = Y$, above rate region reduces to Slepian-Wolf's rate region; and when U, V are constant random variables, it's reduced to Körner-Marton's rate region obtained using linear codes.
2. The multi-letter extensions of the above region tends to the capacity region. To see this, set $U = M_1$ and $V = M_2$ and apply Fano's inequality.
3. The above rate region remains achievable (and multi-letter extension tends to capacity) even if we assume that X, Y take some values in a finite field and Z is the modulo-sum in the field. See for instance Lemma 5 in [32].
4. It has been conjectured in [55], and verified by numerical simulations by different groups of researchers, that the smallest sum-rate yielded by the above region is indeed the minimum of $\{H(XY), 2H(Z)\}$, i.e. the minimum of the Slepian-Wolf region and the Körner-Marton region.
5. It is also known that for weighted sum-rate the region is strictly larger than the convex hull of the Slepian-Wolf region and the Körner-Marton region

Denote the interior of the set of rate pairs (R_1, R_2) satisfying inequalities (1.9) as $\mathscr{A}_{AH}(p_{XY})$.

1.2 Evaluation of Achievable Region

1.2.1 Testing Optimality by Weighted Sum Rate

For the channel coding problem mentioned in the last section 1.1, one could also apply the coding strategies in the achievablity proof directly over the n uses of the channel, and get an achievable rate region for $W_{Y|X}^{\otimes n}$, which is called *n-letter* achievable rate region, denoted as $\mathscr{A}(W_{Y|X}^{\otimes n})$. On the contrary, $\mathscr{A}(W_{Y|X})$ will be referred to as *single-letter* achievable rate region.

It's well-known that testing the optimality of $\mathscr{A}(W)$ is equivalent to comparing the 2-letter achievable rate region with the Minkowski sum of two single-letter achievable rate region, see Lemma 1 in [70]:

Lemma 1.1 (Lemma 1 in [70]). *An achievable rate region $\mathscr{A}(W)$ is optimal for some channel coding problem iff*

$$\mathscr{A}(W^{\otimes 2}) = \mathscr{A}(W) \oplus \mathscr{A}(W) \ \forall \ W \tag{1.10}$$

Due to the time sharing technique, the sets on both sides of equation (1.10) are convex sets. one way to compare them is by comparing their supporting hyperplanes, i.e., the maximized weighed sum rate for a given vector $\vec{\gamma} \in \mathbb{R}_+^d$:

$$S_{\vec{\gamma}\in\mathbb{R}_+^d}(W) := \sup_{(R_1,\cdots,R_d)\in\mathscr{A}(W)} \sum_{i=1}^{d} \gamma_i R_i$$

$$S_{\vec{\gamma}\in\mathbb{R}_+^d}(W^{\otimes 2}) := \sup_{(R_1,\cdots,R_d)\in\mathscr{A}(W^{\otimes 2})} \sum_{i=1}^{d} \gamma_i R_i$$

Equality (1.10) is equivalent to

$$S_{\vec{\gamma}\in\mathbb{R}_+^d}(W^{\otimes 2}) = 2S_{\vec{\gamma}\in\mathbb{R}_+^d}(W) \ \forall\ W, \vec{\gamma} \in \mathbb{R}_+^d \tag{1.11}$$

For a channel coding problem on W, notice that if $(R_1, \cdots, R_d) \in \mathscr{A}(W)$, then $(2R_1, \cdots, 2R_d) \in \mathscr{A}(W^{\otimes 2})$, thus the direction $S_{\vec{\gamma}\in\mathbb{R}_+^d}(W^{\otimes 2}) \geq 2S_{\vec{\gamma}\in\mathbb{R}_+^d}(W)$ always hold for any channel W and vector $\vec{\gamma} \in \mathbb{R}_+^d$. Therefore, the optimality of $\mathscr{A}(W)$ is equivalent to

$$S_{\vec{\gamma}\in\mathbb{R}_+^d}(W^{\otimes 2}) \leq 2S_{\vec{\gamma}\in\mathbb{R}_+^d}(W) \ \forall\ W, \vec{\gamma} \in \mathbb{R}_+^d \tag{1.12}$$

Similar ideas and proofs naturally extends to distributed source coding problems in section 1.1, except that finding the supporting hyperplanes for the achievable rate region in distributed source coding becomes a minimization problem.

The optimality of certain achievable rate regions for distributed source coding problem on a DMS following distribution p, is equivalent to

$$S_{\vec{\gamma}\in\mathbb{R}_+^d}(p^{\otimes 2}) \geq 2S_{\vec{\gamma}\in\mathbb{R}_+^d}(p) \ \forall\ p, \vec{\gamma} \in \mathbb{R}_+^d \tag{1.13}$$

where

$$S_{\vec{\gamma}\in\mathbb{R}_+^d}(p) := \inf_{(R_1,\cdots,R_d)\in\mathscr{A}(p)} \sum_{i=1}^{d} \gamma_i R_i$$

$$S_{\vec{\gamma}\in\mathbb{R}_+^d}(p^{\otimes 2}) := \inf_{(R_1,\cdots,R_d)\in\mathscr{A}(p^{\otimes 2})} \sum_{i=1}^{d} \gamma_i R_i$$

1.2.2 Reducing to Non-convex Functional Family

Observe that one difference between $S_{\vec{\gamma}\in\mathbb{R}_+^d}(W)$ (or $S_{\vec{\gamma}\in\mathbb{R}_+^d}(p)$) and the non-convex functional family (1.1), is that in the non-convex functional family (1.1), there is no constraint on the distribution p_{X^n}; while for $S_{\vec{\gamma}\in\mathbb{R}_+^d}(W)$, the conditional probability of the output random variables given the inputs is fixed by the channel law, and for $S_{\vec{\gamma}\in\mathbb{R}^d_+}(p)$, the joint distribution of DMS random variables must be consistent with the given DMS distribution p.

However, by introducing penalty terms, one could show that $S_{\vec{\gamma}\in\mathbb{R}^d_+}(W)$ and $S_{\vec{\gamma}\in\mathbb{R}^d_+}(p)$ falls into the limiting case of the non-convex functional family (1.1). Take the point-to-point channel coding for instance, the weighted sum rate sub-additive inequality (1.12) simplifies to

$$\max_{p_{X_1X_2}} I(X_1X_2;Y_1Y_2) \leq 2\max_{p_X} I(X;Y)\ \forall\ W_{Y|X} \tag{1.14}$$

Here the conditional distribution $P_{Y|X} = W_{Y|X}$ imposes a constraint on the joint distribution p_{XY}. To show both sides of above equality (1.14) falls into the limiting case of the non-convex functional family (1.1), we will introduce the penalty term in terms of divergence:

$$\begin{aligned}
&D(p_{Y|X=x}||W_{Y|X=x}) := \sum_{y\in\mathcal{Y}} p_{Y|X}(y|x)\log\frac{p_{Y|X}(y|x)}{W_{Y|X}(y|x)},\\
&D(p_{Y_1Y_2|X_1X_2=x_1x_2}||W_{Y|X=x_1}\otimes W_{Y|X=x_2})\\
&:= \sum_{y_1y_2\in\mathcal{Y}^2} p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2)\log\frac{p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2)}{W_{Y|X=x_1}\otimes W_{Y|X=x_2}}.
\end{aligned}$$

Let $c > 0$, observe that the right hand side of equation (1.14) can be rewritten as

$$\begin{aligned}
\max_{p_X} I(X;Y) &= \max_{p_{XY}}\lim_{c\to\infty} I(X;Y) - c\sum_{x\in\mathcal{X}} p_X(x)D(p_{Y|X=x}||W_{Y|X=x})\\
&\overset{(a)}{=} \lim_{c\to\infty}\max_{p_{XY}} I(X;Y) - c\sum_{x\in\mathcal{X}} p_X(x)D(p_{Y|X=x}||W_{Y|X=x})\\
&= \lim_{c\to\infty}\max_{p_{XY}} H(Y) + (c-1)H(Y|X) + cE[\log W_{Y|X}]
\end{aligned}$$

and the left hand side of equation (1.14) can be rewritten as

$$\begin{aligned}
&\max_{p_{X_1X_2}} I(X_1X_2;Y_1Y_2)\\
=&\max_{p_{X_1X_2Y_1Y_2}}\lim_{c\to\infty} I(X_1X_2;Y_1Y_2)\\
&-c\sum_{(x_1,x_2)\in\mathcal{X}^2} p_{X_1X_2}(x_1x_2)D(p_{Y_1Y_2|X_1X_2=x_1x_2}||W_{Y|X=x_1}\otimes W_{Y|X=x_2})\\
\overset{(a)}{=}&\lim_{c\to\infty}\max_{p_{X_1X_2Y_1Y_2}} I(X_1X_2;Y_1Y_2)\\
&-c\sum_{(x_1,x_2)\in\mathcal{X}^2} p_{X_1X_2}(x_1x_2)D(p_{Y_1Y_2|X_1X_2=x_1x_2}||W_{Y|X=x_1}\otimes W_{Y|X=x_2})
\end{aligned}$$

$$= \lim_{c\to\infty} \max_{p_{XY}} H(Y_1Y_2) + (c-1)H(Y_1Y_2|X_1X_2) + cE[\log W_{Y|X}^{\otimes 2}]$$

The exchange of limit and maximum in step (a) can be justified since the functional is bounded from above and is continuous with respect to p_{XY}. Here both $\max_{p_{XY}} H(Y) + (c-1)H(Y|X) + cE[\log W_{Y|X}]$ and $\max_{p_{XY}} H(Y_1Y_2) + (c-1)H(Y_1Y_2|X_1X_2) + cE[\log W_{Y|X}^{\otimes 2}]$ fall into the non-convex functional family (1.1).

Similar arguments extends to evaluation of other $S_{\vec{\gamma}\in\mathbb{R}_+^d}(W)$ and $S_{\vec{\gamma}\in\mathbb{R}_+^d}(p)$ in the communication settings arising in section 1.1, and we will omit the details here to avoid the duplication of arguments.

1.2.3 Auxiliary Random Variable and Dual Representation

The auxiliary random variables have appeared in the achievable rate regions for many communication scenarios, including Broadcast channel, Gray-Wyner source coding, Lossless distributed coding with one helper, and Lossless distributed coding with two helpers. And identifying the optimal auxiliary random variable is critical in the evaluation of the weighted sum rate for these achievable rate regions.

One idea to interpret the auxiliary random variable is to use the so-called upper concave envelope or lower convex envelope, see [47]. Let $f(p_X)$ be a function of p_X defined on a probability simplex $\mathcal{D}$ in $\mathbb{R}^{|\mathcal{X}|}$, the upper concave envelope, denoted by $\mathfrak{C}_{p_X}[f]$, is defined as

$$\mathfrak{C}_{p_X}[f](\hat{p}_X) := \inf\{g(\hat{p}_X) : g(p_X) \text{ is concave in } p_X \in \mathcal{D}, g(p_X) \geq f(p_X) \forall p_X \in \mathcal{D}\}$$

for any $\hat{p}_x \in \mathcal{D}$. And the lower convex envelope, denoted by $\mathfrak{K}_{p_X}[f]$, is defined as

$$\mathfrak{K}_{p_X}[f](\hat{p}_X) := -\mathfrak{C}_{p_X}[-f](\hat{p}_X)$$

for any $\hat{p}_X \in \mathcal{D}$. Intuitively speaking, $\mathfrak{C}_{p_X}[f]$ is taking the convex hull of the set of points $\{(p_X, y) : y \leq f(p_X), p_X \in \mathcal{D}\}$, and $\mathfrak{K}_{p_X}[f]$ is taking the convex hull of the set of points $\{(p_X, y) : y \geq f(p_X), p_X \in \mathcal{D}\}$.

The equivalent characterizations of upper concave envelope and lower convex envelope are given as following, see [47]:

$$\mathfrak{C}_{p_X}[f](\hat{p}_X) = \sup_{p_{U|X}} \sum_{u\in\mathcal{U}} p_U(u) f(\hat{p}_{X|U=u}), \tag{1.15}$$

$$\mathfrak{K}_{p_X}[f](\hat{p}_X) = \inf_{p_{U|X}} \sum_{u\in\mathcal{U}} p_U(u) f(\hat{p}_{X|U=u}) \tag{1.16}$$

where $p_U(u) = \sum_{x\in\mathcal{X}} \hat{p}_X(x)p_{U|X}(u|x)$ and $\hat{p}_{X|U=u} = \left[\frac{\hat{p}_X(x)p_{U|X}(u|x)}{p_U(u)}\right]_{x\in\mathcal{X}}$.

Observe that the upper concave envelope $\mathfrak{C}_{p_X}[f]$ is fully determined by the dual representation of $f(p_X)$, which is defined as

$$f^\dagger(\vec{d}) = \sup_{p_X}\left\{f(p_X) - \sum_{x\in\mathcal{X}} d_x p_X(x)\right\},$$

for any real-valued vector $\vec{d} := (d_x, x\in\mathcal{X})$, see [6]. Similarly, the lower convex envelope is determined by its dual

$$f^\dagger(\vec{d}) = \inf_{p_X}\left\{f(p_X) - \sum_{x\in\mathcal{X}} d_x p_X(x)\right\},$$

for any real-valued vector $\vec{d} := (d_x, x\in\mathcal{X})$.

Take the lossless distributed source coding with one helper for instance, the optimality of $\mathscr{A}(p_{XY})$ is equivalent to:

$$S_\gamma(p_{XY}^{\otimes 2}) \geq 2S_\gamma(p_{XY}) \tag{1.17}$$

Here the single-letter and 2-letter form of the weighted sum rate can be rewritten in terms of the lower convex envelopes:

$$\begin{aligned}
S_\gamma(p_{XY}) &:= \inf_{(R_1,R_2)\in\mathscr{A}(p_{XY})} R_1 + \gamma R_2 \\
&\overset{(a)}{=} \min_{p_{U|X}} H(Y|U) + \gamma I(U;X) \\
&\overset{(b)}{=} H(X) + \mathfrak{K}_{q_X}\left[H(Y) - \gamma H(X)\right](p_X) \\
S_\gamma(p_{XY}^{\otimes 2}) &:= \inf_{(R_1,R_2)\in\mathscr{A}(p_{XY}^{\otimes})} R_1 + \gamma R_2 \\
&\overset{(a)}{=} \min_{p_{U|X_1X_2}} H(Y_1Y_2|U) + \gamma I(U;X_1X_2) \\
&\overset{(b)}{=} H(X_1X_2) + \mathfrak{K}_{q_{X_1X_2}}\left[H(Y_1Y_2) - \gamma H(X_1X_2)\right](p_X^{\otimes 2})
\end{aligned}$$

for some $\gamma \geq 0$.

Here in step (a) the minimum exists since $|\mathcal{U}| \leq |\mathcal{X}|+1$ and thereby $p_{U|X}$ falls into some compact probability simplex space; step (b) follows from the equivalent characterization of lower convex envelope (1.16).

Through the techniques used in the proof of Lemma 2 in [6], equation (1.17) holds if for any real-valued vectors $d_X, \hat{d}_X$,

$$\begin{aligned}
&\min_{q_{X_1X_2}} H(Y_1Y_2) - \gamma H(X_1X_2) - E_{q_{X_1}}[d_X] - E_{q_{X_2}}[\hat{d}_X] \\
&\geq \min_{q_X}\left\{H(Y) - \gamma H(X) - E_{q_X}[d_X]\right\} + \min_{q_X}\left\{H(Y) - \gamma H(X) - E_{q_X}[\hat{d}_X]\right\}
\end{aligned} \tag{1.18}$$

Though the functionals are not convex in general, one could still show that product distribution is the global minimizer of the left-hand side of above equation (1.18) by the following argument:

$$
\begin{aligned}
& H(Y_1Y_2) - \gamma H(X_1X_2) - E_{q_{X_1}}[d_X] - E_{q_{X_2}}[\hat{d}_X] \\
\overset{(a)}{\geq} & H(Y_1) - \gamma H(X_1) - E_{q_{X_1}}[d_X] + H(Y_2|Y_1X_1) \\
& - \gamma H(X_2|X_1Y_1) - E_{q_{X_1Y_1}}\left[E_{q_{X_2|X_1Y_1}}[\hat{d}_X]\right] \\
= & H(Y_1) - \gamma H(X_1) - E_{q_{X_1}}[d_X] + \sum_{x_1,y_1} q_{X_1Y_1}(x_1,y_1) \\
& \left[H(Y_2|X_1=x_1,Y_1=y_1) - \gamma H(X_2|X_1=x_1,Y_1=y_1) - E_{q_{X_2|X_1=x_1,Y_1=y_1}}[\hat{d}_X]\right] \\
\overset{(b)}{\geq} & \min_{q_X}\{H(Y) - \gamma H(X) - E_{q_X}[d_X]\} + \min_{q_X}\left\{H(Y) - \gamma H(X) - E_{q_X}[\hat{d}_X]\right\}
\end{aligned}
$$

Step (a) follows from that conditional reduces entropy, and the markov chain $X_2 \to X_1 \to Y_1$. Step (b) is due to the Markov chain $X_1, Y_1 \to X_2 \to Y_2$ and the fact that taking average will not decrease the functional value below the minimized value. This finishes the optimality proof of $\mathscr{A}(p_{XY})$.

Similar analysis could be applied to other communication problems in section (1.1). For Marton's inner bound, there is a detailed discussion on testing the optimality via the dual of the weighted sum rate in [6] and [48].

1.3 Contributions of this Thesis

This thesis tries to solve several instances in non-convex functional family (1.1), and intends to provide insights to the structure of the optimizers. Some of the results also find applications in other fields including computer science, see [11].

In Chapter 2, we try to evaluate the forward and reverse hypercontractive region for a pair of random variables (X, Y), where a uniform X is passed through a binary erasure channel BEC(ϵ) to produce Y and $0 < \epsilon < 1$. The joint distribution of (X, Y) is denoted as $BIEO(\varepsilon)$. Our technique builds on an equivalent characterization of hypercontractivity using Kullback-Leibler Divergence.

The divergence characterizations are in general non-convex functional optimization problems and belong to the family (1.1). But certain structure of the interior stationary points helps us controlling the behavior of the global optimizers, thus establishing the hypercontractive regime for some non-trivial range of parameters.

A similar analysis also recovers the celebrated results for a pair of variables (X, Y), where a uniform X is passed through a binary symmetric channel $\mathrm{BSC}(\rho)$ with flipping probability $\frac{1-\rho}{2}$ to produce Y and $-1 < \rho < 1$. The joint distribution of (X, Y) is denoted as $\mathrm{DSBS}(\rho)$. This result is also known as the Bonami-Beckner inequality.

Chapter 3 starts from a new non-convex weighted sum rate outer bound for the Körner and Marton's modulo two sum problem. By optimizing over this outer bound, we could show that the optimal sum-rate is given by linear codes for a larger class of binary distributions, thus extending the optimality results for the Körner and Marton's modulo two sum problem.

Chapter 4 is related to the non-convex functional $H(Y_t) - \gamma H(X_t)$, where $X_t := X + \sqrt{t}Z$ is in the set of distributions along the heat flow and Y_t is obtained by passing X_t through an additive Gaussian noise channel. We show that if t is re-scaled so that $H(X_t)$ is linear in t, then $H(Y_t)$ is convex in t. This problem is equivalent to showing the log-convexity of Fisher Information, thus resolving a conjecture in [15] and implicitly in the 1966 paper [41] by McKean. This is a joint work with Michel Ledoux.

Chapter 2

Hypercontractivity Region Evaluation

2.1 Introduction

Forward and reverse hypercontractive inequalities are a family of inequalities that are studied in functional analysis [13, 43], which have also found applications in computer science [11, 42]. The following definitions appeared in [13, 42].

Definition 2.1. A pair of random variables (X, Y) is said to be (λ_1, λ_2) forward hypercontractive, for $\lambda_1, \lambda_2 \in (1, \infty)$, if

$$\mathrm{E}(f(X)g(Y)) \leq \|f(X)\|_{\lambda_1} \|g(Y)\|_{\lambda_2} \tag{2.1}$$

holds for all non-negative functions $f(\cdot) : \mathcal{X} \to \mathbb{R}_+, g(\cdot) : \mathcal{Y} \to \mathbb{R}_+$.

Definition 2.2. A pair of random variables (X, Y) is said to be (λ_1, λ_2) reverse hypercontractive, for $\lambda_1, \lambda_2 \in (-\infty, 1)$, if

$$\mathrm{E}(f(X)g(Y)) \geq \|f(X)\|_{\lambda_1} \|g(Y)\|_{\lambda_2} \tag{2.2}$$

holds for all positive functions $f(\cdot) : \mathcal{X} \to \mathbb{R}_{++}, g(\cdot) : \mathcal{Y} \to \mathbb{R}_{++}$.

Remark 2.1. The following remarks are worth noting:

- In the above, we adopt the following notation for λ-th norm of random variables:

  $$\|Z\|_\lambda := \mathrm{E}(|Z|^\lambda)^{\frac{1}{\lambda}}, \lambda \neq 0.$$

  and $\|Z\|_0 = e^{E(\log |Z|)}$.

- We only consider finite valued random variables in this chapter, though the standard machine (where finite valued random variables are called *simple functions*) enables the extension of the characterizations to families of general random variables.

From Hölder's inequality and monotonicity of norm, it is immediate that if

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \leq 1$$

then the forward hypercontractive inequality (2.1) holds.

Similarly, reverse Hölder's inequality says that the reverse hypercontractive inequality (2.2) holds when

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1,$$

and the monotonicity of the $\|Z\|_\lambda$ in λ yields a trivial region of parameters where (2.2) always holds. This, for instance, includes the region $\lambda_1, \lambda_2 \in (-\infty, 0]$. Therefore the non-trivial region of the reverse hypercontractive region is when at least one of the parameters λ_1 or λ_2 is strictly positive.

A necessary condition for (X, Y) to be (λ_1, λ_2) forward hypercontractive is given in terms of the maximal correlation of (X, Y), see [24, 33, 54].

Definition 2.3 (maximal correlation coefficient)**.** The maximal correlation coefficient between a pair of random variables (X, Y), $\rho_m(X, Y)$, is defined as

$$\rho_m(X, Y) = \sup \mathrm{E}[\psi_1(X)\psi_2(Y)]. \tag{2.3}$$

where $\psi_1(X)$ and $\psi_2(Y)$ are real-valued functions of X and Y such that $E[\psi_1(X)] = E[\psi_2(Y)] = 0$ and $E[\psi_1^2(X)] \leq 1, E[\psi_2^2(Y)] \leq 1$.

Remark 2.2. Observe that the maximal correlation coefficient can also be written as

$$\rho_m(X, Y) = -\inf \mathrm{E}[\psi_1(X)\psi_2(Y)]. \tag{2.4}$$

where $\psi_1(X)$ and $\psi_2(Y)$ are real-valued functions of X and Y such that $E[\psi_1(X)] = E[\psi_2(Y)] = 0$ and $E[\psi_1^2(X)] \leq 1, E[\psi_2^2(Y)] \leq 1$.

Theorem 2.1 (Forward hypercontractive correlation lower bound, [5])**.** *A pair of random variables is* (λ_1, λ_2) *forward hypercontractive for* $\lambda_1, \lambda_2 \in (1, \infty)$*, only if*

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq \rho_m^2(X, Y).$$

Remark 2.3. This is a classical result and we present a proof for completeness.

Proof. In forward hypercontractivity definition (2.1), choose $f(X) = 1+a\varepsilon\psi_1(X)$ and $g(Y) = 1 + b\varepsilon\psi_2(Y)$ where $\psi_1(X)$ and $\psi_2(Y)$ are arbitrary real-valued functions such that $\mathrm{E}[\psi_1(X)] = \mathrm{E}[\psi_2(Y)] = 0$ and $\mathrm{E}[\psi_1^2(X)] \leq 1, \mathrm{E}[\psi_2^2(Y)] \leq 1$. Here $a, b \geq 0$ are parameters to be optimized and $\varepsilon > 0$ is small enough so that both $f(X)$ and $g(Y)$ are nonnegative functions.

Taylor expansion with respect to ε shows that

$$\begin{aligned} \mathrm{E}[f(X)g(Y)] &= 1 + ab\varepsilon^2\,\mathrm{E}[\psi_1(X)\psi_2(Y)] \\ \|f(X)\|_{\lambda_1} &= 1 + \frac{\lambda_1 - 1}{2}\,\mathrm{E}[\psi_1^2(X)]a^2\varepsilon^2 + o(\varepsilon^3) \\ \|g(Y)\|_{\lambda_2} &= 1 + \frac{\lambda_2 - 1}{2}\,\mathrm{E}[\psi_2^2(Y)]b^2\varepsilon^2 + o(\varepsilon^3) \end{aligned} \tag{2.5}$$

So we have for any $\psi_1(X)$ and $\psi_2(Y)$,

$$\begin{aligned} &\mathrm{E}[f(X)g(Y)] \leq \|f(X)\|_{\lambda_1}\|g(Y)\|_{\lambda_2} \\ \Rightarrow &ab\varepsilon^2\,\mathrm{E}[\psi_1(X)\psi_2(Y)] \leq \frac{\lambda_1 - 1}{2}\,\mathrm{E}[\psi_1^2(X)]a^2\varepsilon^2 + \frac{\lambda_2 - 1}{2}\,\mathrm{E}[\psi_2^2(Y)]b^2\varepsilon^2 + o(\varepsilon^3) \\ \Rightarrow &ab\,\mathrm{E}[\psi_1(X)\psi_2(Y)] \leq \frac{\lambda_1 - 1}{2}a^2\,\mathrm{E}[\psi_1^2(X)] + \frac{\lambda_2 - 1}{2}b^2\,\mathrm{E}[\psi_2^2(Y)] \\ \Rightarrow &ab\,\mathrm{E}[\psi_1(X)\psi_2(Y)] \leq \frac{\lambda_1 - 1}{2}a^2 + \frac{\lambda_2 - 1}{2}b^2 \end{aligned} \tag{2.6}$$

where the last step follows from $\lambda_1 > 1, \lambda_2 > 1$ and $\mathrm{E}[\psi_1^2(X)] \leq 1, \mathrm{E}[\psi_2^2(Y)] \leq 1$.

By taking supremum over all possible $\psi_1(X)$ and $\psi_2(Y)$ for the left-hand side of (2.6) and using the definition of maximal correlation coefficient (2.3), we require that

$$\begin{aligned} ab|\rho_m(X,Y)| &\leq \frac{\lambda_1 - 1}{2}a^2 + \frac{\lambda_2 - 1}{2}b^2 \\ \Rightarrow (\lambda_1 - 1)(\lambda_2 - 1) &\geq \rho_m^2(X,Y) \end{aligned}$$

where the last step comes from choosing $a = \frac{|\rho_m(X,Y)|}{\lambda_1 - 1}, b = 1$. □

Similarly, a necessary condition for (X, Y) to be (λ_1, λ_2) reverse hypercontractive is presented in the following theorem via the maximal correlation.

Theorem 2.2 (Reverse hypercontractive correlation bound)**.** *A pair of random variables* $(X, Y) \sim p_{XY}$ *is* (λ_1, λ_2) *reverse hypercontractive for* $\lambda_1, \lambda_2 \in (-\infty, 1)$*, only if*

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq \rho_m^2(X,Y)$$

Remark 2.4. The proof to this theorem, see [42], is similar to forward hypercontractivity and we also present a proof for completeness.

Proof. In reverse hypercontractivity definition (2.2), choose $f(X) = 1 + a\varepsilon\psi_1(X)$ and $g(Y) = 1 + b\varepsilon\psi_2(Y)$ where $\psi_1(X)$ and $\psi_2(Y)$ satisfies $\mathrm{E}[\psi_1(X)] = \mathrm{E}[\psi_2(Y)] = 0$ and $\mathrm{E}[\psi_1^2(X)] \leq 1, \mathrm{E}[\psi_2^2(Y)] \leq 1$.

From equations (2.5), for any $\psi_1(X)$ and $\psi_2(Y)$ we have,

$$\begin{aligned}
&\mathrm{E}[f(X)g(Y)] \geq \|f(X)\|_{\lambda_1}\|g(Y)\|_{\lambda_2} \\
\Rightarrow & ab\varepsilon^2\,\mathrm{E}[\psi_1(X)\psi_2(Y)] \geq \frac{\lambda_1 - 1}{2}\,\mathrm{E}[\psi_1^2(X)]a^2\varepsilon^2 + \frac{\lambda_2 - 1}{2}\,\mathrm{E}[\psi_2^2(Y)]b^2\varepsilon^2 + o(\varepsilon^3) \\
\Rightarrow & -ab\,\mathrm{E}[\psi_1(X)\psi_2(Y)] \leq \frac{1-\lambda_1}{2}a^2\,\mathrm{E}[\psi_1^2(X)] + \frac{1-\lambda_2}{2}b^2\,\mathrm{E}[\psi_2^2(Y)] \\
\Rightarrow & -ab\,\mathrm{E}[\psi_1(X)\psi_2(Y)] \leq \frac{1-\lambda_1}{2}a^2 + \frac{1-\lambda_2}{2}b^2
\end{aligned} \tag{2.7}$$

where the last step follows from $\lambda_1 < 1, \lambda_2 < 1$ and $\mathrm{E}[\psi_1^2(X)] \leq 1, \mathrm{E}[\psi_2^2(Y)] \leq 1$.

By considering all possible $\psi_1(X)$ and $\psi_2(Y)$ and using the alternate definition of maximal correlation coefficient (2.4) for left-hand side of (2.7), we require that

$$\begin{aligned}
ab|\rho_m(X,Y)| &\leq \frac{1-\lambda_1}{2}a^2 + \frac{1-\lambda_2}{2}b^2 \\
\Rightarrow (1-\lambda_1)(1-\lambda_2) &\geq \rho_m^2(X,Y)
\end{aligned}$$

where the last step comes from choosing $a = \frac{|\rho_m(X,Y)|}{1-\lambda_1}, b = 1$.

□

Exact computation of the hypercontractive parameters for certain distributions has been a challenging task with very few exact characterizations. Two well-known cases where exact computations have been feasible are for jointly Gaussian random variables, and when (X, Y) follows a Doubly Symmetric Binary Source (DSBS) distribution, see [12, 13, 31]. In these cases, the hypercontractive parameters matches the correlation lower bound.

Our starting point is the following equivalent characterizations of the forward and reverse hypercontractive region derived in [45] and [7]. One of the characterization using divergence, stated below, can also be inferred from an earlier work [14].

Theorem 2.3 ([45]). *Consider a pair of random variables (X, Y) distributed according to p_{XY}. For any $\lambda_1, \lambda_2 \in (1, \infty)$, the following four assertions are equivalent:*

(i) p_{XY} *is* (λ_1, λ_2) *forward hypercontractive;*

(ii) *For every* $q_{XY}(\ll p_{XY})$ *we have (independently by Carlen et. al. [14])*

$$\frac{1}{\lambda_1} D(q_X \| p_X) + \frac{1}{\lambda_2} D(q_Y \| p_Y) \leq D(q_{XY} \| p_{XY}). \tag{2.8}$$

(iii) *For every extension* $p_{V|XY}$ *such that* $I(V;XY) > 0$ *we have*

$$\frac{1}{\lambda_1} I(U;X) + \frac{1}{\lambda_2} I(U;Y) \leq I(U;XY)$$

(iv)

$$\mathfrak{K}_{q_{XY}} \left[\frac{1}{\lambda_1} H(X) + \frac{1}{\lambda_2} H(Y) - H(XY) \right] \Big|_{p_{XY}} = \frac{1}{\lambda_1} H(X) + \frac{1}{\lambda_2} H(Y) - H(XY) \tag{2.9}$$

In the above, $q_{XY} \ll p_{XY}$ denotes that q_{XY} is absolutely continuous with respect to p_{XY}.

Remark 2.5. In [8], Beigi and Gohari observed that the tensorization property of forward hypercontractivity is equivalent to the optimality of the achievable rate region $\mathscr{A}_{GY}(p_{XY})$ for the Gray-Wyner source coding via using the above characterization (2.9).

More specifically, the weighted sum rate of $\mathscr{A}_{GW}(p_{XY})$ can be written as (W.L.O.G. assume the weight coefficient γ_0 for R_0 equals 1):

$$\begin{aligned}
S_{(\gamma_1,\gamma_2)}(p_{XY}) :=& \inf_{(R_0,R_1,R_2) \in \mathscr{A}_{GW}(p_{XY})} R_0 + \gamma_1 R_1 + \gamma_2 R_2 \\
=& \inf_{p_{V|XY}} I(X,Y;V) + \gamma_1 H(X|V) + \gamma_2 H(Y|V) \\
=& H(X,Y) + \inf_{p_{V|XY}} \gamma_1 H(X|V) + \gamma_2 H(Y|V) - H(XY|V) \\
\stackrel{(a)}{=}& H(X,Y) + \mathfrak{K}_{q_{XY}} \left\{ \gamma_1 H(X) + \gamma_2 H(Y) - H(XY) \right\} |_{p_{XY}}
\end{aligned} \tag{2.10}$$

where step (a) follows from the equivalence between auxiliary random variable V and lower convex envelope (1.16).

To explicitly evaluate the weighted sum rate, notice that the optimal $p_{V|XY}$ is nontrivial happens only when $\gamma_1 < 1, \gamma_2 < 1, \gamma_1 + \gamma_2 > 1$. And determining the lower convex envelope $\mathfrak{K}_{q_{XY}} \{\gamma_1 H(X) + \gamma_2 H(Y) - H(XY)\}$ is essentially determining the set of extreme points:

$$\left\{ p_{XY} : \mathfrak{K}_{q_{XY}} \left[\gamma_1 H(X) + \gamma_2 H(Y) - H(XY) \right] |_{p_{XY}} = \gamma_1 H(X) + \gamma_2 H(Y) - H(XY) \right\},$$

which is the same as the set

$$\{p_{XY} : p_{XY} \text{ is } (\frac{1}{\gamma_1}, \frac{1}{\gamma_2}) \text{ forward hypercontractive}\},$$

by above characterization (2.9)

Theorem 2.4 ([7]). *Depending on the regime of parameters of λ_1, λ_2, the following yields an equivalent characterization of reverse hypercontractive inequality* (2.2) *in terms of divergence.*

(*i*) *When $\lambda_1, \lambda_2 \in (0, 1)$ reverse hypercontractive inequality* (2.2) *holds iff:*
For any q_X and q_Y there exists r_{XY} with $r_X = q_X$ and $r_Y = q_Y$ such that:

$$\frac{1}{\lambda_1} D(q_X||p_X) + \frac{1}{\lambda_2} D(q_Y||p_Y) \geq D(r_{XY}||p_{XY})$$

(*ii*) *When $0 < \lambda_1 < 1$ and $\lambda_2 < 0$ reverse hypercontractive inequality* (2.2) *holds iff:*
For any q_X there exists r_{XY} with $r_X = q_X$ such that:

$$\frac{1}{\lambda_1} D(q_X||p_X) + \frac{1}{\lambda_2} D(r_Y||p_Y) \geq D(r_{XY}||p_{XY})$$

(*iii*) *When $\lambda_1 < 0$ and $0 < \lambda_2 < 1$ reverse hypercontractive inequality* (2.2) *holds iff:*
For any q_Y there exists r_{XY} with $r_Y = q_Y$ such that:

$$\frac{1}{\lambda_1} D(r_X||p_X) + \frac{1}{\lambda_2} D(q_Y||p_Y) \geq D(r_{XY}||p_{XY})$$

Before we state our main results, we state a well-known lemma (mentioned by Mossel to the authors) that already provides some partial results on the first regime of reverse hypercontractive parameters in above Theorem 2.4 for pairs of random variables whose support is not the entire product space $\mathcal{X} \times \mathcal{Y}$.

Lemma 2.1. *Consider a pair of random variables $(X, Y) \sim p_{XY}$. Suppose there exists $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$ such that $p(x_0, y_0) = 0$, then for no pair $(\lambda_1, \lambda_2) \in (0, 1) \times (0, 1)$ will (X, Y) be (λ_1, λ_2) reverse hypercontractive.*

Proof. The simple argument is presented here for completeness. Consider $f(X)$ and $g(Y)$ defined by $f(x_0) = 1, f(x') = \epsilon \forall x' \neq x_0$; $g(y_0) = 1, g(y') = \epsilon \forall y' \neq y_0$. Note that

$$\mathrm{E}(f(X)g(Y)) = p(x_0, y_0) + O(\epsilon) = O(\epsilon).$$

On the other hand $\|f(X)\|_{\lambda_1} \geq p_X(x_0)^{\frac{1}{\lambda_1}}, \|g(Y)\|_{\lambda_2} \geq p_Y(y_0)^{\frac{1}{\lambda_2}}$. Taking $\epsilon \to 0$, we see that reverse hypercontractive inequality (2.2) is violated by a suitably small ϵ. Note that since x_0, y_0 belong to the support of X, Y respectively, $p_X(x_0), p_Y(y_0) > 0$. □

The results of this chapter first appear in [46] and [49]. This is a joint work with Prof. Chandra Nair.

2.2 Main results

2.2.1 Binary Erasure Channel with Uniform Inputs

Consider a uniform binary random variable X passed through a binary erasure channel BEC(ϵ) producing the ternary output Y. Let $p_{XY}^{BEC(\varepsilon)}$ denote the joint distribution of X and Y. From the definition 2.3, one could compute the maximal correlation coefficient for (X, Y).

Proposition 2.1. *Given a pair of random variables (X, Y) following the $p_{XY}^{BEC(\varepsilon)}$ distribution, where $0 \leq \varepsilon \leq 1$. The maximal correlation coefficient $\rho_m(X, Y)$ is $\sqrt{1-\varepsilon}$.*

The correlation lower bound Theorem 2.1 for this setting says that (X, Y) is (λ_1, λ_2) forward hypercontractive for $\lambda_1, \lambda_2 \in (1, \infty)$ only if

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq 1 - \epsilon.$$

The theorem below (first main new result of this chapter) determines the set of parameters for which correlation bound is tight, i.e. yields the hypercontractive region.

Theorem 2.5. *Let (X, Y) distributed according to $p_{XY}^{BEC(\varepsilon)}$ and $\lambda_1, \lambda_2 \in (1, \infty)$ satisfy $(\lambda_1 - 1)(\lambda_2 - 1) = 1 - \epsilon$. Then (X, Y) is (λ_1, λ_2) forward hypercontractive, i.e. the correlation bound is tight, if and only if the following condition is satisfied:*

$$\epsilon - \frac{1}{2} \leq \frac{3}{2}(\lambda_2 - 1).$$

Remark 2.6. If $\epsilon \leq \frac{1}{2}$ then the correlation lower bound is tight; else it turns out to be tight only for a subset of the regime of parameters.

Proof. The proof is divided into two parts. In the first part, we will establish the result for $\lambda_2 \geq 2$ directly using the definition of hypercontractivity, by mimicking Janson's proof [34] for the DSBS case. For $\lambda_2 < 2$ we will use the equivalent characterization using divergences to provide a proof.

Case 1: $\lambda_2 \geq 2$. Let $\lambda_1 = 1 + \frac{1-\varepsilon}{\lambda_2 - 1}$ so that $(\lambda_1 - 1)(\lambda_2 - 1) = 1 - \epsilon$. We wish to show that for all functions $f(\cdot), g(\cdot)$ the inequality

$$\mathrm{E}(f(X)g(Y)) \leq \|f(X)\|_{\lambda_1} \|g(Y)\|_{\lambda_2}$$

holds. Observe that, by Hölder's inequality,

$$\mathrm{E}(f(X)g(Y)) = \mathrm{E}(\mathrm{E}(f(X)|Y)g(Y)) \leq \|\mathrm{E}(f(X)|Y)\|_{\lambda_2'} \|g(Y)\|_{\lambda_2}.$$

Here $\lambda_2' \in (1,2]$ is the Hölder conjugate of λ_2, that is, $\lambda_2' = \frac{\lambda_2}{\lambda_2 - 1}$. Hence showing (in fact this is an equivalent condition) the following suffices

$$\|\mathrm{E}(f(X)|Y)\|_{\lambda_2'} \leq \|f(X)\|_{\lambda_1}.$$

W.l.o.g. let $f(0) = 1 - \delta$, $f(1) = 1 + \delta$. Then the above inequality reduces to

$$\left[\frac{1-\epsilon}{2}(1-\delta)^{\lambda_2'} + \frac{1-\epsilon}{2}(1+\delta)^{\lambda_2'} + \epsilon\right]^{\frac{1}{\lambda_2'}} \leq \left[\frac{1}{2}(1-\delta)^{\lambda_1} + \frac{1}{2}(1+\delta)^{\lambda_1}\right]^{\frac{1}{\lambda_1}}.$$

That is, suffices that

$$1 + (1-\epsilon)\sum_{k=1}^{\infty}\binom{\lambda_2'}{2k}\delta^{2k} \leq \left(1 + \sum_{k=1}^{\infty}\binom{\lambda_1}{2k}\delta^{2k}\right)^{\frac{\lambda_2'}{\lambda_1}}$$

To get the above reduction we use the multiplicative formula extension of binomial co-efficients and the infinite power series

$$(1+x)^{\alpha} = 1 + \sum_{k=1}^{\infty}\binom{\alpha}{k}x^k, |x| < 1.$$

Substituting for λ_1 we see that $\frac{\lambda_2'}{\lambda_1} = \frac{\lambda_2}{\lambda_2 - \epsilon} > 1$. Since $(1+x)^a \geq 1 + ax$ $(a > 1, x > 0)$, it suffices to show that

$$1 + (1-\epsilon)\sum_{k=1}^{\infty}\binom{\lambda_2'}{2k}\delta^{2k} \leq 1 + \frac{\lambda_2'}{\lambda_1}\sum_{k=1}^{\infty}\binom{\lambda_1}{2k}\delta^{2k}$$

Since $1 < \lambda_1 \leq \lambda_2' \leq 2$ the inequality is easily seen to be true by comparing the coefficients of δ^{2k} term by term (all terms are non-negative). Equality holds for $k = 1$ and for all other powers it is an inequality, in general. (See Remark 2.7 at the end of next section.)

Case 2: $\lambda_2 < 2$. We use the equivalent characterization using divergences in this case. Again let $(\lambda_1 - 1)(\lambda_2 - 1) = 1 - \epsilon$. We wish to show that

$$\max_{q_{XY} \ll p_{XY}^{BEC(\varepsilon)}} \frac{1}{\lambda_1} D(q_X \| p_X^{BEC(\varepsilon)}) + \frac{1}{\lambda_2} D(q_Y \| p_Y^{BEC(\varepsilon)})$$

$$- D(q_{XY} \| p_{XY}^{BEC(\varepsilon)}) = \begin{cases} 0 & \epsilon - \frac{1}{2} \leq \frac{3}{2}(\lambda_2 - 1) \\ > 0 & \text{o.w.} \end{cases}$$

It is easy to see that the maximum has to be an interior point by considering the functional behavior at the boundaries. This is primarily because the last term has an infinite slope at the boundaries and since $\lambda_1, \lambda_2 > 1$ this infinite slope cannot be completely canceled by the first two terms. We omit the details of this calculation here.

Thus the main part of the proof is to show that there is only one interior stationary point $q_{XY} = p_{XY}^{BEC(\varepsilon)}$ when $\epsilon - \frac{1}{2} \leq \frac{3}{2}(\lambda_2 - 1)$; and otherwise $q_{XY} = p_{XY}^{BEC(\varepsilon)}$ is not even a local maximum.

For any (strictly) interior stationary points, the Lagrange conditions yield

$$k = \frac{1}{\lambda_1} \ln(q_{00} + q_{0E}) - \frac{1}{\lambda_2'} \ln \frac{q_{00}}{1 - \epsilon} \tag{2.11a}$$

$$k = \frac{1}{\lambda_1} \ln(q_{11} + q_{1E}) - \frac{1}{\lambda_2'} \ln \frac{q_{11}}{1 - \epsilon} \tag{2.11b}$$

$$k = \frac{1}{\lambda_1} \ln(q_{00} + q_{0E}) + \frac{1}{\lambda_2} \ln(q_{0E} + q_{1E}) - \frac{1}{\lambda_2} \ln 2 - \ln q_{0E} + \frac{1}{\lambda_2'} \ln \epsilon \tag{2.11c}$$

$$k = \frac{1}{\lambda_1} \ln(q_{11} + q_{1E}) + \frac{1}{\lambda_2} \ln(q_{0E} + q_{1E}) - \frac{1}{\lambda_2} \ln 2 - \ln q_{1E} + \frac{1}{\lambda_2'} \ln \epsilon \tag{2.11d}$$

Equating (2.11c) and (2.11d) yields

$$\frac{q_{0E}}{q_{1E}} = \left(\frac{q_{00} + q_{0E}}{q_{11} + q_{1E}} \right)^{\frac{1}{\lambda_1}}. \tag{2.12a}$$

Equating (2.11a) and (2.11c) yields

$$q_{00} = \frac{q_{0E}^{\lambda_2'} 2^{\lambda_2' - 1}}{(q_{0E} + q_{1E})^{\lambda_2' - 1}} \frac{1 - \epsilon}{\epsilon}. \tag{2.12b}$$

Equating (2.11b) and (2.11d) yields

$$q_{11} = \frac{q_{1E}^{\lambda_2'} 2^{\lambda_2' - 1}}{(q_{0E} + q_{1E})^{\lambda_2' - 1}} \frac{1 - \epsilon}{\epsilon}. \tag{2.12c}$$

Substituting for q_{00} and q_{11} using (2.12b) and (2.12c) in (2.12a), setting $1 - \delta = \frac{2q_{0E}}{q_{0E} + q_{1E}} \in [0, 2]$, this yields

$$(1 - \varepsilon)(1 - \delta)^{\lambda_2' - \lambda_1} + \varepsilon(1 - \delta)^{1 - \lambda_1} = (1 - \varepsilon)(1 + \delta)^{\lambda_2' - \lambda_1} + \varepsilon(1 + \delta)^{1 - \lambda_1}$$

and using $(\lambda_1 - 1)(\lambda_2 - 1) = 1 - \epsilon$ we obtain

$$(1-\epsilon)(1-\delta)^{\frac{\epsilon}{\lambda_2-1}} + \epsilon(1-\delta)^{\frac{\epsilon-1}{\lambda_2-1}} = (1-\epsilon)(1+\delta)^{\frac{\epsilon}{\lambda_2-1}} + \epsilon(1+\delta)^{\frac{\epsilon-1}{\lambda_2-1}}.$$

From Lemma 2.2 we know that the above equation has exactly one solution, $\delta = 0$, when $(\epsilon - \frac{1}{2}) \leq \frac{3}{2}(\lambda_2 - 1)$. Thus under the above condition on (λ_2, ϵ) every interior stationary point must satisfy $q_{0E} = q_{1E}$. Further from (2.12b) and (2.12c) we can conclude that

$$\frac{q_{00}}{1-\epsilon} = \frac{q_{11}}{1-\epsilon} = \frac{q_{0E}}{\epsilon} = \frac{q_{1E}}{\epsilon},$$

implying that the only stationary point (hence global maximizer) is $q_{XY} = p_{XY}^{BEC(\varepsilon)}$, which yields a maximum value 0 as desired.

for some $\delta > 0$ choose

$$\begin{aligned} q_{XY} &= [q_{00}, q_{0E}, q_{1E}, q_{11}] \\ &= \left[\frac{(1-\delta)^{\lambda_2'}(1-\epsilon)}{A}, \frac{\epsilon(1-\delta)}{A}, \frac{\epsilon(1+\delta)}{A}, \frac{(1-\epsilon)(1+\delta)^{\lambda_2'}}{A}\right] \end{aligned}$$

where $A = 2\epsilon + (1-\epsilon)[(1+\delta)^{\lambda_2'} + (1-\delta)^{\lambda_2'}]$ is the normalizing constant. Taylor series expansion of the term

$$\frac{1}{\lambda_1} D(q_X \| p_X^{BEC(\varepsilon)}) + \frac{1}{\lambda_2} D(q_Y \| p_Y^{BEC(\varepsilon)}) - D(q_{XY} \| p_{XY}^{BEC(\varepsilon)})$$

around $\varepsilon = 0$ yields an expansion

$$\frac{1}{24}\epsilon(1-\epsilon)(\lambda_2'-1)^2((2\epsilon-1)(\lambda_2'-1)-3)\delta^4 + O(\delta^6)$$

which is positive when

$$\epsilon - \frac{1}{2} > \frac{3}{2}(\lambda_2 - 1),$$

yielding that the maximum of the function is strictly positive under these parameter settings. □

Now let us turn to the reverse hypercontractive region for binary erasure channel with uniform inputs. The correlation lower bound for this setting says that (X, Y) is (λ_1, λ_2) reverse hypercontractive for $\lambda_1, \lambda_2 \in (-\infty, 1)$ only if

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq 1 - \epsilon.$$

In all the results mentioned above, the forward hypercontractive or reverse hypercontractive region matches the correlation lower bound (though in general it is known that these two are not the same regions). The computation of reverse hypercontractive region in this setting shows a non-trivial exact characterization where the region is not given by the correlation bound.

The following main new result concerns characterizing the reverse hypercontractive region for the binary erasure channel for certain range of parameters. (This determines the second regime in Theorem 2.4, and leaves the third one as undetermined in Theorem 2.4, since Lemma 2.1 rules out the first regime for $p_{XY}^{BEC(\varepsilon)}$)

Theorem 2.6. *Let (X, Y) be distributed according to $p_{XY}^{BEC(\varepsilon)}, \varepsilon \in (0, 1)$ and $\lambda_1, \lambda_2 \in (-\infty, 1) \setminus \{0\}$. When $\lambda_2 < 0$, (X, Y) is (λ_1, λ_2) reverse-hypercontractive if and only if*

$$\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2 - 1}{\lambda_2} \ln[(1-\epsilon)2^{\frac{1}{\lambda_2 - 1}} + \epsilon]}.$$

Proof. $\lambda_2 < 0$ and $\lambda_1 \leq \lambda_2' (:= \frac{\lambda_2}{\lambda_2 - 1})$ will belong to the reverse hypercontractive region trivially from the Reverse Hölder's inequality and the monotonicity of $\|Z\|_\lambda$ in λ.

From Theorem 2.4 we are left with determining the range of $\lambda_1 \in (\lambda_2', 1)$ satisfying the following: for any q_X there exists r_{XY} with $r_X = q_X$ such that

$$\frac{1}{\lambda_1} D(q_X || p_X^{BEC(\varepsilon)}) + \frac{1}{\lambda_2} D(r_Y || p_Y^{BEC(\varepsilon)}) \geq D(r_{XY} || p_{XY}^{BEC(\varepsilon)}). \tag{2.13}$$

We will show that the above condition holds *if and only if*

$$\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2 - 1}{\lambda_2} \ln[(1-\epsilon)2^{\frac{1}{\lambda_2 - 1}} + \epsilon]}. \tag{2.14}$$

(2.14) $\implies$ (2.13): If r_{XY} is not absolutely continuous with respect to $p_{XY}^{BEC(\varepsilon)}$, $D(r_{XY} || p_{XY}^{BEC(\varepsilon)})$ will become $+\infty$, while $\frac{1}{\lambda_1} D(q_X || p_X^{BEC(\varepsilon)}) + \frac{1}{\lambda_2} D(r_Y || p_Y^{BEC(\varepsilon)})$ are finite; violating (2.13). Thus, it is sufficient to search over r_{XY} that are absolute continuous with respect to $p_{XY}^{BEC(\varepsilon)}$.

Denote $q_X(X = 0) = x$, $r_{XY}(X = 0, Y = 0) = r$, $r_{XY}(X = 1, Y = 1) = s$. Hence $r_{XY}(X = 0, Y = E) = x - r$, $r_{XY}(X = 1, Y = E) = 1 - x - s$, since $r_X(X = 0) = q_X(X = 0) = x$.

Define $f(x, r, s)$ according to

$$f(x, r, s) := \frac{1}{\lambda_1} D(q_X || p_X^{BEC(\varepsilon)}) + \frac{1}{\lambda_2} D(r_Y || p_Y^{BEC(\varepsilon)}) - D(r_{XY} || p_{XY}^{BEC(\varepsilon)})$$

We need to show that when $\lambda_2 < 0$ and λ_1 satisfies (2.14) then

$$\min_{x\in[0,1]} \max_{\substack{0\le r\le x,\\ 0\le s\le 1-x}} f(x,r,s) \ge 0.$$

Define the function

$$g(x) := \max_{0\le r\le x, 0\le s\le 1-x} f(x,r,s).$$

Then suffices that $g(x) \ge 0$ for $x \in [0,1]$. A simple symmetry argument shows that $g(x)$ is symmetric about $x = \frac{1}{2}$.

The idea of the proof is as follows: we will show that $g(x)$ has 3 stationary points in the interval $x \in (0,1)$, with one of them being at $x = \frac{1}{2}$. When $(\lambda_1 - 1)(\lambda_2 - 1) \ge 1 - \epsilon$, we will show that $g(x)$ is a local minimum at $x = \frac{1}{2}$, implying that the other two symmetric stationary points correspond to local maxima. Since $g(\frac{1}{2}) = 0$, it suffices to verify that the boundary condition, i.e. $g(0) \ge 0$. It will turn out that this boundary point is what yields (2.14), the critical condition in this case.

For a fixed $x \in (0,1)$, since $\lambda_2 < 0$, convexity of $D(p||q)$ in p immediately implies that $f(x,r,s)$ is concave in r, s (when viewed as a bivariate function). Further the derivatives at the boundary tend to infinite, implying that the maximum of $f(x,r,s)$ (for a fixed x) is attained strictly in the interior. Thus, from concavity, there is a unique pair of points $r_0(x) \in (0,x)$ and $s_0(x) \in (0,1-x)$ such that

$$g(x) = f(x, r_0(x), s_0(x)).$$

We will first analyze the interior stationary points of $g(x)$. If x^* is a stationary point, then one can check that $f(x^*, r_0(x^*), s_0(x^*))$ is a stationary point of $f(x,r,s)$. This is just a consequence of $f(x,r,s)$ being sufficiently smooth and the details are omitted here.

Setting gradients to be zero, we have

$$\begin{aligned} \frac{1}{\lambda_1}\ln\frac{x}{1-x} - \ln\frac{x-r}{1-x-s} &= 0,\\ \frac{1}{\lambda_2}\ln\frac{2\epsilon r}{(1-\epsilon)(1-r-s)} - \ln\frac{\epsilon r}{(1-\epsilon)(x-r)} &= 0,\\ \frac{1}{\lambda_2}\ln\frac{2\epsilon s}{(1-\epsilon)(1-r-s)} - \ln\frac{\epsilon s}{(1-\epsilon)(1-x-s)} &= 0. \end{aligned}$$

These equations are essentially the same as those Lagrange conditions in forward hypercontractivity Equation (2.11), if we use the parametrization $q_{00} =$

$r, q_{0E} = x - r, q_{1E} = 1 - x - s, q_{11} = s$ in Equation (2.11). So via the same manipulations there (not repeated here), letting $1 - \delta = \frac{2(x-r)}{1-r-s}$, we have

$$\frac{1-\epsilon}{\epsilon}(1-\delta)^{\lambda_2'-\lambda_1} + (1-\delta)^{1-\lambda_1} = \frac{1-\epsilon}{\epsilon}(1+\delta)^{\lambda_2'-\lambda_1} + (1+\delta)^{1-\lambda_1} \tag{2.15}$$

where λ_2' is Hölder conjugate of λ_2. Further every solution of the gradients condition is in one-to-one correspondence to a root of (2.15).

According to Lemma 2.3, under the condition $\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2-1}{\lambda_2}\ln[(1-\epsilon)2^{\frac{1}{\lambda_2-1}}+\epsilon]}$, equation (2.15) has only three roots $\delta = -\gamma, \gamma, 0$ for some $\gamma \in (0,1)$.

Correspondingly, the number of interior stationary points $f(x,r,s)$ is three given by: $x_1^* = \frac{1}{2}$; and two symmetric points $x_2^* = \frac{(1+\gamma)\epsilon + (1+\gamma)^{\lambda_2'}(1-\epsilon)}{2\epsilon + (1-\epsilon)[(1+\gamma)^{\lambda_2'} + (1-\gamma)^{\lambda_2'}]} > \frac{1}{2}$, and $x_3^* = 1 - x_2^* = \frac{(1-\gamma)\epsilon + (1-\gamma)^{\lambda_2'}(1-\epsilon)}{2\epsilon + (1-\epsilon)[(1+\gamma)^{\lambda_2'} + (1-\gamma)^{\lambda_2'}]} < \frac{1}{2}$.

Part (i) of Lemma 2.3 establishes that the condition (2.14) and $\varepsilon \in (0,1)$ implies $(\lambda_1 - 1)(\lambda_2 - 1) > 1 - \epsilon$; and under this case we will show that $x^* = \frac{1}{2}$ is a local minimizer of $g(x)$. Then x_2^* and x_3^* cannot be a local minimizer of $g(x)$ as $g(x)$ is continuously differentiable on $(0,1)$. Thus, x_2^* and x_3^* cannot be global minimizers of $g(x)$.

To show $x^* = \frac{1}{2}$ is a local minimizer of $g(x)$, notice that $g(\frac{1}{2}) = f(\frac{1}{2}, \frac{1-\epsilon}{2}, \frac{1-\epsilon}{2}) = 0$. So suffices to show that for $\delta > 0$ arbitrarily small, $g(\frac{1}{2} + \delta) > 0$.

One can verify that

$$f(\frac{1}{2}+\delta, r_0(\frac{1}{2}+\delta), s_0(\frac{1}{2}+\delta)) = 2(\frac{1}{\lambda_1} - \frac{1-\lambda_2}{\epsilon - \lambda_2})\delta^2 + O(\delta^3).$$

which is strictly positive for small δ precisely when

$$(\lambda_1 - 1)(\lambda_2 - 1) > 1 - \epsilon.$$

Thus the global minimizer of $g(x)$ can only be one of the three points $\{0, \frac{1}{2}, 1\}$. By symmetry $g(0) = g(1)$. Now $g(0) = \max_{s\in[0,1]} f(0,0,s)$, where

$$\begin{aligned} f(0,0,s) =& \frac{1}{\lambda_1}\ln 2 + \frac{1}{\lambda_2}\left[s\ln\frac{2s}{1-\epsilon} + (1-s)\ln\frac{1-s}{\epsilon}\right] \\ & - s\ln\frac{2s}{1-\epsilon} - (1-s)\ln\frac{2(1-s)}{\epsilon} \end{aligned}$$

Notice the above function is concave over s. By taking derivative over s, we get that the maximum point $s_0(0) = \frac{1-\epsilon}{1-\epsilon+2^{\frac{1}{1-\lambda_2}}\epsilon}$.

Thus $f(0,0,s_0(0)) \geq 0$ is equivalent (after re-arranging) to

$$\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2-1}{\lambda_2}\ln[(1-\epsilon)2^{\frac{1}{\lambda_2-1}} + \epsilon]}.$$

This range, from first part of Lemma 2.3, also satisfies $(\lambda_1 - 1)(\lambda_2 - 1) > 1 - \epsilon$, implying that when (2.14) holds, $g(x) \geq 0$ for all $x \in [0, 1]$ and hence (2.13) holds.

(2.13) $\Rightarrow$ (2.14): Let $q_X(X = 0) = 0$. If r_{XY} is not absolutely continuous with respect to $p_{XY}^{BEC(\varepsilon)}$, $D(r_{XY}||p_{XY}^{BEC})$ will become $+\infty$, while $\frac{1}{\lambda_1} D(q_X||p_X^{BEC})$, $\frac{1}{\lambda_2} D(r_Y||p_Y^{BEC})$ are finite, which contradicts the condition. Suffices to consider the case when r_{XY} is absolutely continuous with respect to $p_{XY}^{BEC(\varepsilon)}$.

As before denote $r_{XY}(X = 1, Y = 1) = s, (0 \leq s \leq 1)$. The condition $\frac{1}{\lambda_1} D(q_X||p_X^{BEC(\varepsilon)}) + \frac{1}{\lambda_2} D(r_Y||p_Y^{BEC(\varepsilon)}) \geq D(r_{XY}||p_{XY}^{BEC(\varepsilon)})$ for some r_{XY} with $r_X = q_X$ leads to $f(0, 0, s) \geq 0$ for some $s \in [0, 1]$. But as mentioned in the previous section, this is equivalent to $f(0, 0, s_0(0)) \geq 0$, which leads to

$$\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2 - 1}{\lambda_2} \ln[(1 - \epsilon) 2^{\frac{1}{\lambda_2 - 1}} + \epsilon]}.$$

□

2.2.2 Binary Symmetric Channel with Uniform Inputs

Consider a uniformly distributed binary valued X and Y obtained by passing X through a BSC with crossover probability $\frac{1-\rho}{2}$. Denote the joint distribution as $p_{XY}^{BSC(\rho)}$. The hypercontractivity for this pair of (X, Y) has been established since the 70s and there are various proofs in the literature, see [12,13,31]. The simplest one, according to the authors, is the one due to Janson [34]. This section yields yet another proof of the celebrated Bonami-Beckner inequality starting from the divergence characterization. Friedgut [23] established a proof along the very same lines for a particular choice $\lambda_1 = \lambda_2 = 1 + |\rho|$, and this proof generalizes the proof to all parameters.

Similarly, one could compute the maximal correlation coefficient for $(X, Y) \sim p_{XY}^{BSC(\rho)}$ from the definition (2.3).

Proposition 2.2. *Given a pair of random variables (X, Y) following the DSBS(ρ) distribution, where $-1 \leq \rho \leq 1$. The maximal correlation coefficient $\rho_m(X, Y)$ is ρ.*

For $p_{XY}^{BSC(\rho)}$, both the forward and reverse hypercontractive regimes are characterized by the correlation lower bound.

Theorem 2.7 (Bonami-Beckner; alternate proof provided here)**.** *For (X, Y) distributed according $p_{XY}^{BSC(\rho)}$, the pair (X, Y) is (λ_1, λ_2) forward hypercontractive* *if*

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq \rho^2.$$

Proof. When $\rho = 0$ the result is trivial and follows from the monotonicity of norm. Hence, we assume that $\rho \neq 0$. The proof mimics that of case 2 of the BEC proof. We consider, w.l.o.g. the pair (λ_1, λ_2) satisfying $(\lambda_1 - 1)(\lambda_2 - 1) = \rho^2$. We are required to show that

$$\max_{q_{XY} \ll p_{XY}^{BSC}} \frac{1}{\lambda_1} D(q_X \| p_X^{BSC}) + \frac{1}{\lambda_2} D(q_Y \| p_Y^{BSC}) - D(q_{XY} \| p_{XY}^{BSC}) = 0.$$

It is rather elementary to see that the boundary points cannot be the maximizers; so we will only consider the interior points. The idea is to show that there is only one interior stationary point at $q_{XY} = p_{XY}^{BSC}$.

For any (strictly) interior stationary points, the Lagrange conditions yield

$$k = \frac{1}{\lambda_1} \ln(q_{00} + q_{01}) + \frac{1}{\lambda_2} \ln(q_{00} + q_{10}) - \ln \frac{q_{00}}{1+\rho} \tag{2.16a}$$
$$k = \frac{1}{\lambda_1} \ln(q_{00} + q_{01}) + \frac{1}{\lambda_2} \ln(q_{01} + q_{11}) - \ln \frac{q_{01}}{1-\rho} \tag{2.16b}$$
$$k = \frac{1}{\lambda_1} \ln(q_{10} + q_{11}) + \frac{1}{\lambda_2} \ln(q_{00} + q_{10}) - \ln \frac{q_{10}}{1-\rho} \tag{2.16c}$$
$$k = \frac{1}{\lambda_1} \ln(q_{10} + q_{11}) + \frac{1}{\lambda_2} \ln(q_{01} + q_{11}) - \ln \frac{q_{11}}{1+\rho} \tag{2.16d}$$

By considering equations (2.16a) and (2.16c); and (2.16b) and (2.16d) we obtain

$$\left(\frac{q_{00} + q_{01}}{q_{10} + q_{11}}\right)^{\frac{1}{\lambda_1}} = \frac{q_{00}}{q_{10}} \frac{1-\rho}{1+\rho} = \frac{q_{01}}{q_{11}} \frac{1+\rho}{1-\rho} = x \tag{2.17}$$

Similarly considering equations (2.16a) and (2.16b); and (2.16c) and (2.16d) we obtain

$$\left(\frac{q_{00} + q_{10}}{q_{01} + q_{11}}\right)^{\frac{1}{\lambda_2}} = \frac{q_{00}}{q_{01}} \frac{1-\rho}{1+\rho} = \frac{q_{10}}{q_{11}} \frac{1+\rho}{1-\rho} \tag{2.18}$$

Since $q_{00} + q_{01} + q_{10} + q_{11} = 1$, denoting $\theta = \frac{1-\rho}{1+\rho} \in (0,1) \cup (1,\infty)$ (since $\rho \neq 0$), elementary manipulations show that x satisfies the following equation

$$x^{\lambda_1 - 1} = \frac{(1+\theta x)^{\frac{1}{\lambda_2 - 1}} \theta + (\theta + x)^{\frac{1}{\lambda_2 - 1}}}{(\theta + x)^{\frac{1}{\lambda_2 - 1}} \theta + (1+\theta x)^{\frac{1}{\lambda_2 - 1}}}.$$

Since $(\lambda_1 - 1)(\lambda_2 - 1) = \rho^2 = \left(\frac{1-\theta}{1+\theta}\right)^2$, denoting by $t = \frac{1}{\lambda_2 - 1}$, we obtain that x satisfies

$$x^{t\left(\frac{1-\theta}{1+\theta}\right)^2} = \frac{(1+\theta x)^t \theta + (\theta + x)^t}{(\theta + x)^t \theta + (1+\theta x)^t}.$$

From Lemma 2.4, we could know that the equation above has only one root $x = 1$. Therefore there is exactly one stationary point, $q_{XY} = p_{XY}^{BSC}$. This ensures that the maximum of the divergence expression is zero and completes the proof. $\square$

The same technique that we employed here can be used for the evaluation of reverse hypercontractive region for binary symmetric channel with uniform inputs. In this case, a result due to Borrell [13] already shows that the correlation bound is tight and the technqiue developed here just provides another proof. Since the argument is similar to previous case, we will only provide an outline of this argument. As you will see, this case is considerably simpler than that of the erasure channel.

The information-measure characterization in Theorem 2.4 essentially reduces to checking that a certain min-max expression is non-negative. By analyzing each case (in Theorem 2.4) separately we can show, in a similar fashion, that any interior local minimum must be a stationary point.

Further by analyzing the first derivative conditions, we will arrive that all stationary points are in one-to-one correspondence with the set of y satisfying

$$x^{-t(\frac{1-\theta}{1+\theta})^2} = \frac{(1+\theta x)^t\theta + (\theta + x)^t}{(\theta + x)^t\theta + (1+\theta x)^t}, \tag{2.19}$$

for some appropriately defined $t \in (-\infty, 0)$ and $\theta \in (0, \infty)$. This is identical to the Equation (2.16) in Lemma 2.4 in the forward analysis for $p_{XY}^{BSC(\rho)}$ and the details are omitted. As shown again in the forward case, the above equation has a unique root $y = 1$ in $(0, \infty)$; when $\theta \in (0, \infty) \setminus \{1\}$. This shows that the unique interior stationary point is $r_{XY} = p_{XY}^{BSC}$. Contrary to the binary erasure channel, it turns out that the boundary points do not influence the reverse-hypercontractive region.

2.2.3 Binary Input Symmetric Output Channel with Uniform Inputs

Consider a pair of random variables (X, Y) where X is binary and uniformly distributed, and Y is obtained via a channel $W_{Y|X}$ that satisfies a symmetry property, $W_{Y|X}(Y = i|X = 1) = W_{Y|X}(Y = -i|X = -1) = p_i$, for $-K \leq i \leq K, K \in \mathbb{N}_+$. Denote the joint distribution as $p_{XY}^{BISO(\vec{p})}$ distribution. This class contains both the $p_{XY}^{BEC(\varepsilon)}$ and $p_{XY}^{BSC(\rho)}$. The maximal correlation coefficient for this joint distribution is known by [67].

Proposition 2.3. *Given a pair of random variables (X, Y) following the BISO($\vec{p}$) distribution. The maximal correlation coefficient $\rho_m(X, Y)$ is $\sum_{i=1}^{K} \frac{(p_i - p_{-i})^2}{p_i + p_{-i}}$.*

The correlation inner bound for this setting says that (X, Y) is (λ_1, λ_2) forward hypercontractive *only if*

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq \sum_{i=1}^{K} \frac{(p_i - p_{-i})^2}{p_i + p_{-i}}. \tag{2.20}$$

The following proposition states that the correlation lower bound is tight for forward hypercontractive regime of $p_{XY}^{BISO(\vec{p})}$ when $\lambda_2 \geq 2$.

Proposition 2.4. *For any $\lambda_2 \geq 2$, the pair $(X, Y) \sim p_{XY}^{BISO(\vec{p})}$ is (λ_1, λ_2) forward hypercontractive for any pair of λ_1, λ_2 satisfying the correlation bound* (2.20).

Proof. The proof mimics the proof of Case 1 in the proof of Theorem 2.5. Following the approach we need to show that

$$\| \mathrm{E}(f(X)|Y) \|_{\lambda_2'} \leq \|f(X)\|_{\lambda_1}.$$

Further, by monotonicity of norm, it suffices to restrict to

$$(\lambda_1 - 1)(\lambda_2 - 1) = \sum_{i=1}^{K} \frac{(p_i - p_{-i})^2}{p_i + p_{-i}}.$$

W.l.o.g. let $f(-1) = 1 - \delta$, $f(1) = 1 + \delta$. Then the above inequality reduces to showing

$$\sum_{i=-K}^{K} \frac{p_i + p_{-i}}{2} \left(1 - \delta \frac{p_i - p_{-i}}{p_i + p_{-i}}\right)^{\lambda_2'} \leq \left[\frac{1}{2}(1-\delta)^{\lambda_1} + \frac{1}{2}(1+\delta)^{\lambda_1}\right]^{\frac{\lambda_2'}{\lambda_1}}.$$

Observing that $\frac{\lambda_2'}{\lambda_1} \geq 1$, taking the binomial expansion of both sides (as earlier) and using $(1+x)^a \geq 1 + ax, a > 1, x \geq 0$, it suffices to show

$$1 + \sum_{k=1}^{\infty} \binom{\lambda_2'}{2k} \delta^{2k} \left(\sum_{i=-K}^{K} \frac{p_i + p_{-i}}{2} \left(\frac{p_i - p_{-i}}{p_i + p_{-i}}\right)^{2k} \right) \leq 1 + \frac{\lambda_2'}{\lambda_1} \sum_{k=1}^{\infty} \binom{\lambda_1}{2k} \delta^{2k}.$$

Comparing term by term, we see that equality holds when $k = 1$ and the inequality holds for other terms since $k \geq 2$ implies

$$\sum_{i=-K}^{K} \frac{p_i + p_{-i}}{2} \left(\frac{p_i - p_{-i}}{p_i + p_{-i}}\right)^{2k} \leq \sum_{i=-K}^{K} \frac{p_i + p_{-i}}{2} \left(\frac{p_i - p_{-i}}{p_i + p_{-i}}\right)^{2}.$$

This completes the proof of the proposition. $\square$

Remark 2.7. A key observation in the above argument is that when $1 < \lambda_1 \leq \lambda_2' \leq 2$, the terms $\binom{\lambda_1}{2k}$ and $\binom{\lambda_2'}{2k}$ are non-negative for any $k \geq 1$; $\rho_m(X,Y)^2\binom{\lambda_2'}{2} = \frac{\lambda_2'}{\lambda_1}\binom{\lambda_1}{2}$ (where $\rho_m(X,Y)^2$ is the maximal correlation coefficient); and for $j \geq 2$ the term $j-\lambda_2' \geq j-\lambda_1$ allows one to conclude the term by term relation. This is essentially a borrow of the argument in [34] for the DSBS scenario. .

2.3 Conclusion and Discussion

In this chapter, we derive part of the forward and reverse hypercontractivity region for a pair of variables distributed as the BEC with uniform inputs. The technique employed is essentially a local analysis (identifying local extremal points and comparing the function values between them). The key insight that enables us to do this effectively is that all interior stationary points are in one-to-one correspondence with the roots of certain equation. The Taylor series expansion of this equation has certain patterns on the signs of its coefficients, allowing us to get a control on the number of interior stationary points. We were led to investigating the uniqueness of stationary point after hearing Friedgut present his proof for a particular parameter of the BSC case.

The determination of hypercontractivity parameter for the binary erasure channel was a question posed to us by Jaikumar Radhakrishnan and Venkat Guruswami during the Simon's institute semester long program in information theory. For the binary erasure channel, one can extend the proof technique borrowed from [34] to the forward hypercontractivity parameter regime $\lambda_2 \geq \frac{3}{2}$. However for the rest of the regimes, the only proof we could obtain was using the divergence characterization.

The hypercontractivity parameters for binary symmetric channel with uniform inputs is derived in many regimes. We also obtain a proof of the Bonami-Beckner inequality (the BSC case). An interesting observation is that when correlation inner bound was tight, it turned out that the non-convex optimization problem had only one stationary point.

For the case of binary input symmetric output channels we showed that the correlation inner bound is tight for $\lambda_2 \geq 2$. However numerical simulations indicate that perhaps the correlation inner bound is tight until $\lambda_2 \geq \frac{4}{3}$; indicating yet another example of binary erasure channel being the opposite extremal (the other one is BSC) case among the space of binary input symmetric output channels.

As shown in [8] forward hypercontractive region is same as the Gray-Wyner source coding region. In recent past a variety of computations of capacity regions (or achievable regions) have been performed in network information theory. All of them involve optimizing non-convex functions (1.1) over probability spaces. The functions are linear combinations of information measures and usually satisfy the sub-additivity or super-additivity property. The exact computations have been done in some special cases, where the global maximizer could be identified by a local analysis.

In many cases, for instance [26], there is only a single interior local optimizer; and sometimes it is a competition between the boundary and the interior stationary point, [17]. However, in each case, the proofs are quite complicated and require careful analysis with very few re-use of specific results. There are some other similar problems (conjectures), for example the one in [55], where numerically, there do not exist any other local optimizer other than the conjectured ones. However, a rigorous mathematical proof is lacking for many of these settings.

All the problems being considered can be reduced to the non-convex problem family (1.1) by the technique in section 1.2.2, where a certain set of standard tools could be devised to isolate the global maximizers. This could have far-reaching consequences: for instance a fast approximation algorithm for obtaining the 2 to 4 norm for an arbitrary matrix with non-negative entries.

2.A Binary Erasure Channel with Uniform Inputs

Lemma 2.2. *For $\delta \in [-1,1]$, $\lambda_2 \in (1,2)$, $\epsilon \in (0,1)$ the following equation*

$$(1-\epsilon)(1-\delta)^{\frac{\epsilon}{\lambda_2-1}} + \epsilon(1-\delta)^{\frac{\epsilon-1}{\lambda_2-1}} = (1-\epsilon)(1+\delta)^{\frac{\epsilon}{\lambda_2-1}} + \epsilon(1+\delta)^{\frac{\epsilon-1}{\lambda_2-1}}.$$

has only one root at $\delta = 0$ if $(\epsilon - \frac{1}{2}) \leq \frac{3}{2}(\lambda_2 - 1)$.

Proof. Clearly $\delta = 0$ is a root of this equation. Denote $p-1 = \frac{1}{\lambda_2-1}$ for convenience of writing. Note that $p \in (2,\infty)$. Define the function $g(\delta)$

$$g(\delta) = \frac{1-\epsilon}{\epsilon}(1-\delta)^{(p-1)\epsilon} + (1-\delta)^{(p-1)(\epsilon-1)} - \frac{1-\epsilon}{\epsilon}(1+\delta)^{(p-1)\epsilon} - (1+\delta)^{(p-1)(\epsilon-1)}$$

$g(0) = 0$, $\lim_{\delta\to 1-} g(\delta) = +\infty$. Further $g(\delta) = -g(-\delta)$. The statement follows by showing $g(\delta)$ increases over $(0,1)$ if $(p-1)(\epsilon - \frac{1}{2}) \leq \frac{3}{2}$.

Take the derivative with respect to δ,

$$g'(\delta) = -(1-\epsilon)(p-1)[(1-\delta)^{p\epsilon-\epsilon-1} - (1-\delta)^{p\epsilon-p-\epsilon} + (1+\delta)^{p\epsilon-\epsilon-1} - (1+\delta)^{p\epsilon-p-\epsilon}]$$

Let $r = p\epsilon - \epsilon - \frac{p+1}{2}$, then $g'(\delta) \geq 0$ is equivalent to

$$(1-\delta)^r[(1-\delta)^{-\frac{p-1}{2}} - (1-\delta)^{\frac{p-1}{2}}] \geq (1+\delta)^r[(1+\delta)^{\frac{p-1}{2}} - (1+\delta)^{-\frac{p-1}{2}}]).$$

Observe that $r \leq \frac{1}{2}$ is equivalent to $(\epsilon - \frac{1}{2}) \leq \frac{3}{2}(\lambda_2 - 1)$. So we are done if we show that the above inequality holds for any $r \leq \frac{1}{2}$ and $p > 2$. Further since $\left(\frac{1-\delta}{1+\delta}\right)^r$ decreases in r, it suffices to show the inequality for $r = \frac{1}{2}$ and $p > 2$. Substituting $r = \frac{1}{2}$ and rearranging, we wish to show

$$(1-\delta)^{-\frac{p}{2}+1} + (1+\delta)^{-\frac{p}{2}+1} \geq (1+\delta)^{\frac{p}{2}} + (1-\delta)^{\frac{p}{2}}.$$

Performing a Taylor series expansion, it suffices to show

$$2\left[1 + \sum_{k=1}^{\infty} \binom{1-\frac{p}{2}}{2k} \delta^{2k}\right] \geq 2\left[1 + \sum_{k=1}^{\infty} \binom{\frac{p}{2}}{2k} \delta^{2k}\right]$$

Note that the first term ($k = 1$) is equal for both sides and is positive (in the case that $p > 2$). For $k \geq 2$ it is immediate (by expanding the binomial term) that

$$\binom{1-\frac{p}{2}}{2k} \geq \max\left\{0, \binom{\frac{p}{2}}{2k}\right\}.$$

This completes the proof of the lemma. $\square$

Lemma 2.3. *Let $\lambda_2' < \lambda_1 < 1, \lambda_2 < 0$, where $\lambda_2' := \frac{\lambda_2}{\lambda_2 - 1}$. When $\lambda_1 \leq \frac{\ln 2}{\ln 2 - \frac{\lambda_2 - 1}{\lambda_2}\ln[(1-\epsilon)2^{\frac{1}{\lambda_2-1}} + \epsilon]}$, the following hold:*

(i) $(\lambda_1 - 1)(\lambda_2 - 1) \geq 1 - \epsilon$. Further the inequality is strict if $\epsilon \in (0, 1)$.

(ii) The equation

$$\frac{1-\epsilon}{\epsilon}(1-\delta)^{\lambda_2'-\lambda_1} + (1-\delta)^{1-\lambda_1} = \frac{1-\epsilon}{\epsilon}(1+\delta)^{\lambda_2'-\lambda_1} + (1+\delta)^{1-\lambda_1}$$

has three roots $\delta = -\gamma, 0, \gamma$ for some $\gamma \in (0, 1)$ on the interval $\delta \in (-1, 1)$.

Proof. Note that

$$(\lambda_1 - 1)(\lambda_2 - 1) \geq \frac{\frac{(\lambda_2-1)^2}{\lambda_2}\ln[(1-\epsilon)2^{\frac{1}{\lambda_2-1}} + \epsilon]}{\ln 2 - \frac{\lambda_2-1}{\lambda_2}\ln[(1-\epsilon)2^{\frac{1}{\lambda_2-1}} + \epsilon]}.$$

Therefore it suffices to show that above right-hand side is larger than $1-\epsilon$ when $\lambda_2 < 0$. Setting $r = \frac{1}{1-\lambda_2} \in (0,1)$ and substituting into above right-hand side, it suffices to show that

$$\frac{\frac{1}{r(r-1)}\ln[(1-\epsilon)2^{-r}+\epsilon]}{\ln 2 + \frac{1}{r-1}\ln[(1-\epsilon)2^{-r}+\epsilon]} \geq 1-\epsilon.$$

This can be rearranged as

$$(1-\epsilon)+\epsilon 2^r \leq 2^{\frac{\epsilon r}{1-r+\epsilon r}}.$$

It is a rather immediate exercise to verify that the right-hand-side is strictly concave in ϵ, for $\epsilon \in (0,1)$; and since equality holds at $\epsilon = 0$ and $\epsilon = 1$, we have the desired result. This establishes part (i) of the lemma.

Proof of (ii): Define the function $h(x)$ for $0 < x < 2$

$$h(x) = \frac{1-\epsilon}{\epsilon}x^{\lambda_2'-\lambda_1} + x^{1-\lambda_1} - \frac{1-\epsilon}{\epsilon}(2-x)^{\lambda_2'-\lambda_1} - (2-x)^{1-\lambda_1}.$$

Note that $h(1) = 0$, $\lim_{x\downarrow 0} h(x) = +\infty$. Further $h(x) = -h(2-x)$. Part (ii) follows by showing that there is only one root for $h(x) = 0$ for $x \in (0,1)$.

Take the derivative with respect to x,

$$\begin{aligned} h^{'}(x) = & \frac{(1-\epsilon)(\lambda_2'-\lambda_1)}{\epsilon}x^{\lambda_2'-\lambda_1-1} + (1-\lambda_1)x^{-\lambda_1} \\ & + \frac{(1-\epsilon)(\lambda_2'-\lambda_1)}{\epsilon}(2-x)^{\lambda_2'-\lambda_1-1} + (1-\lambda_1)(2-x)^{-\lambda_1}. \end{aligned}$$

Note that

$$(1-\epsilon)\lambda_2' + \epsilon - \lambda_1 > 0 \iff (\lambda_1 - 1)(\lambda_2 - 1) > 1-\epsilon.$$

Thus $h'(1) = 2\left(\frac{(1-\epsilon)\lambda_2'+\epsilon-\lambda_1}{\epsilon}\right) > 0$ from part (i). Hence $h(x) = 0$ will have at least one root in $(0,1)$ by its continuity.

The claim that $h(x) = 0$ has only one root in $(0,1)$ will follow by showing that $h(x)$ first decreases and then increases on $(0,1)$; in other words $h'(x)$ has only one root in $(0,1)$. Since $\lim_{x\downarrow 0} h'(x) = -\infty$, $h'(1) > 0$, and $h'(x)$ is continuous on $(0,1]$, implies that there is at least one root at $x = 1-y_0$ for $y_0 \in (0,1)$ for $h'(x)$.

Setting $x = 1-y$ and considering the Taylor Series expansion of $h'(x)$ with respect to y about $y = 0$, we obtain

$$h'(1-y) = 2\sum_{k=0}^{\infty}\left[\frac{(1-\epsilon)(\lambda_2'-\lambda_1)}{\epsilon}\binom{\lambda_2'-\lambda_1-1}{2k} + (1-\lambda_1)\binom{-\lambda_1}{2k}\right]y^{2k}.$$

Let $a_k = (1-\lambda_1)\binom{-\lambda_1}{2k}$ and $b_k = \frac{(1-\epsilon)(\lambda_1-\lambda_2')}{\epsilon}\binom{\lambda_2'-\lambda_1-1}{2k}$. Note that $a_k, b_k \geq 0$ and

$$h'(1-y) = 2\sum_{k\geq 0}(a_k - b_k)y^{2k}.$$

Note that $a_0 \geq b_0$ (from part (i) or since this is $h'(1)$).

Suppose there exists $k_0 \in \mathbb{N}$ such that $a_{k_0} \leq b_{k_0}$, then $a_k \leq b_k, \forall k \geq k_0$. This follows basically from an induction argument, since

$$\begin{aligned} a_{k+1} &= a_k \frac{(\lambda_1 + 2k)(\lambda_1 + 2k + 1)}{(2k+1)(2k+2)} \\ b_{k+1} &= b_k \frac{(\lambda_1 + 1 - \lambda_2' + 2k)(\lambda_1 + 1 - \lambda_2' + 2k + 1)}{(2k+1)(2k+2)}. \end{aligned}$$

$1-\lambda_2' > 0$ implies that once $b_k \geq a_k$, the inequality continues to hold for larger k. Since $h'(1-y) = 0$ has a root in $(0,1)$, implies that $\exists\ m \geq 0$ such that $a_k \geq b_k, \forall k \leq m$ and $b_k \geq a_k, \forall k > m$.

Define $c_k = |a_k - b_k|$. Then

$$h'(1-y) = \sum_{k=0}^{m} c_k y^{2k} - \sum_{k\geq m+1}^{\infty} c_k y^{2k}$$

where $c_k \geq 0$ (with at least one c_k in each range, $k \in [1:m]$ and $k \geq m+1$ being strictly positive). Let $y_0 \in (0,1)$ be a root of $h'(1-y) = 0$.

For $y > y_0 > 0$, note that

$$\begin{aligned} \sum_{k=0}^{m} c_k y^{2k} &< \left(\frac{y}{y_0}\right)^{2m} \sum_{k=0}^{m} c_k y_0^{2k} \\ &= \left(\frac{y}{y_0}\right)^{2m} \sum_{k=m+1}^{\infty} c_k y_0^{2k} \\ &< \sum_{k=m+1}^{\infty} c_k y^{2k}. \end{aligned}$$

The equality above is a consequence of y_0 being a root. Thus, no $y > y_0$ can be a root of $h'(1-y) = 0$. Similarly, reversing inequalities above, for $0 < y < y_0$, y cannot be a root for $h'(1-y) = 0$.

Thus $h'(x) = 0$ has only one root in the interval $x \in (0,1)$, and as $\lim_{x\downarrow 0} h'(x) = -\infty$, $h'(1) > 0$, due to the continuity of $h'(x)$, we have $h'(x) < 0$ for $x \in (0, 1-y_0)$ and $h'(x) > 0$ for $x \in (1-y_0, 1)$. Putting this together with $\lim_{x\downarrow 0} h(x) = +\infty$ and $h(1) = 0$ implies that, $h(x) = 0$ has precisely one root, say $x = 1-\gamma$, in the interval $x \in (0,1)$. Since $h(1-\delta)$ is an odd function with respect to δ; the roots are given by $\delta = -\gamma, 0, \gamma$. This completes the proof of part (ii). □

2.B Binary Symmetric Channel with Uniform Inputs

Lemma 2.4. *For any $t \in (0,\infty)$ and $\theta \in (0,1)\cup(1,\infty)$ the equation*

$$x^{t\left(\frac{1-\theta}{1+\theta}\right)^2} = \frac{(1+\theta x)^t\theta + (\theta+x)^t}{(\theta+x)^t\theta + (1+\theta x)^t}$$

has only one root at $x = 1$ for $x \in (0,\infty)$.

Proof. Let $x = e^h$ and define

$$g(h) = \ln\left((1+\theta e^h)^t\theta + (\theta+e^h)^t\right).$$

Taking logarithms of the equation in Lemma 2.4 and making above substitutions, we wish to show that

$$ht\left(\frac{1-\theta}{1+\theta}\right)^2 = g(h) - g(-h) - ht$$

has exactly one zero at $h = 0$. Define

$$r(h) = g(h) - g(-h) - ht - ht\left(\frac{1-\theta}{1+\theta}\right)^2.$$

We will show that $r'(h) \le 0$ implying the desired result.

Note that

$$r'(h) = g'(h) + g'(-h) - t - t\left(\frac{1-\theta}{1+\theta}\right)^2.$$

Observe that

$$\begin{aligned} g'(h) &= t\left(\frac{\theta^2 e^h(1+\theta e^h)^{t-1} + e^h(\theta+e^h)^{t-1}}{(1+\theta e^h)^t\theta + (\theta+e^h)^t}\right) \\ &= t\left(1 - \theta\left(\frac{(1+\theta e^h)^{t-1} + (\theta+e^h)^{t-1}}{(1+\theta e^h)^t\theta + (\theta+e^h)^t}\right)\right). \end{aligned}$$

Substituting this into $r'(h)$, and after performing elementary manipulations, the condition $r'(h) \le 0$ becomes equivalent to verifying

$$\frac{4}{(1+\theta)^2} \le \left(\frac{(1+\theta e^h)^{t-1} + (\theta+e^h)^{t-1}}{(1+\theta e^h)^t\theta + (\theta+e^h)^t}\right) + e^h\left(\frac{(1+\theta e^h)^{t-1} + (\theta+e^h)^{t-1}}{(1+\theta e^h)^t + \theta(\theta+e^h)^t}\right).$$

The above condition can be re-expressed as

$$\begin{aligned} &\left((1+\theta e^h)^{t-1} + (\theta+e^h)^{t-1}\right)\left((1+\theta e^h)^{t+1} + (\theta+e^h)^{t+1}\right) \\ &\quad \ge \frac{4}{(1+\theta)^2}\left((1+\theta e^h)^t\theta + (\theta+e^h)^t\right) \times \left((1+\theta e^h)^t + \theta(\theta+e^h)^t\right). \end{aligned}$$

Elementary algebraic manipulation reduces the above to

$$\left(\frac{1-\theta}{1+\theta}\right)^2 \left((1+\theta e^h)^t - (\theta + e^h)^t\right)^2$$
$$+(1+\theta e^h)^{t-1}(\theta + e^h)^{t-1}(1+\theta e^h - \theta - e^h)^2 \geq 0,$$

which trivially holds. Furthermore, equality holds only at $h = 0$ implying that $r(h) = 0$ only at $h = 0$. □

Chapter 3

Lower Bounds on Distributed Source Coding

3.1 Introduction

Returning to the Körner and Marton's modulo two sum problem in the Introduction chapter, the optimal rate region for the Körner and Marton's modulo two sum problem in general unknown. Recall that we have two achievable rate regions for this problem: Slepian-Wolf region (1.7) and Körner-Marton region (1.8). And the best achievable rate region is given by Ahlswede and Han (1.9) in [2].

Körner showed the following result for the case when Slepian-Wolf region is optimal:

Theorem 3.1 (Exercise 16.23 in [20])**.** *When $H(Z) \geq \min\{H(X), H(Y)\}$, Slepian-Wolf's rate region characterizes the optimal rate region $\mathscr{R}_{KM}(p_{XY})$ for the Körner-Marton sum modulo two problem.*

On the other hand, Körner and Marton gave the following result for the case when Körner-Marton region is optimal in [35]:

Theorem 3.2. *When (X, Y) follows a DSBS distribution, Körner-Marton region characterizes the optimal rate region $\mathscr{R}_{KM}(p_{XY})$ for the Körner-Marton sum modulo two problem.*

Remark 3.1. To the best of the knowledge of the authors, these two theorems are all the collection of joint distributions p_{XY} for which the optimal rate region has been determined. Here we will show that linear codes minimize the sum-capacity for a larger class of distributions that include the DSBS as a special case.

The following is the cut-set lower bound which is rather immediate.

Theorem 3.3 ([35]). *Any achievable rate pair (R_1, R_2) for the modulo sum problem must satisfy*

$$\begin{aligned} R_1 &\geq H(Z|Y) = H(X|Y) \\ R_2 &\geq H(Z|X) = H(Y|X) \\ R_1 + R_2 &\geq H(Z). \end{aligned}$$

In this chapter, we first derive a lower bound for the weighted sum-rate of the optimal rate region for the Körner and Marton's modulo two sum problem. Then we will show that the lower bound is tight for several classes of distributions (including distributions for which the optimality was not known before).

Next, we will present alternate proofs to the converse of the optimal rate regions of quadratic Gaussian CEO and quadratic Gaussian distributed source coding problems. These two proofs are similar. First we will derive some weighted sum rate lower bounds. Then we will use the rotations techniques in [27] to show the Gaussian distribution minimizes the weighted sum rate lower bound, which will imply that the Berger-Tung inner bound is optimal for these two settings.

The results on the Körner and Marton's modulo two sum problem of this chapter first appear in [50]. This is a joint work with Prof. Chandra Nair. To the best knowledge of us, the alternate proofs on the quadratic Gaussian CEO and quadratic Gaussian distributed source coding problems are new in this thesis.

3.2 Main Results on Körner and Marton's Modulo Two Sum Problem

The following tensorization lemma will be used in the proof of the theorem.

Lemma 3.1. *Let $\lambda \geq 1$ and let (X^n, Y^n) be i.i.d distributed according to $p(x, y)$ where X, Y take values in a finite field. Let Z^n be obtained as $Z_i = X_i \oplus Y_i, i = 1, .., n$, i.e. the component-wise modulo sum on the field. Then for any $\lambda \geq 1$ the following holds:*

$$\min_{\hat{U}:\hat{U}\to X^n\to Y^n} \lambda H(Z^n|\hat{U}) - H(Y^n|\hat{U}) = n\left(\min_{U:U\to X\to Y} \lambda H(Z|U) - H(Y|U)\right).$$

Proof. Clearly, by taking i.i.d. copies of the minimizer of the right-hand side, it is immediate that the left-hand side is at most the value of the right-hand side. To show the other direction, observe that

$$\begin{aligned}
&\lambda H(Z^n|\hat{U}) - H(Y^n|\hat{U}) \\
&\quad = \sum_{i=1}^{n} \left[(\lambda-1)H(Z_i|\hat{U}, Z^{i-1}) + H(Z_i|\hat{U}, Z^{i-1}) - H(Y_i|\hat{U}, Y_{i+1}^n)\right] \\
&\quad = \sum_{i=1}^{n} \left[(\lambda-1)H(Z_i|\hat{U}, Z^{i-1}) + H(Z_i|\hat{U}, Z^{i-1}, Y_{i+1}^n) - H(Y_i|\hat{U}, Z^{i-1}, Y_{i+1}^n)\right] \\
&\quad \geq \sum_{i=1}^{n} \lambda H(Z_i|U_i) - H(Y_i|U_i),
\end{aligned}$$

where $U_i = (\hat{U}, Y_{i+1}^n, Z^{i-1})$ and note that $U_i \to X_i \to (Y_i, Z_i)$ is Markov. The second equality above uses the Körner-Marton identity that $\sum_{i=1}^n I(Z^{i-1}; Y_i|\hat{U}, Y_{i+1}^n) = \sum_{i=1}^n I(Y_{i+1}^n; Z_i|\hat{U}, Z^{i-1})$. This completes the proof. □

We now state a lower bound to the optimal rate region, which we believe is new.

Theorem 3.4. *Any achievable rate pair (R_1, R_2) for the modulo sum problem must satisfy the following constraints for any $\lambda \geq 1$:*

$$R_1 + \lambda R_2 \geq H(XY) + \min_{U \to X \to Y} \lambda H(Z|U) - H(Y|U)$$

$$\lambda R_1 + R_2 \geq H(XY) + \min_{V \to Y \to X} \lambda H(Z|V) - H(X|V)$$

Proof. For $\lambda \geq 1$, any sequence of compression schemes that achieves a rate pair (R_1, R_2) will require that

$$\begin{aligned}
&n(R_1 + \lambda R_2) + n(1+\lambda)\varepsilon_n \\
&\overset{(a)}{\geq} I(M_1M_2; X^nY^n) + (\lambda-1)H(M_2|M_1) + (1+\lambda)H(Z^n|M_1M_2) \\
&\overset{(b)}{=} H(X^nY^n) - \underset{\sim}{H(X^nY^nM_1M_2)} + H(M_1M_2) + (\lambda-1)H(M_1M_2) - (\lambda-1)H(M_1) \\
&\qquad + (1+\lambda)H(Z^nM_1M_2) - (\lambda+1)H(M_1M_2) \\
&\overset{(c)}{=} H(X^nY^n) + \lambda H(Z^nM_1M_2) + H(Z^nM_1M_2) - \underset{\sim}{H(Z^nY^nM_1M_2)} - H(M_1M_2) \\
&\qquad - (\lambda-1)H(M_1)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(d)}{=} H(X^nY^n) + \lambda H(Z^nM_1) + \underline{\lambda H(M_2|M_1Z^n)} - \underset{\sim\sim\sim}{H(Y^nM_1M_2)} + \underline{I(Z^n;Y^n|M_1M_2)} \\
&\qquad - (\lambda-1)H(M_1) \\
&\overset{(e)}{\geq} nH(XY) + \lambda H(Z^nM_1) - \underset{\sim\sim\sim}{H(Y^nM_1)} - (\lambda-1)H(M_1) \\
&\overset{(f)}{=} nH(XY) + \lambda H(Z^n|M_1) - H(Y^n|M_1) \\
&\overset{(h)}{\geq} nH(XY) + n\Big(\min_{U\to X\to Y} \lambda H(Z|U) - H(Y|U)\Big)
\end{aligned}
$$

The step (a) is due to the fact that $nR_1 + nR_2 \geq H(M_1M_2) \geq I(M_1M_2; X^nY^n)$, $(\lambda-1)R_2 \geq (\lambda-1)H(M_2) \geq (\lambda-1)H(M_2|M_1)$, and $n\varepsilon_n \geq H(Z^n|M_1M_2)$ by Fano's inequality; step (b) follows from breaking down conditional entropies and mutual informations to entropies; step (c) follows from the indentity $H(X^nY^nM_1M_2) = H(Z^nX^nY^nM_1M_2) = H(Z^nY^nM_1M_2)$; step (d) is trying to get rid of M_2 by chain rules and uses the fact that $I(Z^n;Y^n|M_1M_2) = H(Z^nM_1M_2) + H(Y^nM_1M_2) - H(Z^nY^nM_1M_2) - H(M_1M_2)$; step (e) is dropping the nonnegative terms $H(M_2|M_1Z^n)$ and $I(Z^n;Y^n|M_1M_2)$ (dropping this mutual information is not really a loss since it's upper bounded by $H(Z^n|M_1M_2)$, which is upper bounded by $n\varepsilon_n$); step (f) is using the definitions of conditional entropies; while the last step (h) can be single-letterizied by using Lemma 3.1.

The other lower bound in the Theorem 3.4 follows in a similar manner. □

Remark 3.2. From section 1.2.3 the equivalence characterization of upper concave envelopes (1.15) we can see that

$$
\begin{aligned}
&\min_{U\to X\to Y} \lambda H(Z|U) - H(Y|U) \\
&\quad = -\Big(\max_{U\to X\to Y} H(Y|U) - \lambda H(Z|U)\Big) \\
&\quad = -\mathfrak{C}_{q_X}[H(Y) - \lambda H(Z)](p_X),
\end{aligned}
$$

Hence the lower bound in Theorem 3.4 can be written as

$$
\begin{aligned}
R_1 + \lambda R_2 &\geq H(XY) - \mathfrak{C}_{q_X}[H(Y) - \lambda H(Z)]\big|_{p_X} \\
\lambda R_1 + R_2 &\geq H(XY) - \mathfrak{C}_{q_Y}[H(X) - \lambda H(Z)]\big|_{p_Y}
\end{aligned} \tag{3.1}
$$

for any $\lambda \geq 1$.

The following lemma exhibits two conditions under which the lower bound is tight. A similar statement also holds when the roles of X and Y are interchanged.

Lemma 3.2. *The lower bound for the weighted sum-rate $R_1 + \lambda R_2$, for $\lambda \geq 1$ given in Theorem 3.4 is optimal, i.e. matches the weighted sum-rate of the optimal rate region, if either of the following conditions hold:*

(i) $\mathfrak{C}_{q_X}[H(Y) - \lambda H(Z)]\big|_{p_X} = H(Y) - \lambda H(Z)$ *and* $Y \perp Z$,

(ii) $\mathfrak{C}_{q_X}[H(Y) - \lambda H(Z)]\big|_{p_X} = H(Y|X) - \lambda H(Z|X)$.

Further if condition (i) *holds for some $\lambda_1 > 1$, then it will also hold for $1 \leq \lambda \leq \lambda_1$; and if condition* (ii) *holds for some $\lambda_2 \geq 1$, then it will also hold for $\lambda \geq \lambda_2$.*

Remark 3.3. A relatively easier condition to verify is the following: For a fixed $p_{Y|X}$ (and hence $p_{Z|X}$), if $H(Y) - \lambda H(Z)$ is concave in the distribution of X, q_X, then condition (i) above holds. On the other hand if $H(Y) - \lambda H(Z)$ is convex in the distribution of X, q_X, then condition (ii) above holds.

Proof. If condition (i) holds: we have from (3.1)

$$\begin{aligned} R_1 + \lambda R_2 &\geq H(XY) - H(Y) + \lambda H(Z) \\ &= H(X|Y) + \lambda H(Z) \\ &= (\lambda + 1) H(Z) \end{aligned}$$

where the last equality uses $H(X|Y) = H(Z|Y) = H(Z)$. Note that $R_1 = H(Z), R_2 = H(Z)$ belongs to the Körner-Marton achievable region, thus showing the achievability of this optimal weighted sum-rate using linear codes.

If condition (ii) holds: we have from (3.1)

$$\begin{aligned} R_1 + \lambda R_2 &\geq H(XY) - H(Y|X) + \lambda H(Z|X) \\ &= H(X) + \lambda H(Y|X). \end{aligned}$$

Note that $R_1 = H(X), R_2 = H(Y|X)$ belongs to the Slepian-Wolf achievable region, thus showing the achievability of this optimal weighted sum-rate using random binning.

To show the second part, note that condition (i) is equivalent to

$$H(Y|U) - \lambda H(Z|U) \leq H(Y) - \lambda H(Z) \; \forall U - X - Y.$$

Hence if condition (i) holds for some λ_1 then for $1 \leq \lambda \leq \lambda_1$, we have

$$H(Y|U) - \lambda H(Z|U)$$

$$
\begin{aligned}
&= H(Y|U) - \lambda_1 H(Z|U) + (\lambda_1 - \lambda) H(Z|U) \\
&\leq H(Y) - \lambda_1 H(Z) + (\lambda_1 - \lambda) H(Z) \\
&= H(Y) - \lambda H(Z).
\end{aligned}
$$

Similarly, note that condition (ii) is equivalent to

$$H(Y|U) - \lambda H(Z|U) \leq H(Y|X) - \lambda H(Z|X) \ \forall U - X - Y.$$

Hence if condition (ii) holds for some λ_2 then for $\lambda \geq \lambda_2$, we have

$$
\begin{aligned}
&H(Y|U) - \lambda H(Z|U) \\
&= H(Y|U) - \lambda_2 H(Z|U) - (\lambda - \lambda_2) H(Z|U) \\
&\leq H(Y|X) - \lambda_2 H(Z|X) - (\lambda - \lambda_2) H(Z|X) \\
&= H(Y|X) - \lambda H(Z|X),
\end{aligned}
$$

where we have used $U \to X \to Z$ being Markov in the last inequality, apart from that condition (ii) holds for λ_2. □

Remark 3.4. The conditions for optimality in the lemma is reminiscent of the essentially less noisy condition for broadcast channel in [44].

Corollary 3.1. *The Slepian-Wolf rate region is optimal for the modulo-sum problem if* $\mathfrak{C}_{q_X}[H(Y) - H(Z)]\big|_{p_X} = H(Y|X) - H(Z|X) = 0$. *Similarly, it is optimal if* $\mathfrak{C}_{q_Y}[H(X) - H(Z)]\big|_{p_Y} = H(X|Y) - H(Z|Y) = 0$.

Proof. If $\mathfrak{C}_{q_X}[H(Y) - H(Z)]\big|_{p_X} = H(Y|X) - H(Z|X)$, then we have from Equation (3.1) that

$$R_1 + R_2 \geq H(XY).$$

The constraints $R_1 \geq H(X|Y)$ and $R_2 \geq H(Y|X)$ follow from Theorem 3.3. The other condition follows similarly. □

3.2.1 Application to binary alphabets

In this section we will study distributions over pairs of binary alphabets and determine conditions under which one of the conditions in Lemma 3.2 hold. We will see that we can recover all the previously determined cases as well as recover new distributions from the results listed below.

Notation: We will parameterize the space of distributions over pairs of binary alphabets, p_{XY} as follows: $\mathrm{P}_X(X=0)=x, \mathrm{P}_{Y|X}(Y=0|X=0)=c, \mathrm{P}_{Y|X}(Y=1|X=1)=d$.

Proposition 3.1. *The optimal weighted sum-rate of the optimal rate region is given by the Slepian Wolf region if any of the following conditions hold:*

(1) *For any λ, if $(c-\frac{1}{2})(d-\frac{1}{2})\leq 0$, or*

(2) $\lambda \geq \left(\frac{c-\bar{d}}{c-d}\right)^2, c\neq d$, *and* $(c-\frac{1}{2})(d-\frac{1}{2})>0$.

where $\bar{d}=1-d$.

Proof. If condition (1) holds: then it suffices to show by Corollary 3.1 that $H(Y)-H(Z)$ is convex in q_X, which will then imply that $\mathfrak{C}_{q_X}[H(Y)-H(Z)]\big|_{p_X} = H(Y|X)-H(Z|X)$. Denoting $q(X=0)=u$, we need to show that

$$g(u) := H_2(uc+\bar{u}\bar{d}) - H_2(uc+\bar{u}d)$$

is convex in u, when $(c-\frac{1}{2})(d-\frac{1}{2})\leq 0$. Here $H_2(x)=-x\log_2 x-(1-x)\log_2(1-x)$ denotes the binary entropy function. Elementary calculations show that $g(u)$ is convex for $u\in[0,1]$ if and only if $(c-\frac{1}{2})(d-\frac{1}{2})\leq 0$.

If condition (2) holds: then it suffices to show by Lemma 3.2 that for $\lambda_2 = \left(\frac{c-\bar{d}}{c-d}\right)^2$, we have $\mathfrak{C}_{q_X}[H(Y)-\lambda_2 H(Z)]\big|_{p_X} = H(Y|X)-\lambda_2 H(Z|X)$. As before it suffices to show that

$$g(u) := H_2(uc+\bar{u}\bar{d}) - \lambda_2 H_2(uc+\bar{u}d)$$

is convex in u. This is again verifiable by elementary calculations. □

Remark 3.5. The following points are worth noting:

(*i*) The condition (1) above is already known and stated as exercise 16.23 page 390 of Csiszár and Körner's book [20]. One can verify that $H(Z)\geq H(Y)$ is equivalent to $(c-\frac{1}{2})(d-\frac{1}{2})\leq 0$.

(*ii*) Note that an equivalent proposition can also be stated for the alternate parameterization: $\mathrm{P}(Y=0)=y, \mathrm{P}(X=0|Y=0)=\hat{c}, \mathrm{P}(X=1|Y=1)=\hat{d}$.

The next proposition determines conditions under which the optimal weighted sum-rate is given by the Körner-Marton region, i.e. satisfy the first constraint of Lemma 3.2. Continuing with the same notation $\mathrm{P}(X=0)=x, \mathrm{P}(Y=0|X=0)=c, \mathrm{P}(Y=1|X=1)=d$, since we require Y to be independent of c, we need to restrict to $x=\frac{\sqrt{d\bar{d}}}{\sqrt{d\bar{d}}+\sqrt{c\bar{c}}}$.

Proposition 3.2. *Let* $\mathrm{P}(X=0)=x, \mathrm{P}(Y=0|X=0)=c, \mathrm{P}(Y=1|X=1)=d$ *where* $x=\frac{\sqrt{d\bar{d}}}{\sqrt{d\bar{d}}+\sqrt{c\bar{c}}}$. *The optimal weighted sum-rate of the optimal rate region is given by the Körner-Marton region, i.e. using linear codes, if any of the following conditions hold:*

(*A*) *For any* λ, *if* $c=d$, *or*

(*B*) $1 \leq \lambda \leq \lambda_1, c \neq d$, *and* $(c-\frac{1}{2})(d-\frac{1}{2})>0$, *where* λ_1 *is the larger root of the quadratic equation*

$$\lambda^2(c-d)^2+\lambda\big(2(c-d)(c-\bar{d})-4d\bar{d}(c-\bar{c})^2\big)+(c-\bar{d})^2=0.$$

where $\bar{d}=1-d, \bar{c}=1-c$.

Proof. If condition (A) in Proposition 3.2 holds: then Z is independent of X and $H(Y)-\lambda H(Z)$ is concave in q_X, therefore

$$\mathfrak{C}_{q_X}[H(Y)-\lambda H(Z)]\big|_{p_X}=H(Y)-\lambda H(Z).$$

Therefore Condition (i) in Lemma 3.2 (see (3.1)) is satisfied and we are done. Note that this is precisely the DSBS source whose capacity region was established by Körner and Marton in [35].

If condition (B) in Proposition 3.2 holds: define

$$g(u):=H_2(uc+\bar{u}\bar{d})-\lambda_1 H_2(uc+\bar{u}d)$$

where λ_1 is the larger root of the quadratic equation

$$\lambda^2(c-d)^2+\lambda\big(2(c-d)(c-\bar{d})-4d\bar{d}(c-\bar{c})^2\big)+(c-\bar{d})^2=0.$$

Then elementary calculations can be used to verify that $g(u)$ is concave for $u \in [0,1]$ and hence

$$\mathfrak{C}_{q_X}[H(Y)-\lambda H(Z)]\big|_{p_X}=H(Y)-\lambda H(Z).$$

As before Condition (i) in Lemma 3.2 (see (3.1)) is satisfied and we are done. $\square$

Remark 3.6. The following points are worth noting:

(i) As long as, $(c-\frac{1}{2})(d-\frac{1}{2}) > 0$, we can see that $\lambda_1 > 1$, and hence the optimal sum-rate, will be given by the Körner-Marton region, i.e. using linear codes. Note that we still need $x = \frac{\sqrt{d\bar{d}}}{\sqrt{d\bar{d}}+\sqrt{c\bar{c}}}$. Thus linear coding strategy of Körner-Marton are optimal for some larger class of parameters.

(ii) As before, an equivalent Proposition can also be stated for the alternate parameterization: $\mathrm{P}(Y = 0) = y, \mathrm{P}(X = 0|Y = 0) = \hat{c}, \mathrm{P}(X = 1|Y = 1) = \hat{d}$.

3.2.2 Comparison of the bounds

In [2] Ahlswede and Han chose the following p_{XY} given by

$$p_{XY} = \begin{bmatrix} p_{XY}(0,0) & p_{XY}(0,1) \\ p_{XY}(1,0) & p_{XY}(1,1) \end{bmatrix} = \begin{bmatrix} 0.003920 & 0.019920 \\ 0.976080 & 0.000080 \end{bmatrix}$$

where row index is $x \in \{0,1\}$, column index is $y \in \{0,1\}$, to show that their achievable rate region performs strictly better than both Körner and Marton's rate region and Slepian and Wolf's rate region. It turns out that for this distribution Y is indeed independent of Z. Therefore from Remark 3.6 we already know that the optimal sum-rate is given by the Körner-Marton linear coding region.

Figure 3.1 plots Ahlswede-Han's rate region, the lower bound from Theorem 3.4, and the cut-set lower bound for the above example.

As one can see readily and as established in Proposition 3.2, the lower bound in Theorem 3.4 yields the optimal sum-rate of $2H(Z)$ for this example. By numerical simulations: the largest λ for which the hyperplane of the lower bound passes through the $(H(Z), H(Z))$ point is $\lambda_1^* = 5.253$ (matches, curiously, the sufficient condition established in Proposition 3.2), while that for the Ahlswede-Han region is $\lambda_1^\dagger = 5.338$. Then the largest λ for which the hyperplane of the lower bound passes through the $(H(X), H(Y|X))$ point is $\lambda_2^* = 25.844$ (matches the sufficient condition established in Proposition 3.1), while, by numerical simulations, that for the Ahlswede-Han region is $\lambda_2^\dagger = 6.620$.

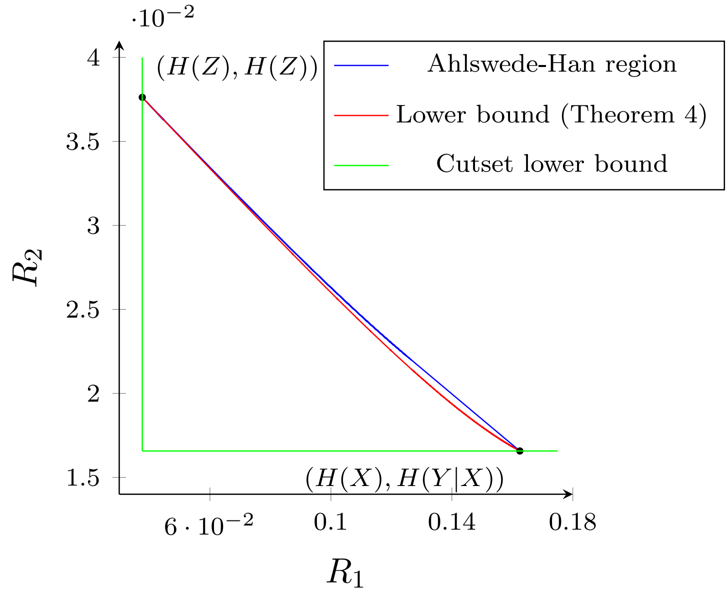

Figure 3.1: Comparison of Ahlswede-Han region and our lower bound

3.2.3 Application to higher alphabet fields

The modulo-sum problem for binary alphabets has a peculiar structure that was exploited in the Exercise 16.23 of [20]. If $H(Z) \geq H(Y)$, then $P_{Y|X}$ was a stochastic degradation of $p_{Z|X}$, and the reverse held if $H(Y) \geq H(Z)$. In general we know that for higher alphabets the above dichotomy does not hold. Hence Lemma 3.2 establishes that a better comparision between the channels $p_{Z|X}$ and $p_{Y|X}$ for obtaining the optimal weighted sum-rate is related to (essentially) less noisy comparison.

Below we provide two examples in $GF(3)$ for which the results in Lemma 3.2 yield optimality. Here $Z = X + Y$ in $GF(3)$.

For $GF(3)$, one instance of p_{XY} satisfying that Z is independent of Y and $\mathfrak{C}_{q_X}[H(Y) - H(Z)]\big|_{p_X} = H(Y) - H(Z)$ is given by the following distribution:

$$p_{XY} = \begin{bmatrix} 0.08 & 0.06 & 0.18 \\ 0.08 & 0.18 & 0.06 \\ 0.24 & 0.06 & 0.06 \end{bmatrix}$$

where row index is $x \in \{0, 1, 2\}$, column index is $y \in \{0, 1, 2\}$.

One can check that for this joint distribution p_{XY}, $P(Y) = [0.4\ 0.3\ 0.3]$, $P(Z) = [0.2\ 0.2\ 0.6]$, so Z is independent of Y.

Besides, one could construct a auxiliary $\hat{Z}$ such that $X \to Y \to \hat{Z}$ and

$p_{\hat{Z}|X} = p_{Z|X}$, by the following choice of $p_{\hat{Z}|Y}$:

$$\begin{array}{lll} P(\hat{Z}=0|Y=0)=\frac{1}{8} & P(\hat{Z}=1|Y=0)=\frac{1}{8} & P(\hat{Z}=2|Y=0)=\frac{3}{4} \\ P(\hat{Z}=0|Y=1)=\frac{1}{6} & P(\hat{Z}=1|Y=1)=\frac{1}{3} & P(\hat{Z}=2|Y=1)=\frac{1}{2} \\ P(\hat{Z}=0|Y=2)=\frac{1}{3} & P(\hat{Z}=1|Y=2)=\frac{1}{6} & P(\hat{Z}=2|Y=2)=\frac{1}{2} \end{array}$$

$X \to Y \to \hat{Z}$ gives that

$$\begin{aligned} & I(U;Y) \geq I(U;\hat{Z}) \ \forall\ p_{U|X} \\ \Leftrightarrow & H(Y) - H(\hat{Z}) \geq H(Y|U) - H(\hat{Z}|U) \ \forall\ p_{U|X} \\ \overset{(a)}{\Rightarrow} & H(Y) - H(Z) \geq H(Y|U) - H(Z|U) \ \forall\ p_{U|X} \\ \overset{(b)}{\Leftrightarrow} & H(Y) - H(Z) \geq \mathfrak{C}_{q_X}\left[H(Y) - H(Z)\right] \end{aligned}$$

The last step (a) follows from $p_{\hat{Z}|X} = p_{Z|X}$. Step (b) follows from the equivalent definition (1.15) of upper concave envelope in 1.2.3.

So when $p_{Y|X}$ is fixed by this joint distribution p_{XY}, $f(q_X) = H(Y) - H(Z) = H(Y) - H(\hat{Z})$ is concave with respect to q_X.

Thus the first constraint (i) of Lemma 3.2 is satisfied for $\lambda = 1$, and thus Körner-Marton rate region is sum rate optimal.

And another instance of p_{XY} satisfying $\mathfrak{C}_{q_X}[H(Y) - H(Z)]\big|_{p_X} = H(Y|X) - H(Z|X)$ is given by the following distribution:

$$p_{XY} = \begin{bmatrix} 0.02 & 0.02 & 0.48 \\ 0.02 & 0.06 & 0.16 \\ 0.06 & 0.02 & 0.16 \end{bmatrix}$$

where row index is $x \in \{0,1,2\}$, column index is $y \in \{0,1,2\}$.

Similar to above, one could construct a auxiliary $\hat{Y}$ such that $X \to Z \to \hat{Y}$ and $p_{\hat{Y}|X} = p_{Y|X}$, by the following choice of $p_{\hat{Y}|Z}$:

$$\begin{array}{lll} P(\hat{Y}=0|Z=0)=\frac{1}{14} & P(\hat{Y}=1|Z=0)=\frac{5}{14} & P(\hat{Y}=2|Z=0)=\frac{4}{7} \\ P(\hat{Y}=0|Z=1)=\frac{5}{14} & P(\hat{Y}=1|Z=1)=\frac{1}{14} & P(\hat{Y}=2|Z=1)=\frac{4}{7} \\ P(\hat{Y}=0|Z=2)=\frac{1}{42} & P(\hat{Y}=1|Z=2)=\frac{1}{42} & P(\hat{Y}=2|Z=2)=\frac{20}{21} \end{array}$$

So when $p_{Z|X}$ is fixed by this joint distribution p_{XY}, one can verify that $f(q_X) = H(Y) - H(Z) = H(\hat{Y}) - H(Z)$ is convex with respect to q_X. So the second constraint (ii) of Lemma 3.2 is satisfied for $\lambda = 1$, thus Slepian-Wolf rate region is sum rate optimal.

3.3 Alternate Proof to Quadratic Gaussian CEO Problem

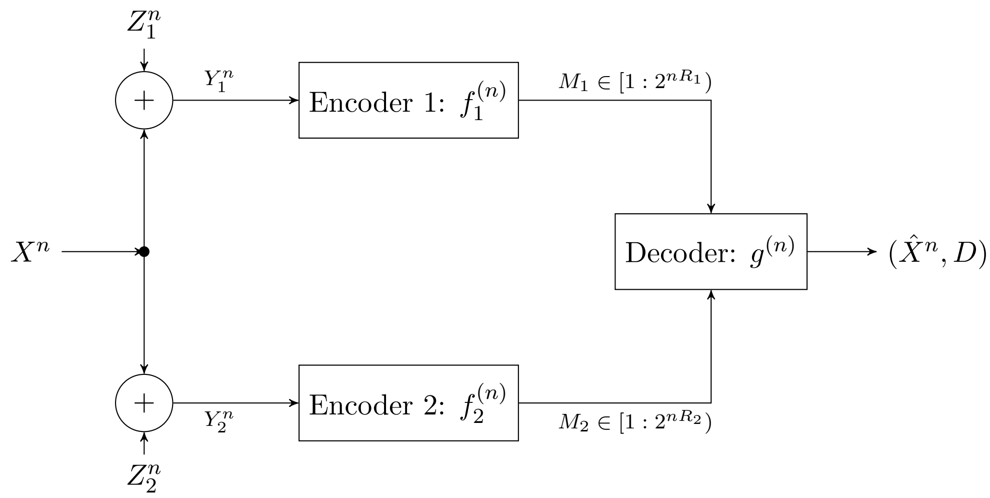

Figure 3.2: Quadratic Guassian CEO distributed source coding

The CEO problem was first introduced by Berger, Zhang, and Viswanathan [10]. The setting for quadratic Gaussian CEO distributed source coding is depicted in figure 3.2: Let X be some source generating a i.i.d. sequence of random variables $X_i \sim N(0, P)$, denoted as WGN(P). The encoder 1 observes $Y_1 = X + Z_1$ where Z_1 is some additive Gaussian noise WGN(N_1) and maps it to $M_1 \in [1 : 2^{nR_1})$ by encoding function $f_1^{(n)}$, the encoder 2 observes $Y_2 = X + Z_2$ where Z_2 is some additive Gaussian noise WGN(N_2) and maps it to $M_2 \in [1 : 2^{nR_2}]$ by encoding function $f_2^{(n)}$. The decoder uses some decoding function $g^{(n)}$ to construct some $\hat{X}^n$ from (M_1, M_2).

Similar to the communication problems in introduction chapter, one could define a (n, R_1, R_2) code $\mathcal{C} := (f_1^{(n)}, f_2^{(n)}, g^{(n)})$ for Quadratic Guassian CEO distributed source coding. A rate-distortion triple (R_1, R_2, D) is said to be achievable if there exists a sequence of codes $\mathcal{C}_n$ such that

$$\limsup_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \hat{X}_i\right)^2\right] \le D$$

And the rate-distortion region $\mathscr{R}_{CEO}(D)$ is defined as the closure of the set of all achievable rate pairs (R_1, R_2) such that (R_1, R_2, D) is achievable.

In 2001, Oohama [52] proved the following single-letter characterization for $\mathscr{R}_{CEO}(D)$, see Chapter 12 in [21]:

Theorem 3.5. *Consider the quadratic Guassian CEO distributed source coding on X, Y_1, Y_2 satisfying that* $P_{XY_1Y_2} \sim N\left(\vec{0}, \begin{bmatrix} P & P & P \\ P & P+N_1 & P \\ P & P & P+N_2 \end{bmatrix}\right)$*, the rate-distortion region* $\mathscr{R}_{CEO}(D)$ *is the set of rate pairs* (R_1, R_2) *such that*

$$R_1 \geq r_1 + \frac{1}{2}\log_+\left(\frac{1}{D}\left(\frac{1}{P} + \frac{1-2^{-2r_2}}{N_2}\right)^{-1}\right)$$

$$R_2 \geq r_2 + \frac{1}{2}\log_+\left(\frac{1}{D}\left(\frac{1}{P} + \frac{1-2^{-2r_1}}{N_1}\right)^{-1}\right)$$

$$R_1 + R_2 \geq r_1 + r_2 + \frac{1}{2}\log_+(\frac{P}{D})$$

for some $r_1, r_2 \geq 0$ *that satisfy the condition*

$$D \geq \left(\frac{1}{P} + \frac{1-2^{-2r_1}}{N_1} + \frac{1-2^{-2r_2}}{N_2}\right)^{-1}.$$

Here $\frac{1}{2}\log_+(x) = \frac{1}{2}\max\{\log x, 0\}$.

The achievablity of above $\mathscr{R}_{CEO}(D)$ can be proven by using the Berger-Tung coding scheme, see [9, 53, 60]. One can check chapter 12 in [21] for details. The converse proof employs the Entropy Power Inequality (EPI), see Oohama [52]. There is also a proof of the sum rate optimality, see [66], for the Gaussian CEO problem without using EPI by exploiting the semidefinite partial order of the distortion covariance matrices associated with the minimum mean squared error (MMSE) estimation and the so-called reduced optimal linear estimation.

Notice that when $D \geq P$, the decoder could choose the mean of X, 0, as the estimate $\hat{X}$, in this case R_1, R_2 can be set to 0. So the interesting case is when $D < P$.

With the re-parameterization $\tilde{N}_j = \frac{N_j}{2^{r_j}-1}, j = 1, 2$ and $\lambda \geq 1$, $\mathscr{R}_{CEO}(D)$ can be equivalently written in terms of weighted sum rate. And here we will present an alternate proof for the converse of the weighted sum rate of $\mathscr{R}_{CEO}(D)$, Theorem 3.6. The main idea is to derive weighted sum rate lower bounds in Theorem 3.7, and then evaluate the weighted sum rate lower bounds using the rotation techniques in [27].

One should notice that the weighted sum rate lower bound derived here is in a similar spirit as the improved lower bound for multiterminal source coding in [64], in terms of the identification of auxiliary random variables. However, to the best knowledge of the authors, applying rotation techinques to the evaluation of these lower bounds should be new.

Theorem 3.6. *For $0 < D < P$ and $\lambda \geq 1$, any rate pairs (R_1, R_2) in $\mathscr{R}_{CEO}(D)$ must satisfy that*

$$\begin{aligned}
R_1 + \lambda R_2 \geq & \min_{\substack{\tilde{N}_1, \tilde{N}_2 \geq 0: \\ \frac{1}{D} \leq \frac{1}{P} + \frac{1}{N_1 + \tilde{N}_1} + \frac{1}{N_2 + \tilde{N}_2}}} \frac{1}{2} \log \frac{P}{D} + \frac{1}{2} \log \frac{N_1 + \tilde{N}_1}{\tilde{N}_1} + \frac{\lambda}{2} \log \frac{N_2 + \tilde{N}_2}{\tilde{N}_2} \\
& + \frac{\lambda - 1}{2} \log_+ \frac{P(N_1 + \tilde{N}_1)}{(P + N_1 + \tilde{N}_1) D} \\
\lambda R_1 + R_2 \geq & \min_{\substack{\tilde{N}_1, \tilde{N}_2 \geq 0: \\ \frac{1}{D} \leq \frac{1}{P} + \frac{1}{N_1 + \tilde{N}_1} + \frac{1}{N_2 + \tilde{N}_2}}} \frac{1}{2} \log \frac{P}{D} + \frac{1}{2} \log \frac{N_2 + \tilde{N}_2}{\tilde{N}_2} + \frac{\lambda}{2} \log \frac{N_1 + \tilde{N}_1}{\tilde{N}_1} \\
& + \frac{\lambda - 1}{2} \log_+ \frac{P(N_2 + \tilde{N}_2)}{(P + N_2 + \tilde{N}_2) D}
\end{aligned} \tag{3.2}$$

3.3.1 Weighted Sum Rate Lower Bounds

Here we state a weighted sum rate lower bounds for a generalized CEO distributed source coding setting depicted in Figure 3.3.

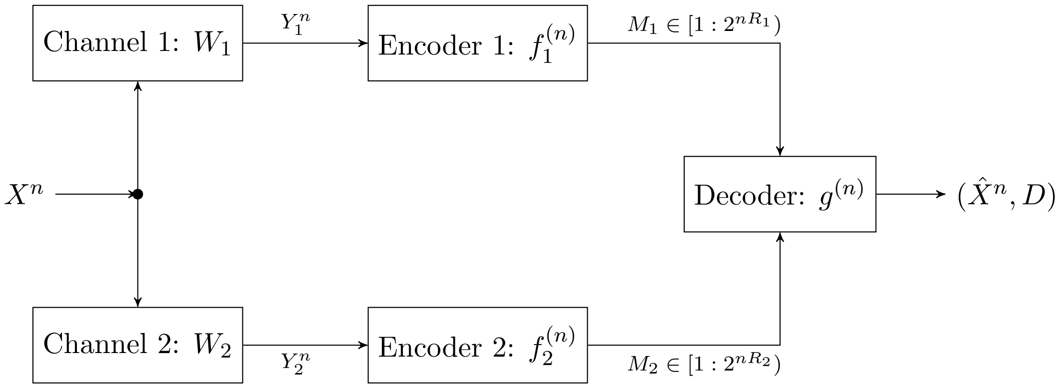

Figure 3.3: generalized CEO distributed source coding

Theorem 3.7. *Consider the generalized CEO distributed source coding on X, Y_1, Y_2 satisfying that X is some source, Y_1 and Y_2 are obtained by passing X through some discrete memoryless channel W_1 and W_2 respectively. The distortion criterion is given by*

$$\limsup_{n \to \infty} \mathrm{E} \left(\frac{1}{n} \sum_{i=1}^{n} d(X_i, \hat{X}_i) \right) \leq D.$$

For any $\lambda \geq 1$, any achievable rate-distortion triple (R_1, R_2, D) must satisfy that

$$
\begin{aligned}
R_1 + \lambda R_2 \geq & H(XY_1) + \lambda H(Y_2|X) + (\lambda - 1)\max\left\{H(X|U_1WQ) - H(X|\hat{X}Q), 0\right\} \\
& - H(X|\hat{X}Q) + H(X|U_1WQ) - H(XY_1|U_1WQ) \\
& + \lambda H(X|U_2WQ) - \lambda H(XY_2|U_2WQ) \\
R_2 + \lambda R_1 \geq & H(XY_2) + \lambda H(Y_1|X) + (\lambda - 1)\max\left\{H(X|U_2WQ) - H(X|\hat{X}Q), 0\right\} \\
& - H(X|\hat{X}Q) + H(X|U_2WQ) - H(XY_2|U_2WQ) \\
& + \lambda H(X|U_1WQ) - \lambda H(XY_1|U_1WQ)
\end{aligned}
$$

subject to the constraints

$$
\begin{aligned}
& U_1 \leftarrow QWY_1 \leftarrow QWX \rightarrow QWY_2 \rightarrow U_2 \\
& QW \perp XY_1Y_2 \\
& \hat{X} \leftarrow QWU_1U_2 \leftarrow XY_1Y_2 \\
& E[d(X, \hat{X})] \leq D.
\end{aligned}
\tag{3.3}
$$

Proof. Observe that for any code $\mathcal{C}$ for generalized CEO distributed source coding problem, we have the long Markov chain $M_1 \leftarrow Y_1^n \leftarrow X^n \rightarrow Y_2^n \rightarrow M_2$.

For any sequence of codes $\mathcal{C}_n$ that achieves the rate pairs (R_1, R_2) for generalized CEO distributed source coding, when $\lambda \geq 1$, we have

$$
\begin{aligned}
& nR_1 + \lambda nR_2 + \lambda H(X^n|M_1M_2) \qquad (3.4) \\
\geq & I(M_1; Y_1^n) + \lambda I(M_2; Y_2^n|M_1) + \lambda H(X^n|M_1M_2) \\
\overset{(a)}{=} & I(M_1; Y_1^n) + \lambda I(M_2; Y_2^n X^n|M_1) + \lambda H(X^n|M_1M_2) \\
\overset{(b)}{=} & \underline{I(M_1; Y_1^n)} + \underline{\lambda I(M_2; Y_2^n|M_1X^n)} + \lambda I(M_2; X^n|M_1) + \lambda H(X^n|M_1M_2) \\
\overset{(c)}{=} & \underline{H(Y_1^n)} - \underline{H(Y_1^n|M_1)} + \underline{\lambda H(Y_2^n|M_1X^n)} - \underline{\lambda H(Y_2^n|M_1M_2X^n)} + \lambda H(X^n|M_1) \\
\overset{(d)}{=} & H(Y_1^n) + \lambda H(X^n|M_1) - H(Y_1^n|M_1) + \lambda H(Y_2^n|M_1X^n) - \lambda H(Y_2^n|M_2X^n) \\
\overset{(e)}{=} & H(Y_1^n) + \cancel{\lambda H(X^n|M_1)} - H(Y_1^n|M_1) + \lambda H(Y_2^n X^n|M_1) - \cancel{\lambda H(X^n|M_1)} \\
& - \lambda H(X^n Y_2^n|M_2) + \lambda H(X^n|M_2) \\
\overset{(f)}{=} & H(Y_1^n) - \underline{H(Y_1^n|M_1)} + \lambda H(Y_2^n|X^n) + \lambda H(X^n|M_1) - \lambda H(X^n Y_2^n|M_2) + \lambda H(X^n|M_2) \\
\overset{(g)}{=} & H(Y_1^n) + \lambda H(Y_2^n|X^n) + \underline{H(X^n|Y_1^n)} + (\lambda - 1)H(X^n|M_1) \\
& + H(X^n|M_1) - \underline{H(X^n Y_1^n|M_1)} + \lambda H(X^n|M_2) - \lambda H(X^n Y_2^n|M_2) \\
\overset{(h)}{=} & nH(XY_1) + \lambda nH(Y_2|X) + (\lambda - 1)H(X^n|M_1) + \sum_{i=1}^{n} \left[H(X_i|M_1X^{n/i}Y_1^{i-1})\right.
\end{aligned}
$$

$$-H(X_iY_{1i}|M_1X^{n/i}Y_1^{i-1})\big] + \sum_{i=1}^{n}\big[\lambda H(X_i|M_2X^{n/i}Y_2^{i-1}) - \lambda H(X_iY_{2i}|M_2X^{n/i}Y_2^{i-1})\big] \tag{3.5}$$

Step (a) is due to $M_2 \to Y_2^n M_1 \to X^n$; step (b) is applying chain rules on the blue term $\lambda I(M_2;Y_2^n|M_1X^n)$; step (c) is applying chain rules on the two underlined terms $I(M_1;Y_1^n)$ and $\lambda I(M_2;Y_2^n|M_1X^n)$; step (d) uses $M_1 \to M_2X^n \to Y_2^n$; step (e) is canceling $\lambda H(X^n|M_1)$ and using chain rule on the red terms $\lambda H(Y_2^n|M_2X^n)$; step (f) is using chain rule on the orange term $\lambda H(Y_2^nX^n|M_1)$; step (g) is using chain rule to break the wavy-underlined term $H(Y_1^n|M_1)$ by chain rules; step (h) follows from applying the well-known Körner Marton identity (see Lemma 3.3) twice on the two purple terms.

Here the term $H(X^n|M_1M_2)$ can be single-letterized in the following ways:

$$H(X^n|M_1M_2) = H(X^n|\hat{X}^nM_1M_2) \le \sum_{i=1}^{n} H(X_i|\hat{X}_i)$$

Observe that left-hand side (3.4) has $\lambda H(X^n|M_1M_2)$, and right-hand side (3.5) has $(\lambda-1)H(X^n|M_1)$, and $\lambda \ge 1$. There are two ways to lower bound the difference $(\lambda-1)\left[H(X^n|M_1) - H(X^n|M_1M_2)\right]$:

$$H(X^n|M_1) - H(X^n|M_1M_2) \ge \max\left\{0, \sum_{i=1}^{n} H(X_i|M_1X^{n/i}Y_1^{i-1}) - \sum_{i=1}^{n} H(X_i|\hat{X}_i)\right\}$$

Thus above weighted sum rate can be rewritten as

$$\begin{aligned}
nR_1 + \lambda nR_2 \ge & nH(XY_1) + \lambda nH(Y_2|X) - \sum_{i=1}^{n} H(X_i|\hat{X}_iX^{n/i}) \\
& + (\lambda-1)\max\left\{0, \sum_{i=1}^{n} H(X_i|M_1X^{n/i}Y_1^{i-1}) - \sum_{i=1}^{n} H(X_i|\hat{X}_i)\right\} \\
& + \sum_{i=1}^{n} H(X_i|M_1X^{n/i}Y_1^{i-1}) - H(X_iY_{1i}|M_1X^{n/i}Y_1^{i-1}) \\
& + \lambda H(X_i|M_2X^{n/i}Y_2^{i-1}) - \lambda H(X_iY_{2i}|M_2X^{n/i}Y_2^{i-1})
\end{aligned}$$

Similarly, by considering $nR_2 + \lambda nR_1$, one could get

$$\begin{aligned}
nR_2 + \lambda nR_1 \ge & nH(XY_2) + \lambda nH(Y_1|X) - \sum_{i=1}^{n} H(X_i|\hat{X}_i) \\
& + (\lambda-1)\max\left\{0, \sum_{i=1}^{n} H(X_i|M_2X^{n/i}Y_2^{i-1}) - \sum_{i=1}^{n} H(X_i|\hat{X}_i)\right\}
\end{aligned}$$

$$+\sum_{i=1}^{n} H(X_i|M_2X^{n/i}Y_1^{i-1}) - H(X_iY_{2i}|M_2X^{n/i}Y_2^{i-1})$$
$$+\lambda H(X_i|M_1X^{n/i}Y_1^{i-1}) - \lambda H(X_iY_{1i}|M_1X^{n/i}Y_1^{i-1})$$

Identify auxiliary random varibles as $W_i = X^{n/i}, U_{1i} = M_1Y_1^{i-1}, U_{2i} = M_2Y_2^{i-1}$. And let Q be the uniform distribution over $i = 1, \cdots, n$. the weighted sum rate satisfies that

$$\begin{aligned}
R_1 + \lambda R_2 \geq & H(XY_1) + \lambda H(Y_2|X) - H(X|\hat{X}Q) \\
& + (\lambda - 1)\max\left\{0, H(X|U_1WQ) - H(X|\hat{X}Q)\right\} \\
& + H(X|U_1WQ) - H(XY_1|U_1WQ) + \lambda H(X|U_2WQ) - \lambda H(XY_2|U_2WQ) \\
R_2 + \lambda R_1 \geq & H(XY_2) + \lambda H(Y_1|X) - H(X|\hat{X}Q) \\
& + (\lambda - 1)\max\left\{0, H(X|U_2WQ) - H(X|\hat{X}Q)\right\} \\
& + H(X|U_2WQ) - H(XY_2|U_2WQ) + \lambda H(X|U_1WQ) - \lambda H(XY_1|U_1WQ)
\end{aligned}$$

And for this set of auxiliary random variables, one can verify the constraints (3.3) holds:

$$\begin{aligned}
& U_1 \leftarrow QWY_1 \leftarrow QWX \rightarrow QWY_2 \rightarrow U_2 \\
& QW \perp XY_1Y_2 \\
& \hat{X} \leftarrow QWU_1U_2 \rightarrow XY_1Y_2 \\
& E[d(X,\hat{X})] \leq D.
\end{aligned}$$

□

Lemma 3.3 (Körner Marton identity, (4.14) in [36].). *For any tuple of random variables (U, Y^n, Z^n) the following equality holds:*

$$H(Y^n|U) - H(Z^n|U) = \sum_{i=1}^{n} H(Y_i|UY^{i-1}Z_{i+1}^n) - H(Z_i|UY^{i-1}Z_{i+1}^n)$$

3.3.2 Optimality of Achievable Weighted Sum Rate

In this section, we will use the weighted sum rate lower bounds derived in Theorem 3.7 to prove Theorem 3.6.

Proof. For quadratic Gaussian CEO distributed source coding, the quadratic distortion measure is $d(x, \hat{x}) = (x - \hat{x})^2$. The weighted sum rate lower bounds in above Theorem 3.7 can be further simplified.

Observe that the quadratic distortion measure impose an upper bound on the term $H(X|\hat{X}Q)$.

$$\begin{aligned}
H(X|\hat{X}Q) \leq& H(X-\hat{X}|Q)\\
=& H(X-\hat{X})\\
\overset{(a)}{\leq}& \frac{1}{2}\log 2\pi e \,\mathrm{E}\left((X-\hat{X})^2\right)\\
\leq& \frac{1}{2}\log 2\pi e D
\end{aligned}$$

where step (a) is from Gaussian maximizes the differential entropy under variance constraint. So we can replace $H(X|\hat{X}Q)$ with $\frac{1}{2}\log 2\pi e D$ in the weighted sum rate lower bounds in Theorem 3.7.

On the other hand, the Markov chain $\hat{X} \leftarrow QWU_1U_2 \rightarrow XY_1Y_2$ in constraints (3.3) implies that

$$H(X|U_1U_2QW) \leq H(X|\hat{X}Q)$$

So we have

$$H(X|U_1U_2QW) \leq \frac{1}{2}\log 2\pi e D.$$

So the constraints (3.3) can be relaxed to

$$\begin{aligned}
H(X|U_1U_2QW) \leq \frac{1}{2}\log 2\pi e D\\
U_1 \leftarrow QWY_1 \leftarrow QWX \rightarrow QWY_2 \rightarrow U_2\\
QW \perp XY_1Y_2.
\end{aligned}$$

Write $Q = QW$, the weighted sum rate lower bounds in above Theorem 3.7 can be simplified to be

$$\begin{aligned}
R_1 + \lambda R_2 \geq& H(XY_1) + \lambda H(Y_2|X) + (\lambda-1)\max\left\{H(X|U_1Q) - \frac{1}{2}\log 2\pi e D, 0\right\}\\
&- \frac{1}{2}\log 2\pi e D + H(X|U_1Q) - H(XY_1|U_1Q) + \lambda H(X|U_2Q) - \lambda H(XY_2|U_2Q)\\
R_2 + \lambda R_1 \geq& H(XY_2) + \lambda H(Y_1|X) + (\lambda-1)\max\left\{0, H(X|U_2Q) - \frac{1}{2}\log 2\pi e D\right\}\\
&- \frac{1}{2}\log 2\pi e D + H(X|U_2Q) - H(XY_2|U_2Q) + \lambda H(X|U_1Q) - \lambda H(XY_1|U_1Q)
\end{aligned} \tag{3.6}$$

subject to the constraints:

$$
\begin{aligned}
H(X|U_1U_2Q) &\le \frac{1}{2}\log 2\pi e D \\
U_1 \leftarrow QY_1 \leftarrow QX &\to QY_2 \to U_2 \\
Q &\perp XY_1Y_2
\end{aligned}
\tag{3.7}
$$

Above constraint 3.7 implies that the distribution $p_{QU_1U_2|XY_1Y_2}$ can be explicitly written in the form of $p_{U_1|QY_1}p_{U_2|QY_2}p_Q$.

The weighted sum rate lower bounds in Equation (3.6) can be written as a infmax problem.

$$
\begin{aligned}
R_1 + \lambda R_2 \ge & \inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} H(XY_1) + \lambda H(Y_2|X) \\
& + (\lambda - 1)\max\left\{H(X|U_1Q) - \frac{1}{2}\log 2\pi e D, 0\right\} - \frac{1}{2}\log 2\pi e D \\
& + H(X|U_1Q) - H(XY_1|U_1Q) + \lambda H(X|U_2Q) - \lambda H(XY_2|U_2Q) \\
\overset{(a)}{=} & \inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e P N_1 + \frac{\lambda}{2}\log N_2 - \frac{\alpha(\lambda-1)+1}{2}\log 2\pi e D \\
& + ((\lambda-1)\alpha+1)\, H(X|U_1Q) - H(XY_1|U_1Q) \\
& + \lambda H(X|U_2Q) - \lambda H(XY_2|U_2Q) \\
\overset{(b)}{=} & \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e P N_1 + \frac{\lambda}{2}\log N_2 - \frac{\alpha(\lambda-1)+1}{2}\log 2\pi e D \\
& + \inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} ((\lambda-1)\alpha+1)\, H(X|U_1Q) - H(XY_1|U_1Q) \\
& + \lambda H(X|U_2Q) - \lambda H(XY_2|U_2Q)
\end{aligned}
$$

$$
\begin{aligned}
R_2 + \lambda R_1 \ge & \inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} H(XY_2) + \lambda H(Y_1|X) \\
& + (\lambda - 1)\max\left\{0, H(X|U_2Q) - \frac{1}{2}\log 2\pi e D\right\} - \frac{1}{2}\log 2\pi e D \\
& + H(X|U_2Q) - H(XY_2|U_2Q) + \lambda H(X|U_1Q) - \lambda H(XY_1|U_1Q) \\
\overset{(a)}{=} & \inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e P N_2 + \frac{\lambda}{2}\log N_1 - \frac{\alpha(\lambda-1)+1}{2}\log 2\pi e D \\
& + ((\lambda-1)\alpha+1)\, H(X|U_2Q) - H(XY_2|U_2Q) \\
& + \lambda H(X|U_1Q) - \lambda H(XY_1|U_1Q) \\
\overset{(b)}{=} & \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e P N_2 + \frac{\lambda}{2}\log N_1 - \frac{\alpha(\lambda-1)+1}{2}\log 2\pi e D \\
& + \inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} ((\lambda-1)\alpha+1)\, H(X|U_2Q) - H(XY_2|U_2Q) \\
& + \lambda H(X|U_1Q) - \lambda H(XY_1|U_1Q)
\end{aligned}
$$

where $p_{U_1|QY_1}p_{U_2|QY_2}p_Q$ satisfies the constraint (3.7).

Step (a) comes from $\max\left\{H(X|U_1Q) - \frac{1}{2}\log 2\pi eD, 0\right\} = \max_{\alpha\in[0,1]} \alpha H(X|U_1Q) - \frac{\alpha}{2}\log 2\pi eD$. Step (b) comes from exchanging inf and max by Theorem 5 in Appendix of [25].

Thus to evaluate above weighted sum rate lower bounds, suffices to compute the following functional

$$\inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \kappa H(X|U_1Q) - H(XY_1|U_1Q) + \lambda H(X|U_2Q) - \lambda H(XY_2|U_2Q) \tag{3.8}$$

where $p_{U_1|QY_1}p_{U_2|QY_2}p_Q$ satisfies the constraints 3.7 and $\kappa \geq 1, \lambda \geq 1$,

Lemma 3.4 shows that the infimum of (3.8) is attained by $Q = \emptyset$ and $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \perp Y_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), U_2 = Y_2 + \tilde{U}_2, \tilde{U}_2 \perp Y_2, \tilde{U}_2 \sim N(0, \tilde{N}_2)$ subject to the constraints (3.9).

Thus the weighted sum rate can be written

$$\begin{aligned}
R_1 + \lambda R_2 \geq \max_{\alpha\in[0,1]} & \min_{\substack{\tilde{N}_1,\tilde{N}_2\geq 0:\\ \frac{1}{D}\leq\frac{1}{P}+\frac{1}{N_1+\tilde{N}_1}+\frac{1}{N_2+\tilde{N}_2}}} \frac{1}{2}\log 2\pi ePN_1 + \frac{\lambda}{2}\log N_2 \\
& - \frac{\alpha(\lambda-1)+1}{2}\log 2\pi eD + ((\lambda-1)\alpha+1)\frac{1}{2}\log 2\pi e\frac{P(N_1+\tilde{N}_1)}{P+N_1+\tilde{N}_1} \\
& - \frac{1}{2}\log 2\pi e\frac{PN_1\tilde{N}_1}{P+N_1+\tilde{N}_1} + \frac{\lambda}{2}\log 2\pi e\frac{P(N_2+\tilde{N}_2)}{P+N_2+\tilde{N}_2} - \frac{\lambda}{2}\log 2\pi e\frac{PN_2\tilde{N}_2}{P+N_2+\tilde{N}_2} \\
R_2 + \lambda R_1 \geq \max_{\alpha\in[0,1]} & \min_{\substack{\tilde{N}_1,\tilde{N}_2\geq 0:\\ \frac{1}{D}\leq\frac{1}{P}+\frac{1}{N_1+\tilde{N}_1}+\frac{1}{N_2+\tilde{N}_2}}} \frac{1}{2}\log 2\pi ePN_2 + \frac{\lambda}{2}\log N_1 \\
& - \frac{\alpha(\lambda-1)+1}{2}\log 2\pi eD + \frac{(\lambda-1)\alpha+1}{2}\log 2\pi e\frac{P(N_2+\tilde{N}_2)}{P+N_2+\tilde{N}_2} \\
& - \frac{1}{2}\log 2\pi e\frac{PN_2\tilde{N}_2}{P+N_2+\tilde{N}_2} + \frac{\lambda}{2}\log 2\pi e\frac{P(N_1+\tilde{N}_1)}{P+N_1+\tilde{N}_1} - \frac{\lambda}{2}\log 2\pi e\frac{PN_1\tilde{N}_1}{P+N_1+\tilde{N}_1}
\end{aligned}$$

For this, again by Theorem 5 in Appendix of [25], we could exchange max and min here. Then we will reach the weighted sum rate Equation (3.2).

□

Lemma 3.4. *Given* $P_{XY_1Y_2} \sim N\left(\vec{0}, \begin{bmatrix} P & P & P \\ P & P+N_1 & P \\ P & P & P+N_2 \end{bmatrix}\right)$, *for any* $\kappa, \lambda \geq 1$,

$$\inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \kappa H(X|U_1Q) - H(XY_1|U_1Q) + \lambda H(X|U_2Q) - \lambda H(XY_2|U_2Q)$$

subject to the constraints 3.7, is attained by $Q = \emptyset$ *and* $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \perp Y_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), U_2 = Y_2 + \tilde{U}_2, \tilde{U}_2 \perp Y_2, \tilde{U}_2 \sim N(0, \tilde{N}_2)$ *subject to:*

$$H(X|U_1U_2) \leq \frac{1}{2}\log 2\pi eD \tag{3.9}$$

Proof. This proof is essentially using the rotation trick in [27] with along some perturbation ideas for establishing strict sub-additivity, see [28].

For any small enough $\varepsilon_1, \varepsilon_2 > 0$, consider the following perturbed optimization problem:

Given $P_{XY_1Y_2} \sim N\left(\vec{0}, \begin{bmatrix} P & P & P \\ P & P+N_1 & P \\ P & P & P+N_2 \end{bmatrix}\right)$, for any $\kappa, \lambda \geq 1$, want to find the infimum of the following function:

$$\begin{aligned}\Theta_{\kappa,\lambda}^{\varepsilon_1,\varepsilon_2}(p_{U_1|QY_1}p_{U_2|QY_2}p_Q) :=& (\kappa+\varepsilon_2)H(X|U_1Q) - H(XY_1|U_1Q) \\ &+ (\lambda+\varepsilon_1)H(X|U_2Q) - \lambda H(XY_2|U_2Q)\end{aligned}$$

subject to the constraints

$$H(X|U_1U_2Q) \leq \frac{1}{2}\log 2\pi eD \tag{3.10}$$

From Lemma 3.5, we could know that for $\varepsilon_1, \varepsilon_2 > 0$, infimum value of $\Theta_{\kappa,\lambda}^{\varepsilon_1,\varepsilon_2}$ is attained by $Q = \emptyset$ and $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \perp Y_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), U_2 = Y_2 + \tilde{U}_2, \tilde{U}_2 \perp Y_2, \tilde{U}_2 \sim N(0, \tilde{N}_2)$ subject to the constraints 3.9.

Use $\mathcal{G}$ to denote the set of $p_{U_1|QY_1}p_{U_2|QY_2}p_Q$ satisfying $Q = \emptyset$ and $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \perp Y_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), U_2 = Y_2 + \tilde{U}_2, \tilde{U}_2 \perp Y_2, \tilde{U}_2 \sim N(0, \tilde{N}_2)$ subject to the constraints 3.9.

It remains to show that when $\varepsilon_1 = \varepsilon_2 = 0$, the minimizing distribution of $\Theta_{\kappa,\lambda}^{0,0}$ is also attained by some distribution in $\mathcal{G}$. This can be done by a continuity argument.

For any $\varepsilon_1, \varepsilon_2 > 0$ close to 0, observe that for any distribution $p_{U_1|QY_1}p_{U_2|QY_2}p_Q$ satisfying the constraints 3.10, we have

$$\begin{aligned}\Theta_{\kappa,\lambda}^{0,0} =& \Theta_{\kappa,\lambda}^{\varepsilon_1,\varepsilon_2} - \varepsilon_1 H(X|U_2Q) - \varepsilon_2 H(X|U_1Q) \\ \geq& \Theta_{\kappa,\lambda}^{\varepsilon_1,\varepsilon_2} - \frac{\varepsilon_1+\varepsilon_2}{2}\log 2\pi eP\end{aligned}$$

so we have

$$\inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \Theta_{\kappa,\lambda}^{0,0} \geq \min_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \Theta_{\kappa,\lambda}^{\varepsilon_1,\varepsilon_2} - \frac{\varepsilon_1+\varepsilon_2}{2}\log 2\pi eP$$

Take $\varepsilon_1, \varepsilon_2 \to 0$, we get

$$\inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \Theta^{0,0}_{\kappa,\lambda} \geq \liminf_{\varepsilon_1,\varepsilon_2\to 0} \min_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$$

On the other hand, when $\varepsilon_1, \varepsilon_2 \to 0$, pick the minimizing distribution $p^*_{\varepsilon_1,\varepsilon_2} \in \mathcal{G}$ for $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ to construct a sequence so that $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}(p^*_{\varepsilon_1,\varepsilon_2}) \to \liminf_{\varepsilon_1,\varepsilon_2\to 0} \min_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$.

Since $\mathcal{G}$ is compact, there exists some subsequence that will tend to some limit $p^* \in \mathcal{G}$. Since $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ is continuous with respect to $p_{U_1|QY_1}p_{U_2|QY_2}p_Q$, so

$$\Theta^{0,0}_{\kappa,\lambda}(p^*) = \lim_{\varepsilon_1,\varepsilon_2\to 0} \Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}(p^*_{\varepsilon_1,\varepsilon_2}) = \liminf_{\varepsilon_1,\varepsilon_2\to 0} \min_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$$

Therefore, $p^* \in \mathcal{G}$ attain the minimizing value of $\Theta^{0,0}_{\kappa,\lambda}$ subject to the constraints (3.7). $\square$

Lemma 3.5. *For $\varepsilon_1, \varepsilon_2 > 0$, and $\kappa, \lambda \geq 1$, infimum of $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$*

$$\begin{aligned}\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}(p_{U_1|QY_1}p_{U_2|QY_2}p_Q) =&(\kappa+\varepsilon_2)H(X|U_1Q) - H(XY_1|U_1Q)\\ &+ (\lambda+\varepsilon_1)H(X|U_2Q) - \lambda H(XY_2|U_2Q)\end{aligned}$$

subject to the constraints (3.10), *is attained by $Q = \emptyset$ and $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \perp Y_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), U_2 = Y_2 + \tilde{U}_2, \tilde{U}_2 \perp Y_2, \tilde{U}_2 \sim N(0, \tilde{N}_2)$ subject to the constraints 3.9.*

Proof. Since scaling U_1, U_2, Q doesn't affect $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ and the constraints (3.10), one could truncate U_1, U_2, Q to some random variables with support on $[0, 1]$. This will give tightness of the joint distribution $P_{XY_1Y_2U_1U_2Q}$. By routine arguments in Appendix II of [27] one can show that there is a minimizer from the tightness of the sequence of distributions.

So we assume infimum of $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ is attained by some minimizing distribution $p^*_{U_1|QY_1}p^*_{U_2|QY_2}p^*_Q$, and write the joint distribution of U_1, U_2, X, Y_1, Y_2, Q by $p^*_{U_1|QY_1}$ $p^*_{U_2|QY_2}p^*_Q p_{XY_1Y_2}$. Take two i.i.d. copies of the joint distribution at the minimizer and denote them using subscripts a, b respectively. Let $(\cdot)_+ = \frac{(\cdot)_a + (\cdot)_b}{\sqrt{2}}$ and $(\cdot)_- = \frac{(\cdot)_a - (\cdot)_b}{\sqrt{2}}$, where $(\cdot)$ can be replaced with X, Y_1, Y_2.

Denote infimum of $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ as V. We have, by the rotation trick in [27]:

$$\begin{aligned}2V =&(\kappa+\varepsilon_2)H(X_aX_b|U_{1a}U_{1b}Q_aQ_b) - H(X_aX_bY_{1a}Y_{1b}|U_{1a}U_{1b}Q_aQ_b)\\ &+ (\lambda+\varepsilon_1)H(X_aX_b|U_{2a}U_{2b}Q_aQ_b) - \lambda H(X_aX_bY_{2a}Y_{2b}|U_{2a}U_{2b}Q_aQ_b)\end{aligned}$$

$$
\begin{aligned}
=&(\kappa+\varepsilon_2)H(X_+X_-|U_{1a}U_{1b}Q_aQ_b) - H(X_+X_-Y_{1+}Y_{1-}|U_{1a}U_{1b}Q_aQ_b)\\
&+(\lambda+\varepsilon_1)H(X_+X_-|U_{2a}U_{2b}Q_aQ_b) - \lambda H(X_+X_-Y_{2+}Y_{2-}|U_{2a}U_{2b}Q_aQ_b)\\
=&(\kappa+\varepsilon_2)H(X_+|U_{1a}U_{1b}Q_aQ_bX_-) + (\kappa+\varepsilon_2)H(X_-|U_{1a}U_{1b}Q_aQ_bY_{1+}X_+)\\
&+(\kappa+\varepsilon_2)I(X_-;Y_{1+}X_+|U_{1a}U_{1b}Q_aQ_b)\\
&-H(X_+Y_{1+}|U_{1a}U_{1b}Q_aQ_bX_-) - H(X_-Y_{1-}|U_{1a}U_{1b}Q_aQ_bY_{1+}X_+)\\
&-I(X_-;X_+Y_{1+}|U_{1a}U_{1b}Q_aQ_b)\\
&+(\lambda+\varepsilon_1)H(X_+|U_{2a}U_{2b}Q_aQ_bX_-) + (\lambda+\varepsilon_1)H(X_-|U_{2a}U_{2b}Q_aQ_bY_{2+}X_+)\\
&+(\lambda+\varepsilon_1)I(X_-;Y_{2+}X_+|U_{2a}U_{2b}Q_aQ_b)\\
&-\lambda H(X_+Y_{2+}|U_{2a}U_{2b}Q_aQ_bX_-) - \lambda H(X_-Y_{2-}|U_{2a}U_{2b}Q_aQ_bY_{2+}X_+)\\
&-\lambda I(X_-;X_+Y_{2+}|U_{2a}U_{2b}Q_aQ_b)\\
=&(\kappa+\varepsilon_2)H(X_+|U_{1a}U_{1b}Q_aQ_bX_-) - H(X_+Y_{1+}|U_{1a}U_{1b}Q_aQ_bX_-)\\
&+(\lambda+\varepsilon_1)H(X_+|U_{2a}U_{2b}Q_aQ_bX_-) - \lambda H(X_+Y_{2+}|U_{2a}U_{2b}Q_aQ_bX_-)\\
&+\lambda H(X_-|U_{1a}U_{1b}Q_aQ_bY_{1+}X_+) - H(X_-Y_{1-}|U_{1a}U_{1b}Q_aQ_bY_{1+}X_+)\\
&+(\lambda+\varepsilon_1)H(X_-|U_{2a}U_{2b}Q_aQ_bY_{2+}X_+) - \lambda H(X_-Y_{2-}|U_{2a}U_{2b}Q_aQ_bY_{2+}X_+)\\
&+(\kappa+\varepsilon_2-1)I(X_-;Y_{1+}X_+|U_{1a}U_{1b}Q_aQ_b) + \varepsilon_1 I(X_-;Y_{2+}X_+|U_{2a}U_{2b}Q_aQ_b)
\end{aligned}
$$

Set Q_0 to be uniform binary random variable with support $\{0,1\}$:

when $Q_0 = 0$ we set $\hat{Q} = (Q_0, Q_a, Q_b, X_-)$, $\hat{U}_1 = (U_{1a}U_{1b})$, $\hat{U}_2 = (U_{2a}U_{2b})$, $\hat{X} = X_+$, $\hat{Y}_1 = Y_{1+}$ and $\hat{Y}_2 = Y_{2+}$;

when $Q_0 = 1$ we set $\hat{Q} = (Q_0, Q_a, Q_b, X_+)$, $\hat{U}_1 = (U_{1a}U_{1b}Y_{1+})$, $\hat{U}_2 = (U_{2a}U_{2b}Y_{2+})$, $\hat{X} = X_-$, $\hat{Y}_1 = Y_{1-}$ and $\hat{Y}_2 = Y_{2-}$.

In this way we construct a new joint distribution $\hat{p}_{\hat{Q}\hat{U}_1\hat{U}_2\hat{X}\hat{Y}_1\hat{Y}_2}$. Observe that since $p_{X_aY_{1a}Y_{2a}}$ and $p_{X_bY_{1b}Y_{2b}}$ are i.i.d jointly Gaussian random variables, so are $p_{X_+Y_{1+}Y_{2+}}$ and $p_{X_-Y_{1-}Y_{2-}}$. Thus $\hat{p}_{\hat{X}\hat{Y}_1\hat{Y}_2}$ follows the same distribution as $p_{XY_1Y_2}$.

One could verify that this construction $\hat{p}_{\hat{Q}\hat{U}_1\hat{U}_2\hat{X}\hat{Y}_1\hat{Y}_2}$ is a candidate satisfying constraints 3.7:

$$
\begin{aligned}
&I(\hat{X};\hat{U}_1\hat{U}_2|\hat{Q})\\
=&\frac{1}{2}I(X_+;U_{1a}U_{1b}U_{2a}U_{2b}|Q_aQ_bX_-) + \frac{1}{2}I(X_-;U_{1a}U_{1b}U_{2a}U_{2b}Y_{1+}Y_{2+}|Q_aQ_bX_+)\\
\geq&\frac{1}{2}H(X_+X_-|Q_aQ_b) - \frac{1}{2}H(X_+X_-|U_{1a}U_{1b}U_{2a}U_{2b}Q_aQ_b)\\
=&\frac{1}{2}\left[H(X_aX_b|Q_aQ_b) - H(X_aX_b|U_{1a}U_{1b}U_{2a}U_{2b}Q_aQ_b)\right]
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{1}{2}\log\frac{P}{D}\\
&\hat{U}_1 \leftarrow \hat{Q}\hat{Y}_1 \leftarrow \hat{Q}\hat{X} \rightarrow \hat{Q}\hat{Y}_2 \rightarrow \hat{U}_2\\
&\hat{Q} \perp \hat{X}\hat{Y}_1\hat{Y}_2
\end{aligned}$$

This implies that this joint distribution $\hat{p}_{\hat{Q}\hat{U}_1\hat{U}_2|\hat{X}\hat{Y}_1\hat{Y}_2}$ constructed above is a feasible choice for the optimization problem $\inf_{p_{U_1|QY_1}p_{U_2|QY_2}p_Q} \Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$. So we build the following inequality

$$\begin{aligned}
V =&(\kappa+\varepsilon_2)H(\hat{X}|\hat{U}_1\hat{Q}) - H(\hat{X}\hat{Y}_1|\hat{U}_1\hat{Q}) + (\lambda+\varepsilon_1)H(\hat{X}|\hat{U}_2\hat{Q}) - \lambda H(\hat{X}\hat{Y}_2|\hat{U}_2\hat{Q})\\
&+\frac{\kappa+\varepsilon_2-1}{2}I(X_-;Y_{1+}X_+|U_{1a}U_{1b}Q_aQ_b) + \frac{\varepsilon_1}{2}I(X_-;Y_{2+}X_+|U_{2a}U_{2b}Q_aQ_b)\\
\geq& V + \frac{\kappa+\varepsilon_2-1}{2}I(X_-;Y_{1+}X_+|U_{1a}U_{1b}Q_aQ_b) + \frac{\varepsilon_1}{2}I(X_-;Y_{2+}X_+|U_{2a}U_{2b}Q_aQ_b)
\end{aligned} \tag{3.11}$$

Thus, the term $I(X_-;Y_{1+}X_+|U_{1a}U_{1b}Q_aQ_b)$ in the right-hand side (3.11) will be forced to be 0 due to $\kappa+\varepsilon_2 > 1$. It implies that given any value assignments of $U_{1a}U_{1b}Q_aQ_b$, X_+ is independent of X_-. In the following we will argue that this implies given any value assignments of $U_{1a}U_{1b}Q_aQ_b$, Y_{1+} is independent of Y_{1-}.

Notice that $Y_1 = X + Z_1$, one can compute the linear MMSE estimate of X given Y_1, Y_2 (see [21] Appendix B Minimum mean square error estimation), which will gives

$$X = \frac{P}{P+N_1}Y_1 + G$$

where $G \sim N(0, \frac{PN_1}{P+N_1})$ and is independent of Y_1.

From the constraints 3.7, we have the Markov chain $U_1 \leftarrow QY_1 \leftarrow X$ and $I(Q;XY_1Y_2) = 0$, which leads to $I(QU_1;X|Y_1) = 0$, i.e., $QU_1 \rightarrow Y_1 \rightarrow X$. In other words, we have $QU_1 \rightarrow Y_1 \rightarrow G$.

Thus, we know G is independent of QU_1Y_1. So for the two letter copies of the minimizer, we have G_{1a} and G_{1b} are both Gaussians, $G_{1a} \perp G_{1b}$ and they are independent of $Q_a, Q_b, U_{1a}, U_{1b}, Y_{1a}, Y_{1b}$.

For the rotated version, we could write

$$\begin{aligned}
X_+ &= \frac{P}{P+N_1}Y_{1+} + G_+\\
X_- &= \frac{P}{P+N_1}Y_{1-} + G_-
\end{aligned}$$

where G_+ and G_- are both Gaussians, $G_+ \perp G_-$ and they are independent of $Q_a, Q_b, U_{1a}, U_{1b}, Y_{1a}, Y_{1b}$.

So we could apply the Proposition 2 in [27]. Treat Y_{1+} and Y_{1-} as the "channel input", and treat X_+ and X_- as the "channel output", one can conclude that given any value assignments of $U_{1a}U_{1b}Q_aQ_b$, Y_{1+} is independent of Y_{1-}.

By applying Corollary 3 in [27], this implies that at the minimizing distribution $p^*_{U_1|QY_1}p^*_{U_2|QY_2}p^*_Q$, conditioned on U_1Q, Y_1 is Gaussian and that the conditional variance is invariant over choices of U_1Q.

Similarly, $\varepsilon_1 > 0$ will force $I(X_-; Y_{2+}X_+|_{2a}U_{2b}Q_aQ_b) = 0$, similarly we could argue that the minimizing distribution $p^*_{U_1|QY_1}p^*_{U_2|QY_2}p^*_Q$ satisfies that conditioned on U_2Q, Y_2 is Gaussian and the conditional covariance is independent of U_2Q.

So the minimizing distribution $p^*_{U_1|QY_1}p^*_{U_2|QY_2}p^*_Q$ satisfies that:

$$\begin{aligned} Y_1 - E[Y_1|U_1Q] &\sim N(0, K_1), \text{ where } K_1 > 0, K_1 \perp U_1Q \\ Y_2 - E[Y_2|U_2Q] &\sim N(0, K_2), \text{ where } K_2 > 0, K_2 \perp U_2Q \end{aligned} \tag{3.12}$$

Denote $U_1^\dagger := E[Y_1|U_1Q], U_2^\dagger := E[Y_2|U_2Q]$. We can show that the minimizing value of $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ is attained by $p^*_{U_1^\dagger U_2^\dagger|XY_1Y_2Q}p^*_Q$, which also satisfies the constraints (3.7) by Lemma 3.6, and thereby can be rewritten in the form of $p^*_{U_1^\dagger|QY_1}p^*_{U_2^\dagger|QY_2}p^*_Q$. The proof is natural but messy, so we put it in the appendix of this chapter.

Conditioned on Q, notice that $Y_1 \sim N(0, P + N_1)$ and $Y_1 - U_1^\dagger \sim N(0, K_1)$, so $U_1^\dagger \sim N(0, P + N_1 - K_1)$. On the other hand, $Y_1 - U_1^\dagger, U_1^\dagger$ are independent, thus $Y_1 - U_1^\dagger, U_1^\dagger$ are jointly Gaussian with mean zeros, so are Y_1 and $U_1^\dagger$. What's more, the covariance matrix of $Y_1, U_1^\dagger$ are $\begin{bmatrix} P + N_1 - K_1 & P + N_1 - K_1 \\ P + N_1 - K_1 & P + N_1 \end{bmatrix}$ and independent of Q. Similarly, one could argue that $U_2^\dagger$ and Y_2 are jointly Gaussian with mean zeros, and covariance matrix independent of Q.

Since conditioned on Q we have the Markov chain $U_1^\dagger \leftarrow Y_1 \leftarrow X \rightarrow Y_2 \rightarrow U_2^\dagger$, and $p_{XY_1Y_2|Q}$, $p_{Y_1U_1^\dagger|Q}$, and $p_{Y_2U_2^\dagger|Q}$ are all joint Gaussian distributions with mean zeros and covariance matrices independent of Q. Thus at the minimizing distribution $p^*_{U_1^\dagger|QY_1}p^*_{U_2^\dagger|QY_2}p^*_Q$, $U_1^\dagger, U_2^\dagger, Y_1, Y_2, X$ are jointly Gaussian and independent of Q. So to attain the minimizing value of $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$, Q could be set to constant.

Since scaling $U_1^\dagger, U_2^\dagger$ doesn't affect the functional $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ and the constraint, we could choose $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \perp Y_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), U_2 = Y_2 + \tilde{U}_2, \tilde{U}_2 \perp Y_2, \tilde{U}_2 \sim N(0, \tilde{N}_2)$, as long as constraints 3.9 are satisfied. □

3.4 Alternate Proof to Quadratic Gaussian Distributed Source Coding

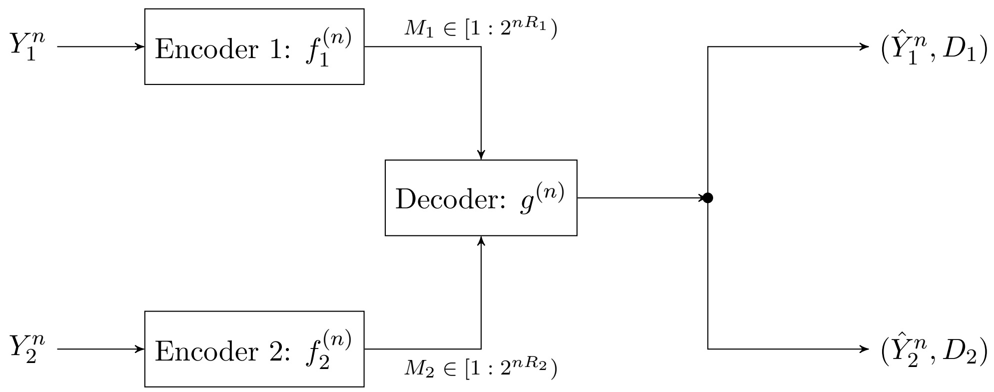

Figure 3.4: Quadratic Gaussian Distributed Source Coding

THe quadratic Gaussian distributed source coding was studied by Oohama in [51]. The setting for quadratic Gaussian distributed source coding is depicted in Figure 3.4: Let (Y_1, Y_2) be a 2-DMS generating i.i.d. sequences of random variables $(Y_{1i}, Y_{2i}) \sim N\left(\vec{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$. The encoder 1 observes Y_1^n and maps it to $M_1 \in [1 : 2^{nR_1})$ by encoding function $f_1^{(n)}$, the encoder 2 observes Y_2^n and maps it to $M_2 \in [1 : 2^{nR_2})$ by encoding function $f_2^{(n)}$. The decoder use some decoding function $g^{(n)}$ to construct some $\hat{Y}_1^n$ and $\hat{Y}_2^n$ from (M_1, M_2).

Similar to the communication problems in introduction chapter, one could define a (n, R_1, R_2) code $\mathcal{C} := (f_1^{(n)}, f_2^{(n)}, g^{(n)})$ for Quadratic Guassian CEO distributed source coding. A rate-distortion triple (R_1, R_2, D_1, D_2) is said to be achievable if there exists a sequence of codes $\mathcal{C}_n$ such that

$$\limsup_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n}\left(Y_{1i} - \hat{Y}_{1i}\right)^2\right] \le D_1$$
$$\limsup_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n}\left(Y_{2i} - \hat{Y}_{2i}\right)^2\right] \le D_2$$

And the rate-distortion region $\mathscr{R}_{QDS}(D_1, D_2)$ is defined as the closure of the set of all achievable rate pairs (R_1, R_2) such that (R_1, R_2, D_1, D_2) is achievable.

Observe that when $\rho = 0$, $Y_1^n \perp Y_2^n$, the problem will be reduced to two separate lossy source coding on two independent Gaussian sources. When $\rho = 1$, $Y_1^n = Y_2^n$, the problem is reduced to one lossy source coding on one Gaussian source. So the interesting case is that $1 > \rho > 0$, since if $\rho < 0$, we could flip Y_1^n to $-Y_1^n$ and do the compression.

Besides, by symmetry of Y_1, Y_2, one could assume that $D_1 \leq D_2$, and we can assume that $D_1 \leq 1$. Since if $D_1 \geq 1, D_2 \geq 1$, one could pick 0 for $\hat{Y}_1$ and $\hat{Y}_2$.

Wagner, Tavildar, and Viswanath in [65] proved the following single-letter characterization of $\mathscr{R}_{QDS}(D_1, D_2)$:

Theorem 3.8. *Consider the quadratic Gaussian distributed source coding on 2-DMS (Y_1, Y_2) satisfying that $p_{Y_1Y_2} \sim N\left(\vec{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, the rate-distortion $\mathscr{R}_{QDS}(D_1, D_1)$ is the set of rate pairs (R_1, R_2) such that*

$$\begin{aligned} R_1 \geq & \frac{1}{2} \log_+ \frac{1-\rho^2+\rho^2 2^{-2R_2}}{D_1} \\ R_2 \geq & \frac{1}{2} \log_+ \frac{1-\rho^2+\rho^2 2^{-2R_1}}{D_2} \\ R_1 + R_2 \geq & \frac{1}{2} \log_+ \frac{1-\rho^2+\sqrt{(1-\rho^2)^2+4\rho^2 D_1 D_2}}{2D_1D_2} \end{aligned}$$

The achievablity of above $\mathscr{R}_{QDS}(D_1, D_2)$ can be proven by using the Berger-Tung coding scheme, see Berger [9], Tung [60], and [53]. One can check chapter 12 in [21] for details. And the converse proof is mainly by using auxiliary random variable X such that $Y_1 \to X \to Y_2$ and results from estimation theory, see [21].

We will express $\mathscr{R}_{QDS}(D_1, D_2)$ in terms of the weighted sum rates, and then present an alternate proof for the converse of the weighted sum rates in a similar way as the previous section. Still we need to use the idea of matching two lower bounds and auxiliary random variable $X = \frac{1}{\sqrt{D_1}} Y_1 + \frac{1}{\sqrt{D_2}} Y_2 + Z$ where $Z \sim N(0, \frac{1-\rho^2}{\rho\sqrt{D_1D_2}}), Z \perp (Y_1, Y_2)$ such that $Y_1 \leftarrow X \to Y_2$.

Similar as before, one should notice that the weighted sum rate lower bound derived here is in a similar spirit as the improved lower bound for multiterminal source coding in [64], in terms of the identification of auxiliary random variables. However, to the best knowledge of the authors, applying rotation techinques to the evaluation of these lower bounds should be new.

Theorem 3.9. *For $0 < \rho < 1, D_1 \leq D_2, D_1 \leq 1$, and $\lambda \geq 1$, any rate pairs (R_1, R_2) in $\mathscr{R}_{QDS}(D_1, D_2)$ must satisfy that*

$$\lambda R_1 + R_2 \geq \min_{x \geq 0} x + \frac{\lambda}{2} \log_+ \frac{1-\rho^2+\rho^2 2^{-2x}}{D_1} \tag{3.13}$$

$$R_1 + \lambda R_2 \geq \min_{x \geq 0} x + \frac{\lambda}{2} \log_+ \frac{1-\rho^2+\rho^2 2^{-2x}}{D_2} \tag{3.14}$$
$$R_1 + R_2 \geq \frac{1}{2} \log_+ \frac{1-\rho^2+\sqrt{(1-\rho^2)^2+4\rho^2 D_1 D_2}}{2D_1 D_2} \tag{3.15}$$

3.4.1 Weighted Sum Rate Lower Bound

Similarly, we could derive the following weighted sum rate lower bounds for a generalized distributed source coding depicted in Figure 3.5:

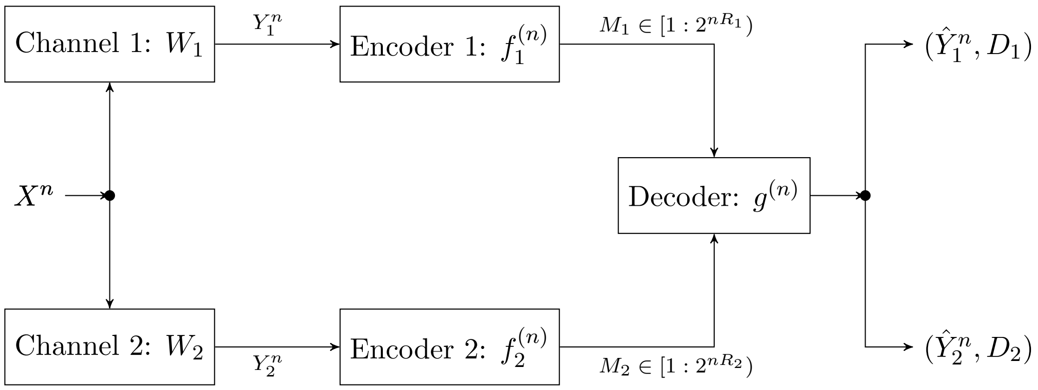

Figure 3.5: Generalized Distributed Source Coding

Theorem 3.10. *Consider the generalized quadratic distributed source coding on 2-DMS (Y_1, Y_2), assume there exists some auxiliary source X such that Y_1 and Y_2 are obtained by passing X through some discrete memoryless channel W_1 and W_2 respectively. The distortion criterion is given by*

$$\limsup_{n\to\infty} \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} d(Y_{1i}, \hat{Y}_{1i})\right) \leq D_1;$$
$$\limsup_{n\to\infty} \mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n} d(Y_{2i}, \hat{Y}_{2i})\right) \leq D_2.$$

For any $\lambda \geq 1$, any achievable rate-distortion triple (R_1, R_2, D_1, D_2) must satisfy that

$$\begin{aligned} R_1 + \lambda R_2 \geq & H(XY_1) + \lambda H(Y_2|X) - H(X|\hat{Y}_1\hat{Y}_2Q) \\ & + (\lambda-1)\max\left\{H(X|U_1QW) - H(X|\hat{Y}_1\hat{Y}_2Q), 0\right\} \\ & + H(X|U_1QW) - H(XY_1|U_1QW) + \lambda H(X|U_2QW) - \lambda H(XY_2|U_2QW) \\ R_2 + \lambda R_1 \geq & H(XY_2) + \lambda H(Y_1|X) - H(X|\hat{Y}_1\hat{Y}_2Q) \\ & + (\lambda-1)\max\left\{H(X|U_2QW) - H(X|\hat{Y}_1\hat{Y}_2Q), 0\right\} \end{aligned}$$

$$+ H(X|U_2QW) - H(XY_2|U_2QW) + \lambda H(X|U_1QW) - \lambda H(XY_1|U_1QW)$$

subject to the constraint

$$\begin{aligned}
&U_1 \leftarrow QWY_1 \leftarrow QWX \rightarrow QWY_2 \rightarrow U_2 \\
&QW \perp XY_1Y_2 \\
&\hat{Y}_1\hat{Y}_2 \leftarrow QWU_1U_2 \leftarrow XY_1Y_2 \\
&\mathrm{E}[d(Y_1,\hat{Y}_1)] \le D_1, \mathrm{E}[d(Y_2,\hat{Y}_2)] \le D_2
\end{aligned} \tag{3.16}$$

.

Proof. The proof of this theorem is essentially the same as the proof to Theorem 3.7 with the same auxiliary random variable identifications, i.e., $Q = i, W_i = X^{n/i}, U_{1i} = M_1Y_1^{i-1}, U_{2i} = M_2Y_2^{i-1}$. The term $\frac{1}{n}H(X^n|M_1M_2)$ is single-letterized in the following way:

$$H(X^n|M_1M_2) = H(X^n|\hat{Y}_1^n\hat{Y}_2^nM_1M_2) \le \sum_{i=1}^{n} H(X_i|\hat{Y}_{1i}\hat{Y}_{2i}).$$

□

3.4.2 Optimality of Achievable Weighted Sum Rate

In this section, we will use the weighted sum rates lower bounds in Theorem 3.10 to prove the converse of weighted sum rate in quadratic Gaussian distributed source coding, Theorem 3.9.

Proof. To show any $(R_1, R_2) \in \mathscr{R}_{QDS}(D_1, D_2)$ satisfy inequality (3.13). In Theorem 3.10, pick $X = Y_2$ for $R_1 + \lambda R_2$, we have

$$\begin{aligned}
R_1 + \lambda R_2 \ge & H(Y_1Y_2) + \lambda H(Y_2|Y_2) - H(Y_2|\hat{Y}_1\hat{Y}_2Q) \\
& + (\lambda - 1)\max\{H(Y_2|U_1QW) - H(Y_2|\hat{Y}_1\hat{Y}_2Q), 0\} \\
& + H(Y_2|U_1QW) - H(Y_2Y_1|U_1QW) + \lambda H(Y_2|U_2QW) - \lambda H(Y_2|U_2QW)
\end{aligned}$$

subject to the constraints (3.16).

Here we could bound $H(Y_2|\hat{Y}_1\hat{Y}_2Q)$ in a similar way as before

$$H(Y_2|\hat{Y}_1\hat{Y}_2Q) \le H(Y_2 - \hat{Y}_2|Q) \le \frac{1}{2}\log 2\pi e D_2.$$

Write $Q = QW$, above weighted sum rate lower bound can be relaxed to

$$R_1 + \lambda R_2 \ge H(Y_1Y_2) - \frac{1}{2}\log 2\pi e D_2 + (\lambda - 1)\max\{H(Y_2|U_1Q) - \frac{1}{2}\log 2\pi e D_2, 0\}$$

$$+ H(Y_2|U_1Q) - H(Y_1Y_2|U_1Q)$$

subject to the constraints

$$\begin{aligned} U_1 \leftarrow QY_1 \rightarrow Y_2 \\ Q \perp Y_1Y_2 \end{aligned} \tag{3.17}$$

Here we drop the constraints involving $\hat{Y}_1$ and $\hat{Y}_2$ in constraints (3.16).

From the constraints (3.17), $p_{U_1Q|Y_1Y_2}$ can be written in the form of $p_{U_1|QY_1}p_Q$. Thus we have:

$$\begin{aligned}
R_1 + \lambda R_2 \geq & \inf_{p_{U_1|QY_1}p_Q} \frac{1}{2}\log 2\pi e\frac{1-\rho^2}{D_2} + (\lambda-1)\max\{H(Y_2|U_1Q) - \frac{1}{2}\log 2\pi e D_2, 0\} \\
& + H(Y_2|U_1Q) - H(Y_1Y_2|U_1Q) \\
= & \inf_{p_{U_1|QY_1}p_Q} \frac{1}{2}\log 2\pi e\frac{1-\rho^2}{D_2} + (\lambda-1)\max_{\alpha\in[0,1]} \alpha\left[H(Y_2|U_1Q) - \frac{1}{2}\log 2\pi e D_2\right] \\
& + H(Y_2|U_1Q) - H(Y_1Y_2|U_1Q) \\
\overset{(a)}{=} & \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e\frac{1-\rho^2}{D_2} - \frac{(\lambda-1)\alpha}{2}\log 2\pi e D_2 \\
& + \inf_{p_{U_1|QY_1}p_Q} ((\lambda-1)\alpha+1)\, H(Y_2|U_1Q) - H(Y_1Y_2|U_1Q) \\
\overset{(b)}{=} & \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e\frac{1-\rho^2}{D_2} - \frac{(\lambda-1)\alpha}{2}\log 2\pi e D_2 \\
& + \inf_{p_{U_1Q|Y_1}} ((\lambda-1)\alpha+1)\, H(Y_2|U_1Q) - H(Y_1Y_2|U_1Q) \\
\overset{(c)}{=} & \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e\frac{1-\rho^2}{D_2} - \frac{(\lambda-1)\alpha}{2}\log 2\pi e D_2 \\
& + \inf_{p_{U_1|Y_1}} ((\lambda-1)\alpha+1)\, H(Y_2|U_1) - H(Y_1Y_2|U_1)
\end{aligned}$$

Step (a) follows from the inf max exchange via Theorem 5 in Appendix of [25]. Step (b) is due to $Q \perp Y_1Y_2$, so $p_{U_1|QY_1}p_Q = p_{U_1|QY_1}p_{Q|Y_1} = p_{U_1Q|Y_1}$. Step (c) is from replacing U_1Q with U_1.

Thus we needs to compute for $\kappa \geq 1$

$$\inf_{p_{U_1|Y_1}} \kappa H(Y_2|U_1) - H(Y_1Y_2|U_1)$$

By Lemma 3.7, above value is attained by $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), \tilde{U}_1 \perp Y_1$. So the weighted sum rate can be explicitly written as

$$R_1 + \lambda R_2 \geq \max_{\alpha\in[0,1]} \frac{1}{2}\log 2\pi e\frac{1-\rho^2}{D_2} - \frac{(\lambda-1)\alpha}{2}\log 2\pi e D_2$$

$$
\begin{aligned}
&+ \min_{\tilde{N}_1 \geq 0} \frac{(\lambda-1)\alpha+1}{2} \log 2\pi e \frac{1+\tilde{N}_1-\rho^2}{1+\tilde{N}_1} - \frac{1}{2}\log(2\pi e)^2 \frac{(1-\rho^2)\tilde{N}_1}{1+\tilde{N}_1} \\
= &\max_{\alpha\in[0,1]} \min_{\tilde{N}_1>0} -\frac{1}{2}\log\frac{\tilde{N}_1}{1+\tilde{N}_1} + \frac{(\lambda-1)\alpha+1}{2}\log\frac{1+\tilde{N}_1-\rho^2}{D_2(1+\tilde{N}_1)}
\end{aligned}
$$

On the other hand, we have $R_1+\lambda R_2 \geq 0$, thus we know that

$$
\begin{aligned}
R_1+\lambda R_2 \geq &\max\left\{0, \max_{\alpha\in[0,1]} \min_{\tilde{N}_1>0} -\frac{1}{2}\log\frac{\tilde{N}_1}{1+\tilde{N}_1} + \frac{(\lambda-1)\alpha+1}{2}\log\frac{1+\tilde{N}_1-\rho^2}{D_2(1+\tilde{N}_1)}\right\} \\
\overset{(a)}{\geq} &\min_{\tilde{N}_1>0} -\frac{1}{2}\log\frac{\tilde{N}_1}{1+\tilde{N}_1} + \frac{\lambda}{2}\log_+\frac{1+\tilde{N}_1-\rho^2}{D_2(1+\tilde{N}_1)}
\end{aligned}
$$

Step (a) follows from Lemma 3.8.

Reparamterize $x = -\frac{1}{2}\log\frac{\tilde{N}_1}{1+\tilde{N}_1}$, so we will get back to the first weighted sum rate (3.13).

To show any $(R_1, R_2) \in \mathscr{R}_{QDS}(D_1, D_2)$ satisfy inequality (3.14). In Theorem 3.10, pick $X = Y_1$ for $R_2+\lambda R_1$, we could show the second weighed sum rate (3.14) similarly.

To prove the converse for the third sum rate lower bound (3.15). The main framework of proof is still the same as the converse proof in Chapter 12 of book [21]: we need to derive the Cooperative lower bound, and another lower bound from an auxiliary random variable X, which is slightly different from the μ-Sum lower bound in the book. Then taking the minmax of the two lower bounds will give the third sum rate lower bound (3.15).

In Theorem 3.10, for the constraints (3.16), given the distortion measure $d(x,\hat{x}) = (x-\hat{x})^2$ in quadratic Gaussian distributed source coding, we could introduce some $\theta \in [-1,1]$ such that:

$$
\begin{bmatrix}
\mathrm{E}\left[\left(Y_1-\hat{Y}_1\right)^2\right] & \mathrm{E}\left[\left(Y_1-\hat{Y}_1\right)\left(Y_2-\hat{Y}_2\right)\right] \\
\mathrm{E}\left[\left(Y_1-\hat{Y}_1\right)\left(Y_2-\hat{Y}_2\right)\right] & \mathrm{E}\left[\left(Y_2-\hat{Y}_2\right)^2\right]
\end{bmatrix}
\preceq
\begin{bmatrix}
D_1 & \theta\sqrt{D_1D_2} \\
\theta\sqrt{D_1D_2} & D_2
\end{bmatrix}
\tag{3.18}
$$

Let $\lambda = 1$, first pick $X = (Y_1, Y_2)$ for R_1+R_2 in Theorem 3.10, we will obtain

$$
\begin{aligned}
R_1+R_2 \geq &H(Y_1Y_2) - H(Y_1Y_2|\hat{Y}_1\hat{Y}_2Q) \\
= &H(Y_1Y_2) - H(Y_1-\hat{Y}_1, Y_2-\hat{Y}_2|\hat{Y}_1\hat{Y}_2Q) \\
\geq &H(Y_1Y_2) - H(Y_1-\hat{Y}_1, Y_2-\hat{Y}_2|Q) \\
\geq &\frac{1}{2}\log\frac{1-\rho^2}{D_1D_2(1-\theta^2)},
\end{aligned}
\tag{3.19}
$$

which recovers the Cooperative lower bound in Chapter 12 of [21].

On the other hand, in Theorem 3.10, fix $\lambda = 1$ and $X = \frac{1}{\sqrt{D_1}} Y_1 + \frac{1}{\sqrt{D_2}} Y_2 + Z$ where $Z \sim N(0, \frac{1-\rho^2}{\rho\sqrt{D_1 D_2}})$, $Z \perp (Y_1, Y_2)$. One can verify $Y_1 \leftarrow X \leftarrow Y_2$.

For convenience of writing, denote $\mu_1 = \frac{1}{\sqrt{D_1}}, \mu_2 = \frac{1}{\sqrt{D_2}}, N = \frac{1-\rho^2}{\rho\sqrt{D_1 D_2}}$.

Notice that the covariance distortion constraint (3.18) gives a bound on $H(X|\hat{Y}_1\hat{Y}_2Q)$

$$\begin{aligned} H(X|\hat{Y}_1\hat{Y}_2Q) =& H(X - \mu_1\hat{Y}_1 - \mu_2\hat{Y}_2|\hat{Y}_1\hat{Y}_2Q) \\ \leq& H\left(\mu_1(Y_1 - \hat{Y}_1) + \mu_2(Y_2 - \hat{Y}_2) + Z\right) \\ \leq& \frac{1}{2}\log(2\pi e)\left(2 + 2\theta + N\right) \end{aligned}$$

Besides, the constraints (3.16) could be relaxed to

$$\begin{aligned} & U_1 \leftarrow QWY_1 \leftarrow QWX \rightarrow QWY_2 \rightarrow U_2 \\ & QW \perp XY_1Y_2 \\ & H(Y_1|U_1U_2QW) \leq \frac{1}{2}\log 2\pi e D_1 \\ & H(Y_2|U_1U_2QW) \leq \frac{1}{2}\log 2\pi e D_2 \end{aligned}$$

where the last two equations come from

$$\begin{aligned} H(Y_1|U_1U_2QW) \leq H(Y_1|\hat{Y}_1\hat{Y}_2Q) \leq H(Y_1 - \hat{Y}_1) \leq \frac{1}{2}\log 2\pi e D_1 \\ H(Y_2|U_1U_2QW) \leq H(Y_2|\hat{Y}_1\hat{Y}_2Q) \leq H(Y_2 - \hat{Y}_2) \leq \frac{1}{2}\log 2\pi e D_2 \end{aligned}$$

Similar to the proof in quadratic Gaussian CEO problem, write $Q = QW$, we will get the following lower bounds for weighted sum rates:

$$\begin{aligned} R_1 + R_2 \geq& \frac{1}{2}\log(2\pi e)^3(1-\rho^2)N - \frac{1}{2}\log(2\pi e)\left(2 + 2\theta + N\right) \\ &+ H(X|U_1Q) - H(XY_1|U_1Q) + H(X|U_2Q) - H(XY_2|U_2Q) \end{aligned}$$

subject to the constraints:

$$\begin{aligned} & U_1 \leftarrow QY_1 \leftarrow QX \rightarrow QY_2 \rightarrow U_2 \\ & Q \perp XY_1Y_2 \\ & H(Y_1|U_1U_2Q) \leq \frac{1}{2}\log 2\pi e D_1 \\ & H(Y_2|U_1U_2Q) \leq \frac{1}{2}\log 2\pi e D_2 \end{aligned} \tag{3.20}$$

Notice the constraint (3.20) implies that $p_{U1U_2Q|Y_1Y_2X}$ can be written in the form of $p_{U_1|Y_1Q}p_{U_2|Y_2Q}p_Q$. So we get

$$
\begin{aligned}
R_1 + R_2 \geq& \frac{1}{2}\log(2\pi e)^3(1-\rho^2)N - \frac{1}{2}\log(2\pi e)\left(2+2\theta+N\right) \\
&+ \inf_{p_{U_1|Y_1Q}p_{U_2|Y_2Q}p_Q} H(X|U_1Q) - H(XY_1|U_1Q) + H(X|U_2Q) - H(XY_2|U_2Q)
\end{aligned}
$$

where $p_{U_1|Y_1Q}p_{U_2|Y_2Q}p_Q$ needs to satisfy

$$
\begin{aligned}
H(Y_1|U_1U_2Q) &\leq \frac{1}{2}\log 2\pi e D_1 \\
H(Y_2|U_1U_2Q) &\leq \frac{1}{2}\log 2\pi e D_2
\end{aligned}
\tag{3.21}
$$

Similar to the proof to Lemma 3.4, one could show that given X, Y_1, Y_2 jointly Gaussians and $Y_1 \to X \to Y_2$, for $\lambda \geq 1$, the minimizer of

$$
\inf_{p_{U_1|Y_1Q}p_{U_2|Y_2Q}p_Q} H(X|U_1Q) - H(XY_1|U_1Q) + H(X|U_2Q) - H(XY_2|U_2Q)
$$

subject to the constraints (3.20), is attained by $Q = \emptyset$ and $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \perp Y_1, \tilde{U}_1 \perp N(0, N_1), U_2 = Y_2 + \tilde{U}_2, \tilde{U}_2 \perp Y_2, \tilde{U}_2 \perp N(0, N_2)$ subject to:

$$
\begin{aligned}
H(Y_1|U_1U_2) &\leq \frac{1}{2}\log 2\pi e D_1 \\
H(Y_2|U_1U_2) &\leq \frac{1}{2}\log 2\pi e D_2
\end{aligned}
\tag{3.22}
$$

From $X = \mu_1 Y_1 + \mu_2 Y_2 + N$ and $p_{Y_1Y_2} \sim N\left(\vec{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, we could write:

$$
\begin{aligned}
Y_1 &= \frac{\mu_1 + \mu_2\rho}{\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\rho + N}X + \sqrt{\frac{\mu_2^2(1-\rho^2)+N}{\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\rho + N}}G_1 \\
Y_2 &= \frac{\mu_2 + \mu_1\rho}{\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\rho + N}X + \sqrt{\frac{\mu_1^2(1-\rho^2)+N}{\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\rho + N}}G_1
\end{aligned}
$$

where $G_1 \sim N(0,1), G_1 \perp (X, G_2), G_2 \sim N(0,1), G_2 \perp (X, G_1)$.

For convenience of writing, denote

$$
\begin{aligned}
p_x =& \mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\rho + N \\
A_1 =& \mu_2^2(1-\rho^2) + N \\
A_2 =& \mu_1^2(1-\rho^2) + N
\end{aligned}
$$

Then the sum rate lower bound becomes:

$$
R_1 + R_2 \geq \frac{1}{2}\log(2\pi e)^3(1-\rho^2)N - \frac{1}{2}\log 2\pi e\left(2+2\theta+N\right)
$$

$$
\begin{aligned}
&+\min_{N_1,N_2\geq 0}\frac{1}{2}\log(2\pi e)(\frac{A_1}{p_x}+N_1)-\frac{1}{2}\log(2\pi e)^2N_1\frac{A_1}{p_x}\\
&+\frac{1}{2}\log(2\pi e)(\frac{A_2}{p_x}+N_2)-\frac{1}{2}\log(2\pi e)^2N_2\frac{A_2}{p_x}\\
=&\frac{1}{2}\log\frac{(1-\rho^2)N}{(2+2\theta+N)}+\min_{N_1,N_2\geq 0}\frac{1}{2}\log\frac{(A_1+N_1p_x)(A_2+N_2p_x)}{N_1N_2A_1A_2}
\end{aligned}
$$

subject to the constraints:

$$
\begin{aligned}
\frac{N_1(1+N_2-\rho^2)}{(1+N_1)(1+N_2)-\rho^2}\leq D_1\\
\frac{N_2(1+N_1-\rho^2)}{(1+N_1)(1+N_2)-\rho^2}\leq D_2
\end{aligned}
\tag{3.23}
$$

By Lemma 3.9, the above sum rate lower bound becomes

$$
R_1+R_2\geq\frac{1}{2}\log\frac{(1-\rho^2)N}{2(1+\theta)+N}\left(\frac{\rho}{1-\rho^2}+\frac{1}{2}\mu_1\mu_2+\frac{1}{2}\sqrt{\mu_1^2\mu_2^2+\frac{4\rho^2}{(1-\rho^2)^2}}\right)^2
\tag{3.24}
$$

Denote the right-hand side of above lower bound (3.24) as $S_1(\theta)$.

Notice that the Cooperative lower bound (3.19) can be written in terms of μ_1,μ_2:

$$
R_1+R_2\geq\frac{1}{2}\log\frac{(1-\rho^2)\mu_1^2\mu_2^2}{1-\theta^2}
\tag{3.25}
$$

Denote the right-hand side of the lower bound (3.25) as $S_2(\theta)$.

Observe that $S_1(\theta)$ is decreasing on $\theta\in[-1,1]$ and $S_2(\theta)$ is first decreasing and then increasing on $\theta\in[-1,1]$. And by Lemma 3.10 there are two roots $\theta_1<\theta_2=\frac{\sqrt{4\rho^2+\mu_1^2\mu_2^2(1-\rho^2)^2}-\mu_1\mu_2(1-\rho^2)}{2\rho}$ and $\theta_2\in(0,1]$ such that $S_1(\theta)=S_2(\theta)$.

So we have

$$
R_1+R_2\geq\min_{\theta\in[-1,1]}\max\{S_1(\theta),S_2(\theta)\}=S_2(\theta_2)
$$

which will lead to the sum rate lower bound (3.15).

□

3.5 Discussion and Conclusion

In this chapter we established that linear coding strategy of Körner and Marton [35] yields the optimal sum-rate for pairs of distributions outside the doubly symmetric binary source. This was shown by developing a lower bound and identifying sufficient conditions when the lower bound is tight. The ideas and results are applicable to larger fields as well.

Via using the idea in deriving weighted sum rate lower bounds for Körner and Marton's modulo two sum problem, we could derive similar weighted sum rate lower bounds for quadratic Gaussian CEO problem and quadratic Gaussian distributed source coding, and thereby provide alternate proofs for the optimality of Berger-Tung coding scheme in these two settings.

3.A Quadratic Gaussian CEO

Lemma 3.6. *Given the minimizing distribution* $p^*_{U_1|QY_1}p^*_{U_2|QY_2}p^*_Q$ *for* $\Theta^{\varepsilon_1;\varepsilon_2}_{\kappa,\lambda}$ *satisfying the two properties* (3.12)*, denote* $U_1^\dagger := E[Y_1|U_1Q], U_2^\dagger := E[Y_2|U_2Q]$*, then* $p^*_{U_1^\dagger U_2^\dagger|XY_1Y_2Q}p^*_Q$ *satisfies the constraints* (3.7) *and also attain the minimizing value of* $\Theta^{\varepsilon_1;\varepsilon_2}_{\kappa,\lambda}$.

Proof. First we will prove some Markov structures on the joint distribution $Q, U_1, U_2, U_1^\dagger, U_2^\dagger$, X, Y_1, Y_2, which will be useful in the following proof.

From the two properties (3.12) of the minimizing distribution $p^*_{U_1U_2|QXY_1Y_2}p^*_Q$, we could write $Y_1 = U_1^\dagger + V_1$ where $V_1 \sim N(0, K_1), V_1 \perp U_1Q$ and $Y_2 = U_2^\dagger + V_2$ where $V_2 \sim N(0, K_2), V_2 \perp U_2Q$.

Since $V_1 \perp U_1Q, Y_1 \perp Q$, so $U_1^\dagger = Y_1 - V_1 \perp Q$, $U_1^\dagger$ is a function of U_1; similarly one could argue that $U_2^\dagger$ is a function of U_2.

The conditions that $Y_1 = U_1^\dagger + V_1$ where $V_1 \sim N(0, K_1), V_1 \perp U_1Q$ and $Y_2 = U_2^\dagger + V_2$ where $V_2 \sim N(0, K_2), V_2 \perp U_2Q$ also give the following Markov chain

$$Y_1 \to U_1^\dagger Q \to U_1 \tag{3.26}$$

$$Y_2 \to U_2^\dagger Q \to U_2 \tag{3.27}$$

With these two Markov chains, the Markov chain $U_1 \leftarrow QY_1 \leftarrow XQ \to QY_2 \to U_2$, and $U_1^\dagger$ is a function of U_1 and $U_2^\dagger$ is a function of U_2, one can verify the following long Markov chain:

$$U_1 \leftarrow QU_1^\dagger \leftarrow QY_1 \leftarrow QX \to QY_2 \to QU_2^\dagger \to U_2 \tag{3.28}$$

The verification is as following:

1. $U_1 \leftarrow QU_1^\dagger \leftarrow QY_1$ follows from Markov chain (3.26);

2. $U_1QU_1^\dagger \leftarrow QY_1 \leftarrow QX$ follows from $U_1^\dagger$ is a function of U_1 and $U_1 \leftarrow QY_1 \leftarrow X$;

3. $U_1QU_1^\dagger Y_1 \leftarrow QX \leftarrow QY_2$ follows from $U_1^\dagger$ is a function of U_1 and $U_1Y_1 \leftarrow QX \leftarrow Y_2$;

4. $U_1QU_1^\dagger Y_1X \leftarrow QY_2 \leftarrow QU_2^\dagger$ follows from $U_1Y_1X \leftarrow QY_2 \leftarrow U_2$ and $U_1^\dagger$ is a function of U_1 and $U_2^\dagger$ is a function of U_2;

5. $U_1QU_1^\dagger Y_1XY_2 \leftarrow QU_2^\dagger \leftarrow U_2$ follows from $U_1^\dagger U_1Y_1X \leftarrow QY_2 \leftarrow U_2U_2^\dagger$ and the Markov chain (3.27).

First we will use this long Markov chain to verify that $p^*_{U_1^\dagger U_2^\dagger|XY_1Y_2Q}p^*_Q$ satisfies the constraints (3.7).

- $p^*_{U_1U_2|QXY_1Y_2}p^*_Q$ satisfies the Markov chain in the constraints (3.7).

$$U_1^\dagger \leftarrow QY_1 \leftarrow QX \rightarrow QY_2 \rightarrow U_2^\dagger,$$

  which is implied by the long Markov chain (3.28).

- Observe that

$$\begin{aligned} H(X|U_1^\dagger U_2^\dagger Q) &\stackrel{(a)}{\leq} H(X|U_1U_2Q) \\ &\stackrel{(b)}{\leq} \frac{1}{2}\log 2\pi eD \end{aligned}$$

  Inequality (a) follows from the markov chain $U_1U_2Q \leftarrow U_1^\dagger U_2^\dagger Q \leftarrow X$, which is implied by the long Markov chain (3.28). Inequality (c) follows from that $p^*_{U_1U_2|QXY_1Y_2}p^*_Q$ satisfies $H(X|U_1U_2Q) \geq \frac{1}{2}\log\frac{P}{D}$.

- $Q \perp XY_1Y_2$ is satisfied since the joint distribution of $U_1^\dagger, U_2^\dagger, X, Y_1, Y_2$ is given by $p_{XY_1Y_2}p^*_{U_1^\dagger U_2^\dagger|XY_1Y_2Q}p^*_Q$.

Second we will show that the minimizing value of $\Theta^{\varepsilon_1,\varepsilon_2}_{\kappa,\lambda}$ is attained by this $p^*_{U_1^\dagger U_2^\dagger|QXY_1Y_2}p^*_Q$.

Observe that

$$\begin{aligned} H(X|U_1Q) &\stackrel{(a)}{=} H(X|U_1QU_1^\dagger) \stackrel{(b)}{=} H(X|U_1^\dagger Q) \\ H(XY_1|U_1Q) &\stackrel{(a)}{=} H(XY_1|U_1QU_1^\dagger) \stackrel{(c)}{=} H(XY_1|U_1^\dagger Q) \end{aligned} \tag{3.29}$$

Equality (a) follows from the fact that $U_1^\dagger$ is a function U_1; Equality (b) is due to the Markov chain $U_1 \leftarrow U_1^\dagger Q \leftarrow X$ implied by the long Markov chain (3.28); Equality (c) is due to the markov chain $U_1 \leftarrow U_1^\dagger Q \leftarrow XY_1$, which also follows from the long Markov chain (3.28).

Similarly one could argue that

$$\begin{aligned} H(X|U_2Q) &= H(X|U_2QU_2^\dagger) = H(X|U_2^\dagger Q) \\ H(XY_2|U_2Q) &= H(XY_2|U_2QU_2^\dagger) = H(XY_2|U_2^\dagger Q) \end{aligned} \tag{3.30}$$

With these equalities (3.29) and (3.30), we have

$$\begin{aligned} &(\kappa+\varepsilon_2)H(X|U_1U_2Q) - H(XY_1|U_1Q) + (\lambda+\varepsilon_1)H(X|U_2Q) - \lambda H(XY_2|U_2Q) \\ =&(\kappa+\varepsilon_2)H(X|U_1^\dagger U_2^\dagger Q) - H(XY_1|U_1^\dagger Q) + (\lambda+\varepsilon_1)H(X|U_2^\dagger Q) - \lambda H(XY_2|U_2^\dagger Q) \end{aligned}$$

This proves that the minimizing value of $\Theta_{\kappa,\lambda}^{\varepsilon_1,\varepsilon_2}$ is attained by this $p^*_{U_1^\dagger U_2^\dagger|QXY_1Y_2}p^*_Q$.

□

3.B Quadratic Gaussian Distributed Source Coding

Lemma 3.7. *Given* $p_{Y_1Y_2} \sim N\left(\vec{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, *for any* $\kappa \geq 1$,

$$\inf_{p_{U_1|Y_1}} \kappa H(Y_2|U_1) - H(Y_1Y_2|U_1)$$

is attained by $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), \tilde{U}_1 \perp Y_1$.

Proof. This proof is similar to the proof to Lemma 3.4. Consider the following perturbed optimization problem:

Given $p_{Y_1Y_2} \sim N\left(\vec{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, for any $\varepsilon > 0, \kappa \geq 1$, want to find the infimum of the following function:

$$\Gamma_\kappa^\varepsilon(p_{U_1|Y_1}) := (\kappa+\varepsilon)H(Y_2|U_1) - H(Y_1Y_2|U_1)$$

Since scaling U_1 doesn't affect $\Gamma_\kappa^\varepsilon$, one could truncate U_1 to some random variable with support on $[0,1]$. This will give tightness of the joint distribution $P_{Y_1U_1U_2}$. By routine arguments in Appendix II of [27] one can show that there is a minimizer from the tightness of the sequence of distributions.

So we assume that infimum of $\Gamma^{\varepsilon}_{\kappa}(p_{U_1|Y_1})$ is attained by some minimizing distribution $p^*_{U_1|Y_1}$, and write the joint distribution of U_1, Y_1, Y_2 by $p^*_{U_1|Y_1}p_{Y_1Y_2}$. Take two i.i.d. copies of the joint distribution at the minimizer and denote them using subscripts a, b respectively. Let $(\cdot)_+ = \frac{(\cdot)_a+(\cdot)_b}{\sqrt{2}}$ and $(\cdot)_- = \frac{(\cdot)_a-(\cdot)_b}{\sqrt{2}}$, where $(\cdot)$ can be replaced with Y_1, Y_2.

Denote minimum of $\Gamma^{\varepsilon}_{\kappa}$ as V. Now we have, by the rotation trick in [27]:

$$
\begin{aligned}
2V =&(\kappa+\varepsilon)H(Y_{2a}Y_{2b}|U_{1a}U_{1b}) - H(Y_{1a}Y_{1b}Y_{2a}Y_{2b}|U_{1a}U_{1b})\\
=&(\kappa+\varepsilon)H(Y_{2+}Y_{2-}|U_{1a}U_{1b}) - H(Y_{1+}Y_{1-}Y_{2+}Y_{2-}|U_{1a}U_{1b})\\
=&(\kappa+\varepsilon)H(Y_{2+}|U_{1a}U_{1b}Y_{1-}Y_{2-}) + (\kappa+\varepsilon)H(Y_{2-}|U_{1a}U_{1b}Y_{2+}) + (\kappa+\varepsilon)I(Y_{2+};Y_{1-}Y_{2-}|U_{1a}U_{1b})\\
&- H(Y_{1+}Y_{2+}|U_{1a}U_{1b}Y_{1-}Y_{2-}) - H(Y_{1-}Y_{2-}|U_{1a}U_{1b}Y_{2+}) - I(Y_{2+};Y_{1-}Y_{2-}|U_{1a}U_{1b})\\
=&(\kappa+\varepsilon)H(Y_{2+}|U_{1a}U_{1b}Y_{1-}Y_{2-}) - H(Y_{1+}Y_{2+}|U_{1a}U_{1b}Y_{1-}Y_{2-})\\
&+ (\kappa+\varepsilon)H(Y_{2-}|U_{1a}U_{1b}Y_{2+}) - H(Y_{1-}Y_{2-}|U_{1a}U_{1b}Y_{2+})\\
&+ (\kappa+\varepsilon-1)I(Y_{2+};Y_{1-}Y_{2-}|U_{1a}U_{1b})\\
\overset{(a)}{=}&\Gamma^{\varepsilon}_{\kappa}(p_{U_{1a}U_{1b}Y_{1-}Y_{2-}|Y_{1+}}) + \Gamma^{\varepsilon}_{\kappa}(p_{U_{1a}U_{1b}Y_{2+}|Y_{1-}}) + (\kappa+\varepsilon-1)I(Y_{2+};Y_{1-}Y_{2-}|U_{1a}U_{1b})\\
\geq&2V + (\kappa+\varepsilon-1)I(Y_{2+};Y_{1-}Y_{2-}|U_{1a}U_{1b})
\end{aligned}
$$

Observe that since $p_{Y_{1a}Y_{2a}}$ and $p_{Y_{1b}Y_{2b}}$ are i.i.d. Gaussians, so are $p_{Y_{1+}Y_{2+}}$ and $p_{Y_{1-}Y_{2-}}$. Besides, $U_{1a}U_{1b} \to Y_{1-}Y_{1+} \to Y_{2-}Y_{2+}$ and $Y_{1-}Y_{2-} \perp Y_{1+}Y_{2+}$ implies that $Y_{1-}Y_{2-}U_{1a}U_{1b} \to Y_{1+} \to Y_{2+}$ and $Y_{1+}Y_{2+}U_{1a}U_{1b} \to Y_{1-} \to Y_{2-}$. Thus we have step (a).

Therefore, $\kappa+\varepsilon-1>0$ will force $I(Y_{2+};Y_{1-}Y_{2-}|U_{1a}U_{1b})=0$. It implies that given any value assignments of $U_{1a}U_{1b}$, $Y_{2+} \perp Y_{2-}$. Observe that for the two i.i.d. copies of the joint distribution at the minimizer, we could have

$$
\begin{aligned}
Y_{2a} = \rho Y_{1a} + \sqrt{1-\rho^2}G_a, G_a \sim N(0,1), G_a \perp U_{1a}Y_{1a}\\
Y_{2b} = \rho Y_{1b} + \sqrt{1-\rho^2}G_b, G_b \sim N(0,1), G_b \perp U_{1b}Y_{1b}
\end{aligned}
$$

Thus for the rotated version,

$$
\begin{aligned}
Y_{2+} = \rho Y_{1+} + \sqrt{1-\rho^2}G_+, G_+ \sim N(0,1), G_+ \perp U_{1a}U_{1b}Y_{1+}Y_{1+}\\
Y_{2-} = \rho Y_{1-} + \sqrt{1-\rho^2}G_-, G_- \sim N(0,1), G_- \perp U_{1a}U_{1b}Y_{1-}Y_{1-}
\end{aligned}
$$

Again we apply the Proposition 2 in [27], treat Y_{1+} and Y_{1-} as the "channel input", and treat Y_{2+} and Y_{2-} as the "channel output". One can conclude that given any value assignments of $U_{1a}U_{1b}Q_aQ_b$, Y_{1+} is independent of Y_{1-}.

By applying Corollary 3 in [27], we know that the minimizing distribution $p^*_{U_1U_2|QXY_1Y_2}p^*_Q$ satisfies that:

$$Y_1 - E[Y_1|U_1] \sim N(0, K_1), \text{ where } K_1 > 0, K_1 \perp U_1 \tag{3.31}$$

Denote $U_1^\dagger := E[Y_1|U_1]$. We can show that the minimizing value of $\Gamma_\kappa^\varepsilon$ is attained by $p^*_{U_1^\dagger|Y_1}$ from the Markov chain $U_1 \to U_1^\dagger \to Y_1$ and $U_1^\dagger$ is a function of U_1. Since Y_1 and $Y_1 - U_1^\dagger$ are both Gaussians with mean zeros and fixed variance, and $Y_1 \perp Y_1 - U_1^\dagger$, thus they are jointly Gaussian with mean zeros and fixed covariance matrix, so are $Y_1, U_1^\dagger$.

Observe that scaling U_1 doesn't affect $\Gamma_\kappa^\varepsilon$. Thus for $\varepsilon > 0, \kappa \geq 1$, the minimizing value of $\Gamma_\kappa^\varepsilon$ is attained by $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), \tilde{U}_1 \perp Y_1$.

Notice that

$$\begin{aligned}\Gamma_\kappa^0 =& \Gamma_\kappa^\varepsilon - \varepsilon H(Y_1|U_1) \\ \geq& \Gamma_\kappa^\varepsilon - \varepsilon \log 2\pi e\end{aligned}$$

By a similar continuity argument as that in quadratic Gaussian CEO alternate proof, one could argue that for $\kappa \geq 1$, the minimizing value of Γ_κ^0 is also attained by $U_1 = Y_1 + \tilde{U}_1, \tilde{U}_1 \sim N(0, \tilde{N}_1), \tilde{U}_1 \perp Y_1$.

□

Lemma 3.8. *For $D_2 > 0, 0 < \rho < 1, \lambda \geq 1$,*

$$\begin{aligned}&\max\left\{0, \max_{\alpha\in[0,1]} \min_{\tilde{N}_1>0} -\frac{1}{2}\log\frac{\tilde{N}_1}{1+\tilde{N}_1} + \frac{(\lambda-1)\alpha+1}{2}\log\frac{1+\tilde{N}_1-\rho^2}{D_2(1+\tilde{N}_1)}\right\} \\ &\geq \min_{\tilde{N}_1>0} -\frac{1}{2}\log\frac{\tilde{N}_1}{1+\tilde{N}_1} + \frac{\lambda}{2}\log_+\frac{1+\tilde{N}_1-\rho^2}{D_2(1+\tilde{N}_1)}\end{aligned}$$

Proof. Re-paramterize in $k = \frac{\tilde{N}_1}{1+\tilde{N}_1} \in (0, 1]$. We want to show that

$$\begin{aligned}&\max\left\{0, \max_{\alpha\in[0,1]} \min_{k\in(0,1]} -\frac{1}{2}\ln k + \frac{(\lambda-1)\alpha+1}{2}\ln\frac{1-\rho^2+\rho^2 k}{D_2}\right\} \\ &\geq \min_{k\in(0,1]} -\frac{1}{2}\ln k + \frac{\lambda}{2}\ln_+\frac{1-\rho^2+\rho^2 k}{D_2}\end{aligned} \tag{3.32}$$

where $\ln_+ x = \max\{\ln x, 0\}$.

Denote the functions

$$f_{\lambda,\alpha}(k) := -\frac{1}{2}\ln k + \frac{(\lambda-1)\alpha+1}{2}\log\frac{1-\rho^2+\rho^2 k}{D_2}$$

Then inequality 3.32 can be written as

$$\max\{0, \max_{\alpha\in[0,1]} \min_{k\in(0,1]} f_{\lambda,\alpha}(k)\} \geq \min_{k\in(0,1]} \max\{-\frac{1}{2}\ln k, f_{\lambda,1}(k)\} \tag{3.33}$$

The first derivative of $f_{\lambda,\alpha}(k)$ gives that

$$f'_{\lambda,\alpha}(k) = \frac{(\lambda-1)\alpha\rho^2 k - (1-\rho^2)}{2k(1-\rho^2+\rho^2 k)}$$

When $\lambda = 1$, for the right-hand side of inequality (3.33), since both $-\frac{1}{2}\ln k$ and $f_{1,1}(k)$ decreases on $k \in (0,1]$, so $\max\{-\frac{1}{2}\ln k, f_{1,1}(k)\}$ minimizes at $k = 1$, which evaluates to $\max\{0, \frac{1}{2}\log\frac{1}{D_2}\}$; On the other hand, for the left-hand side of inequality (3.33), pick $\alpha = 1$, $f_{1,1}(k)$ minimizes at $k = 1$, so left-hand side becomes $\max\{0, \frac{1}{2}\ln\frac{1}{D_2}\}$. So the inequality (3.33) holds.

When $\lambda > 1$, notice that

$$\begin{aligned} f_{\lambda,1}(k) =& -\frac{1}{2}\ln k + \frac{\lambda}{2}\ln\frac{1-\rho^2+\rho^2 k}{D_2} \\ f'_{\lambda,1}(k) =& \frac{(\lambda-1)\rho^2 k - (1-\rho^2)}{2k(1-\rho^2+\rho^2 k)} \end{aligned}$$

- When $(\lambda-1)\rho^2 \leq 1-\rho^2$, both $-\frac{1}{2}\ln k$ and $f_{\lambda,1}(k)$ decreases over $k \in (0,1]$, so the right-hand side of inequality (3.33) has

  $$\min_{k\in(0,1]} \max\{-\frac{1}{2}\ln k, f_{\lambda,1}(k)\} = \max\{0, f_{\lambda,1}(1)\} = \max\{0, \frac{1}{2}\log\frac{1}{D_2}\};$$

  For the left-hand side, pick $\alpha = 0$, then we get $\max\{0, \frac{1}{2}\log\frac{1}{D_2}\}$, so inequality (3.33) holds trivially;

- When $(\lambda-1)\rho^2 > 1-\rho^2$, observe that $f_{\lambda,1}(k)$ first decrease from $+\infty$ to $f_{\lambda,1}(\frac{1-\rho^2}{(\lambda-1)\rho^2})$ on $k \in (0, \frac{1-\rho^2}{(\lambda-1)\rho^2})$ and then increase to $\frac{\lambda}{2}\ln\frac{1}{D_2}$ on $k \in (\frac{1-\rho^2}{(\lambda-1)\rho^2}, 1]$, but $-\frac{1}{2}\ln k$ decreases from $+\infty$ to 0 over $k \in (0,1]$. And their intersection point (root of $f_{\lambda,1}(k) = -\frac{1}{2}\ln k$) either doesn't exist or happens at $k = \frac{D_2-1+\rho^2}{\rho^2} \in (0,1]$.

  If $D_2 \geq 1$, then $-\frac{1}{2}\ln k \geq f_{\lambda,1}(k), \forall k \in (0,1]$. The right-hand side of inequality (3.33) is 0, so the inequality (3.33) holds trivially;

  If $D_2 \leq 1-\rho^2$, then $-\frac{1}{2}\ln k \leq f_{\lambda,1}(k), \forall k \in (0,1]$. The right-hand side of inequality (3.33) is $f_{\lambda,1}(\frac{1-\rho^2}{(\lambda-1)\rho^2})$. For the left-hand side, pick $\alpha = 1$, it becomes $\max\{0, f_{\lambda,1}(\frac{1-\rho^2}{(\lambda-1)\rho^2})\}$. Thus the inequality (3.33) holds;

  If $0 < \frac{1-\rho^2}{(\lambda-1)\rho^2} \leq \frac{D_2-1+\rho^2}{\rho^2} < 1$, then the right-hand side of inequality (3.33) is $-\frac{1}{2}\ln\frac{D_2-1+\rho^2}{\rho^2}$. For the left-hand side, pick $\alpha = \frac{1-\rho^2}{(\lambda-1)(D_2-1+\rho^2)}$, then $f_{\lambda,\alpha}(k)$

minimizes at $k = \frac{D_2-1+\rho^2}{\rho^2}$, whose value is equal to $-\frac{1}{2}\ln\frac{D_2-1+\rho^2}{\rho^2}$. So the inequality (3.33) holds;

If $0 < \frac{D_2-1+\rho^2}{\rho^2} < \frac{1-\rho^2}{(\lambda-1)\rho^2} < 1$, then the right-hand side of inequality (3.33) is $f_{\lambda,1}(\frac{1-\rho^2}{(\lambda-1)\rho^2})$. For the left-hand side, pick $\alpha = 1$, then $f_{\lambda,1}(k)$ minimizes at $k = \frac{1-\rho^2}{(\lambda-1)\rho^2}$. So the inequality (3.33) holds.

□

Lemma 3.9. *For $1 > \rho \geq 0, 0 < \mu_1 \leq \mu_2$, denote $N = \frac{(1-\rho^2)\mu_1\mu_2}{\rho}$ and $p_x = \mu_1^2+\mu_2^2+2\mu_1\mu_2\rho+N, A_1 = \mu_2^2(1-\rho^2)+N, A_2 = \mu_1^2(1-\rho^2)+N$, the following quantity*

$$\min_{N_1,N_2\geq 0}\frac{(A_1+N_1p_x)(A_2+N_2p_x)}{N_1N_2A_1A_2} \tag{3.34}$$

subject to constraints:

$$\begin{aligned}\frac{N_1(1+N_2-\rho^2)}{(1+N_1)(1+N_2)-\rho^2} &\leq \frac{1}{\mu_1^2}\\ \frac{N_2(1+N_1-\rho^2)}{(1+N_1)(1+N_2)-\rho^2} &\leq \frac{1}{\mu_2^2}\end{aligned} \tag{3.35}$$

is lower bounded by $\left(\frac{\rho}{1-\rho^2}+\frac{1}{2}\mu_1\mu_2+\frac{1}{2}\sqrt{\mu_1^2\mu_2^2+\frac{4\rho^2}{(1-\rho^2)^2}}\right)^2$.

Proof. Reparameterize in $x = \frac{1}{N_1}+\frac{1}{1-\rho^2}, y = \frac{1}{N_2}+\frac{1}{1-\rho^2}$, then $x, y \geq \frac{1}{1-\rho^2}$. For the constraints (3.35) we have:

$$\begin{cases}\frac{\frac{1-\rho^2}{N_2}+1}{\frac{1-\rho^2}{N_1N_2}+\frac{1}{N_1}+\frac{1}{N_2}+1} \leq \frac{1}{\mu_1^2}\\ \frac{\frac{1-\rho^2}{N_1}+1}{\frac{1-\rho^2}{N_1N_2}+\frac{1}{N_1}+\frac{1}{N_2}+1} \leq \frac{1}{\mu_2^2}\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{(1-\rho^2)y}{(1-\rho^2)(x-\frac{1}{1-\rho^2})(y-\frac{1}{1-\rho^2})+x-\frac{1}{1-\rho^2}+y-\frac{1}{1-\rho^2}+1} \leq \frac{1}{\mu_1^2}\\ \frac{(1-\rho^2)x}{(1-\rho^2)(x-\frac{1}{1-\rho^2})(y-\frac{1}{1-\rho^2})+x-\frac{1}{1-\rho^2}+y-\frac{1}{1-\rho^2}+1} \leq \frac{1}{\mu_2^2}\end{cases}$$

$$\Leftrightarrow\begin{cases}\frac{(1-\rho^2)y}{(1-\rho^2)xy-\frac{1}{1-\rho^2}+1} \leq \frac{1}{\mu_1^2}\\ \frac{(1-\rho^2)x}{(1-\rho^2)xy-\frac{1}{1-\rho^2}+1} \leq \frac{1}{\mu_2^2}\end{cases}$$

$$\Leftrightarrow\begin{cases}xy-\mu_1^2y \geq \frac{\rho^2}{(1-\rho^2)^2}\\ xy-\mu_2^2x \geq \frac{\rho^2}{(1-\rho^2)^2}\end{cases}$$

And the minimization functional can be simplified as following:

$$\frac{1}{A_1A_2}\left[p_x^2+p_xA_1(x-\frac{1}{1-\rho^2})+p_xA_2(y-\frac{1}{1-\rho^2})+A_1A_2(x-\frac{1}{1-\rho^2})(y-\frac{1}{1-\rho^2})\right]$$

$$
\begin{aligned}
=&\frac{1}{A_1A_2}\left[p_x^2 - p_xA_1\frac{1}{1-\rho^2} - p_xA_2\frac{1}{1-\rho^2} + p_xA_1x + p_xA_2y + A_1A_2xy\right.\\
&\qquad\left. -A_1A_2\frac{x+y}{1-\rho^2} + \frac{A_1A_2}{(1-\rho^2)^2}\right]\\
=&\frac{1}{A_1A_2}\left[p_x(p_x - A_1\frac{1}{1-\rho^2} - A_2\frac{1}{1-\rho^2}) + (p_x - \frac{A_2}{1-\rho^2})A_1x + (p_x - \frac{A_1}{1-\rho^2})A_2y\right.\\
&\qquad\left. +A_1A_2xy + \frac{A_1A_2}{(1-\rho^2)^2}\right]\\
=&\frac{1}{A_1A_2}\left[p_x(\rho - \frac{1}{\rho})\mu_1\mu_2 + (\mu_2^2 + \rho\mu_1\mu_2)\frac{1-\rho^2}{\rho}\mu_2(\rho\mu_2 + \mu_1)x\right.\\
&\qquad +(\mu_1^2 + \rho\mu_1\mu_2)\frac{1-\rho^2}{\rho}\mu_1(\rho\mu_1 + \mu_2)y + \frac{(1-\rho^2)^2}{\rho^2}\mu_1\mu_2(\rho\mu_1 + \mu_2)(\rho\mu_2 + \mu_1)xy\\
&\qquad\left. +\frac{1}{\rho^2}\mu_1\mu_2(\rho\mu_1 + \mu_2)(\rho\mu_2 + \mu_1)\right]\\
=&\frac{1}{A_1A_2}\left[\left(\mu_1^2 + \mu_2^2 + (\rho + \frac{1}{\rho})\mu_1\mu_2\right)(\rho - \frac{1}{\rho})\mu_1\mu_2 + (\mu_1 + \rho\mu_2)\frac{1-\rho^2}{\rho}\mu_2^2(\rho\mu_1 + \mu_2)x\right.\\
&\qquad + (\mu_2 + \rho\mu_1)\frac{1-\rho^2}{\rho}\mu_1^2(\rho\mu_2 + \mu_1)y + \frac{(1-\rho^2)^2}{\rho^2}\mu_1\mu_2(\rho\mu_1 + \mu_2)(\rho\mu_2 + \mu_1)xy\\
&\qquad\left. +\frac{1}{\rho^2}\mu_1\mu_2(\rho\mu_1 + \mu_2)(\rho\mu_2 + \mu_1)\right]\\
=&\frac{1}{A_1A_2}\left[(\mu_1 + \rho\mu_2)\frac{1-\rho^2}{\rho}\mu_2^2(\rho\mu_1 + \mu_2)x + (\mu_2 + \rho\mu_1)\frac{1-\rho^2}{\rho}\mu_1^2(\rho\mu_2 + \mu_1)y\right.\\
&\left. +\frac{(1-\rho^2)^2}{\rho^2}\mu_1\mu_2(\rho\mu_1 + \mu_2)(\rho\mu_2 + \mu_1)xy + \mu_1\mu_2\left[\rho\mu_1^2 + \rho\mu_2^2 + \mu_1\mu_2(1+\rho^2)\right]\right]\\
=&\frac{1}{A_1A_2}\frac{(\rho\mu_1 + \mu_2)(\rho\mu_2 + \mu_1)(1-\rho^2)}{\rho^2}\left[(1-\rho^2)\mu_1\mu_2xy + \rho\mu_2^2x + \rho\mu_1^2y + \mu_1\mu_2\frac{\rho^2}{1-\rho^2}\right]\\
=&xy + \frac{\rho}{1-\rho^2}\frac{\mu_2}{\mu_1}x + \frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2}y + \frac{\rho^2}{(1-\rho^2)^2}\\
=&\left(x + \frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2}\right)\left(y + \frac{\rho}{1-\rho^2}\frac{\mu_2}{\mu_1}\right)
\end{aligned}
$$

So the original quantity (3.34) is equal to

$$\min_{x,y}\left(x + \frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2}\right)\left(y + \frac{\rho}{1-\rho^2}\frac{\mu_2}{\mu_1}\right)$$

subject to the constraints

$$
\begin{aligned}
x \geq& \frac{1}{1-\rho^2}, y \geq \frac{1}{1-\rho^2}\\
xy - \mu_1^2y \geq& \frac{\rho^2}{(1-\rho^2)^2}\\
xy - \mu_2^2x \geq& \frac{\rho^2}{(1-\rho^2)^2}
\end{aligned}
$$

When $\rho = 0$, this minimization problem is simplified to

$$\min_{x,y} xy$$

subject to the constraints

$$\begin{aligned} x \geq & 1, y \geq 1 \\ x \geq & \mu_1^2, y \geq \mu_2^2, \end{aligned}$$

which will be lower bounded by $\mu_1^2\mu_2^2$. So suffices to consider the case when $\rho \neq 0$.

When $0 < \rho < 1$, Observe that the previous constraints implies that $x > \mu_1^2, y > \mu_2^2$. So we could replace the constraints $x, y \geq \frac{1}{1-\rho^2}$ with $x > \mu_1^2, y > \mu_2^2$, the minimizing value will not increase as the domain enlarges. The original quantity (3.34) is lower bounded by

$$\min_{x,y} \left(x + \frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2}\right)\left(y + \frac{\rho}{1-\rho^2}\frac{\mu_2}{\mu_1}\right) \tag{3.36}$$

subject to the constraints

$$\begin{aligned} x > & \mu_1^2, y > \mu_2^2 \\ xy - \mu_1^2 y \geq & \frac{\rho^2}{(1-\rho^2)^2} \\ xy - \mu_2^2 x \geq & \frac{\rho^2}{(1-\rho^2)^2} \end{aligned} \tag{3.37}$$

Since $\rho > 0$, the target (3.36) is increasing when x or y increases. For any feasible x, y satisfying (3.37), fix y, if neither constraints involving x are tight, one could always fix y and decrease x until one of the constraints become tight, and the target functional (3.36) also decreases. And the tight constraint can not be $x = \mu_1^2$, in this case $xy - \mu_1^2 y = 0 < \frac{\rho^2}{(1-\rho^2)^2}$ for $\rho > 0$.

Assume the constraint $xy - \mu_1^2 y \geq \frac{\rho^2}{(1-\rho^2)^2}$ is tight, then we have

$$y = \frac{\rho^2}{(1-\rho^2)^2}\frac{1}{x-\mu_1^2}$$

Above minimization functional (3.36) can be rewritten as a function of x:

$$\begin{aligned} F(x) := & \left(x + \frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2}\right)\left(\frac{\rho^2}{(1-\rho^2)^2}\frac{1}{x-\mu_1^2} + \frac{\rho}{1-\rho^2}\frac{\mu_2}{\mu_1}\right) \\ = & \frac{\rho^2}{(1-\rho^2)^2}\frac{x}{x-\mu_1^2} + \frac{\rho}{1-\rho^2}\frac{\mu_2}{\mu_1}x + \frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2}\frac{\rho^2}{(1-\rho^2)^2}\frac{1}{x-\mu_1^2} + \frac{\rho^2}{(1-\rho^2)^2} \\ = & \frac{\rho^2}{(1-\rho^2)^2}\left[2 + \frac{1-\rho^2}{\rho}\frac{\mu_2}{\mu_1}x + \left(\frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2} + \mu_1^2\right)\frac{1}{x-\mu_1^2}\right] \end{aligned}$$

When $x > \mu_1^2$, above functional $F(x)$ is first decreasing on (μ_1^2, x_0) and then increasing on (x_0, ∞), where $x_0 = \mu_1^2 + \frac{\mu_1}{\mu_2}\sqrt{\frac{\rho}{1-\rho^2}\mu_1\mu_2 + \frac{\rho^2}{(1-\rho^2)^2}}$.

And the constraint (3.37) pose a constraint on x's range:

$$\begin{aligned}
&\frac{\rho^2}{(1-\rho^2)^2}\frac{x}{x-\mu_1^2} - \mu_2^2 x \geq \frac{\rho^2}{(1-\rho^2)^2} \\
\Leftrightarrow & \mu_2^2 x^2 - \mu_2^2\mu_1^2 x - \frac{\rho^2}{(1-\rho^2)^2}\mu_1^2 \leq 0 \\
\Leftrightarrow & \frac{\mu_1^2\mu_2 - \mu_1\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}}}{2\mu_2} \leq x \leq \frac{\mu_1^2\mu_2 + \mu_1\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}}}{2\mu_2} \\
\Leftrightarrow & \frac{\mu_1^2}{2} - \frac{\mu_1}{2\mu_2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} \leq x \leq \frac{\mu_1^2}{2} + \frac{\mu_1}{2\mu_2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} \qquad (3.38)
\end{aligned}$$

One can check that the following holds:

$$\begin{aligned}
&\frac{\mu_1^2}{2} + \frac{\mu_1}{2\mu_2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} \leq x_0 \\
\Leftrightarrow & \frac{1}{2\mu_2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} \leq \frac{\mu_1}{2} + \frac{1}{\mu_2}\sqrt{\frac{\rho}{1-\rho^2}\mu_1\mu_2 + \frac{\rho^2}{(1-\rho^2)^2}} \\
\Leftrightarrow & \sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} - \mu_1\mu_2 \leq 2\sqrt{\frac{\rho}{1-\rho^2}\mu_1\mu_2 + \frac{\rho^2}{(1-\rho^2)^2}} \\
\Leftrightarrow & 2\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2} - 2\mu_1\mu_2\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} \leq \frac{4\rho}{1-\rho^2}\mu_1\mu_2 + \frac{4\rho^2}{(1-\rho^2)^2} \\
\Leftrightarrow & \mu_1\mu_2 - \frac{2\rho}{1-\rho^2} \leq \sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}}
\end{aligned}$$

Thus the minimizer of $F(x)$ subject to the constraint (3.38) happens at the

$$x_1 := \frac{\mu_1^2}{2} + \frac{\mu_1}{2\mu_2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}}.$$

which will lead to the corresponding $y_1 = \frac{\mu_2^2}{2} + \frac{\mu_2}{2\mu_1}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}}$.

At this choice, one could compute $F(x_1)$ as:

$$\begin{aligned}
&\left(\frac{\mu_1^2}{2} + \frac{\mu_1}{2\mu_2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} + \frac{\rho}{1-\rho^2}\frac{\mu_1}{\mu_2}\right)\left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2\mu_1}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}} + \frac{\rho}{1-\rho^2}\frac{\mu_2}{\mu_1}\right) \\
=&\left(\frac{\rho}{1-\rho^2} + \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}}\right)^2
\end{aligned}$$

If we assume the other constraint $xy - \mu_2^2 x \geq \frac{\rho^2}{(1-\rho^2)^2}$ in constraints (3.37) is tight, by a similar argument to above, one could get to the same minimizing value as $F(x_1)$.

So the original quantity (3.34) subject to constraints (3.35) is lower bounded by $\left(\frac{\rho}{1-\rho^2} + \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}\sqrt{\mu_1^2\mu_2^2 + \frac{4\rho^2}{(1-\rho^2)^2}}\right)^2$.

$\square$

Lemma 3.10. *For $0 < \rho < 1, \mu_1 > 0, \mu_2 > 0$. There are two roots $\theta_1 \leq 0 < \theta_2 \leq 1$ to following equation*

$$S_1(\theta) = S_2(\theta)$$

and $\theta_2 = \frac{\sqrt{4\rho^2+\mu_1^2\mu_2^2(1-\rho^2)^2}-\mu_1\mu_2(1-\rho^2)}{2\rho}$.

Proof. The equation $S_1(\theta) = S_2(\theta)$ can be simplified as

$$\frac{(1-\rho^2)\frac{1-\rho^2}{\rho}\mu_1\mu_2}{2(1+\theta)+\frac{1-\rho^2}{\rho}\mu_1\mu_2}\left(\frac{\rho}{1-\rho^2}+\frac{1}{2}\mu_1\mu_2+\frac{1}{2}\sqrt{\mu_1^2\mu_2^2+\frac{4\rho^2}{(1-\rho^2)^2}}\right)^2 = \frac{(1-\rho^2)\mu_1^2\mu_2^2}{1-\theta^2}$$

$$\Leftrightarrow \frac{1-\rho^2}{2\rho(1+\theta)+(1-\rho^2)\mu_1\mu_2}\left(\frac{\rho}{1-\rho^2}+\frac{1}{2}\mu_1\mu_2+\frac{1}{2}\sqrt{\mu_1^2\mu_2^2+\frac{4\rho^2}{(1-\rho^2)^2}}\right)^2 = \frac{\mu_1\mu_2}{1-\theta^2}$$

Denote $\alpha = \frac{2\rho}{1-\rho^2} > 0, \beta = \mu_1\mu_2 > 0$, then we could rewrite above equation as

$$\frac{1}{\alpha(1+\theta)+\beta}\left(\alpha+\beta+\sqrt{\beta^2+\alpha^2}\right)^2 = \frac{4\beta}{1-\theta^2}$$

$$\Leftrightarrow 4\alpha^2\beta^2(1-\theta^2) = 4\beta(\alpha(1+\theta)+\beta)\left(\alpha+\beta-\sqrt{\alpha^2+\beta^2}\right)^2$$

$$\Leftrightarrow \alpha^2\beta\theta^2 + \alpha\left(\alpha+\beta-\sqrt{\alpha^2+\beta^2}\right)^2\theta + (\alpha+\beta)\left(\alpha+\beta-\sqrt{\alpha^2+\beta^2}\right)^2 - \alpha^2\beta = 0$$

$$\Leftrightarrow (\alpha\theta+\beta-\sqrt{\alpha^2+\beta^2})\left(\alpha\beta\theta+\alpha^2+(\alpha+\beta)^2-(2\alpha+\beta)\sqrt{\alpha^2+\beta^2}\right) = 0$$

The last step follows from the following verification: For the coefficient of θ,

$$\begin{aligned}&\alpha\left(\alpha^2+(\alpha+\beta)^2-(2\alpha+\beta)\sqrt{\alpha^2+\beta^2}\right)+\alpha\beta\left(\beta-\sqrt{\alpha^2+\beta^2}\right)\\ =&\alpha\left(\alpha^2+\beta^2+(\alpha+\beta)^2-2(\alpha+\beta)\sqrt{\alpha^2+\beta^2}\right) = \alpha\left(\alpha+\beta-\sqrt{\alpha^2+\beta^2}\right)^2;\end{aligned}$$

For the constant part,

$$\begin{aligned}&(\beta-\sqrt{\alpha^2+\beta^2})\left(\alpha^2+(\alpha+\beta)^2-(2\alpha+\beta)\sqrt{\alpha^2+\beta^2}\right)\\ =&\beta\left(\alpha^2+(\alpha+\beta)^2\right)-\sqrt{\alpha^2+\beta^2}\left(\alpha^2+(\alpha+\beta)^2+2\alpha\beta+\beta^2\right)+(2\alpha+\beta)(\alpha^2+\beta^2)\\ =&\beta\alpha^2+\beta(\alpha+\beta)^2+(2\alpha+\beta)(\alpha^2+\beta^2)-2(\alpha+\beta)^2\sqrt{\alpha^2+\beta^2}\\ =&\beta\alpha^2+\alpha(\alpha^2+\beta^2)-\alpha(\alpha+\beta)^2+(\alpha+\beta)\left[\alpha^2+\beta^2+(\alpha+\beta)^2-2(\alpha+\beta)\sqrt{\alpha^2+\beta^2}\right]\\ =&(\alpha+\beta)\left(\alpha+\beta-\sqrt{\alpha^2+\beta^2}\right)-\alpha^2\beta.\end{aligned}$$

So there are two roots $\theta_1 = -\frac{\alpha^2+(\alpha+\beta)^2-(2\alpha+\beta)\sqrt{\alpha^2+\beta^2}}{\alpha\beta} \in [-1,1], \theta_2 = \frac{-\beta+\sqrt{\alpha^2+\beta^2}}{\alpha}$ to equation $S_1(\theta) = S_2(\theta)$.

And one can verify that $\theta_1 \leq 0 < \theta_2 \leq 1$ from $\alpha > 0, \beta > 0$

$$
\begin{aligned}
\theta_1 &= -\frac{\alpha^2 + (\alpha+\beta)^2 - (2\alpha+\beta)\sqrt{\alpha^2+\beta^2}}{\alpha\beta} \leq 0 \\
\Leftarrow & 2\alpha^2 + \beta(2\alpha+\beta) \geq (2\alpha+\beta)\sqrt{\alpha^2+\beta^2} \\
\Leftarrow & 2\alpha^2 \geq (2\alpha+\beta)\left(\sqrt{\alpha^2+\beta^2} - \beta\right) \\
\Leftarrow & 2(\sqrt{\alpha^2+\beta^2} + \beta) \geq 2\alpha + \beta \\
1 \geq \theta_2 &= \frac{-\beta + \sqrt{\alpha^2+\beta^2}}{\alpha} > 0 \\
\Leftarrow & \alpha + \beta \geq \sqrt{\alpha^2+\beta^2} > \beta
\end{aligned}
$$

When we put $\alpha = \frac{2\rho}{1-\rho^2} > 0, \beta = \mu_1\mu_2 > 0$ into θ_2, we will get back to $\theta_2 = \frac{\sqrt{4\rho^2+\mu_1^2\mu_2^2(1-\rho^2)^2}-\mu_1\mu_2(1-\rho^2)}{2\rho}$. This finishes the proof.

□

Chapter 4

Log-Convexity of Fisher Information

4.1 Introduction

The primary motivation for this chapter comes from one special case of non-convex problems (1.1), which occurs often times in evaluation of achievable rate regions or outer bounds to the capacity regions or optimal rate regions in network information theory settings. Let $W_{Y|X}$ denote a channel that maps input random variable X with distribution μ_X into output random variable Y with distribution μ_Y. If X and Y takes values in a finite alphabet space, then consider the problem of computing the maximum, over μ_X, of

$$F_\lambda(\mu_X) := \lambda H(X) - H(Y),$$

where $\lambda \geq 0$ is a fixed constant. When $\lambda \geq 1$, it is immediate from the data-processing inequality that the functional $F_\lambda(\mu_X)$ is concave in μ_X. However for $\lambda \in [0, 1)$, this is not necessarily true. In particular for $\lambda = 0$, $F_0(\mu_X)$ is convex in μ_X. Therefore, from a optimization perspective, computing the optimizers of $F_\lambda(\mu_X)$ becomes a non-convex optimization problem at least for some values of λ in the range $[0, 1)$.

For example, consider the lossless source coding with one helper 1.1.6 in Introduction chapter, recall that the weighted sum rate for the optimal rate region $\mathscr{A}_{p_{XY}}$ is given by

$$S_\lambda(p_{XY}) = H(Y) + \mathfrak{K}_{q_X}\left[H(Y) - \lambda H(X)\right](p_X)$$

for some $\lambda \geq 0$. Here the "channel law" $W_{Y|X}$ is fixed by $p_{Y|X}$. To explicitly evaluate $S_\lambda(p_{XY})$, one needs to determine all its dual representations, that is, for any real-valued vectors d_X:

$$\min_{q_X} H(Y) - \lambda H(X) - E_{q_X}[d_X] = -\max_{q_X} \{F_\lambda(q_X) + E_{q_X}[d_X]\} \tag{4.1}$$

When the channel $W_{Y|X}$ is the binary-symmetric-channel (BSC), say with crossover probability p, consider the following reparameterization of μ_X, defined by $\mu_X(u) = H_2^{-1}(u)$, where $H_2^{-1} : [0,1] \mapsto [0, \frac{1}{2}]$ denotes the inverse binary entropy function. Under this reparameterization, for BSC, observe that

$$F_\lambda(u) = \lambda u - H_2(p * H_2^{-1}(u)).$$

It was shown in [68] that $H_2(p * H_2^{-1}(u))$ is convex in u and hence $\lambda u - H_2(p * H_2^{-1}(u))$ is a concave function in u for any λ. Therefore this non-linear parameterization converted the non-convex optimization problem to a convex-optimization problem. It is also worth remarking that the convexity of $H_2(p * H_2^{-1}(u))$ was developed by Wyner and Ziv in the context of evaluating the superposition-coding region for a degraded binary symmetric broadcast channel.

Additive White Gaussian Noise channels are in many ways the continuous analogue of Binary Symmetric Channels. Therefore it is natural to see if there is an analogous result in the additive Gaussian noise setting, where under a suitable parameterization of μ_X, $h(\mu_X)$ - the differential entropy - becomes linear in the parameter and $h(T_G \mu_X)$ becomes convex in the parameter, where T_G refers to the channel with additive Gaussian noise W.

For distributions on binary alphabets, there is only one degree of freedom and hence the parameterization of $\mu_X(u) = H_2^{-1}(u)$ is forced on us, if we wish to make $H_2(\mu_X)$ linear. In the continuous world we assume that μ_X evolves along the heat flow, i.e. $X_t := X + \sqrt{t}Z, t > 0$, where Z is the standard Gaussian and independent of X. Therefore we seek a parameterization $t = \phi(u)$ such that $h(X + \sqrt{\phi(u)}Z)$ is linear in u and investigate whether, the output entropy, $h(\mu_Y) = h(X + \sqrt{\phi(u)}Z + W)$ is convex in u, where W is some Gaussian independent of X and Z.

Let μ_t^X denote the probability density function of $X_t = X + \sqrt{t}Z$. A bit of algebra immediately shows that this question is equivalent to asking whether the

Fisher information $I(\mu_t^X)$ is log-convex in t, for all random variables X (see the following remark 4.1).

Remark 4.1. Let $\phi(u) : [0,1] \to [0,1]$, with $\phi(0) = 0$ and $\phi(1) = 1$, be the uniquely defined increasing function of u such that $h(X + \sqrt{\phi(u)}Z)$ is linear in u. Then we have

$$0 = \frac{d^2}{du^2} h(X + \sqrt{\phi(u)}Z) \overset{(a)}{=} \frac{1}{2}\left(\frac{d^2\phi(u)}{du^2} I(\mu_{\phi(u)}^X) + \left(\frac{d\phi(u)}{du}\right)^2 \frac{d}{d\phi(u)} I(\mu_{\phi(u)}^X)\right).$$

Here $\mu_{\phi(u)}^X$ is the probability density function of the random variable $X + \sqrt{\phi(u)}Z$. Step (a) follow from Equation (4.3). Now, showing that $\frac{d^2}{du^2} h(X + \sqrt{\phi(u)}Z + W) \geq 0$, for $W \sim \mathcal{N}(0, \sigma^2)$ independent of (X, Z), is equivalent to showing that

$$0 \leq \frac{1}{2}\left(\frac{d^2\phi(u)}{du^2} I(\mu_{\phi(u)}^{X+W}) + \left(\frac{d\phi(u)}{du}\right)^2 \frac{d}{d\phi(u)} I(\mu_{\phi(u)}^{X+W})\right).$$

Here $\mu_{\phi(u)}^{X+W}$ is the probability density function of the random variable $X + \sqrt{\phi(u)}Z + W$. This can be rewritten using the equalities above as requiring

$$\frac{\frac{d}{d\phi(u)} I(\mu_{\phi(u)}^{X+W})}{I(\mu_{\phi(u)}^{X+W})} \geq \frac{\frac{d}{d\phi(u)} I(\mu_{\phi(u)}^{X})}{I(\mu_{\phi(u)}^{X})}.$$

Since $I(\mu_{\phi(u)}^{X+W}) = I(\mu_{\phi(u_1)}^X)$ for some $u_1 \geq u$, the above inequality is equivalent to showing that

$$\frac{\frac{d}{dt} I(\mu_t^X)}{I(\mu_t^X)}$$

is increasing in t or equivalently, that $\log I(f_t^X)$ is convex in t. Thus, the result we showed can be considered as a continuous analogue of the convexity result for BSC established by Wyner and Ziv.

4.1.1 An independent motivation

Let X be a random variable with a finite variance. Let $g_X^{(k)}(t) := \frac{\partial^k}{\partial t^k} h(\mu_t^X)$. Notice that Fisher information $I(\mu_t^X) = 2g_X^{(1)}(t)$, see Eq. (4.3) in the next section. Further, let us denote $g_X^{(0)}(t) = h(X)$. Let Z be a Gaussian random variable with the same variance as X. In Section 12 of [41], McKean observes that $g_Z^{(0)}(t) \geq g_X^{(0)}(t) \geq 0$, $g_Z^{(1)}(t) \leq g_X^{(1)}(t) \leq 0$, and $g_Z^{(2)}(t) \geq g_X^{(2)}(t) \geq 0$. Therefore he conjectures that

$$(-1)^k g_Z^{(k)}(t) \geq (-1)^k g_X^{(0)}(t) \geq 0$$

holds for every $k \geq 3$.

The above conjecture and similar ones on the alternative signs of derivatives (which characterize *completely monotone* functions) has attracted a fair amount of attention in mathematics. See [61], [63].

In [15], the authors study the signs of the higher order derivatives of $g_X(t) := h(\mu_t^X)$. They establish that $g_X^{(3)}(t) \geq 0$, and $g_X^{(4)}(t) \leq 0$. The techniques used follow the ideas in [62], which was in turn motivated by calculations of Bakry. The authors further conjectured that $g_X^{(k)}(t) \geq 0$, if k is odd and $g_X^{(k)}(t) \leq 0$ if k is even; or equivalently that $I(\mu_t^X) = 2g_X^{(1)}(t)$ is a completely monotone function of t, for all X. Note that this conjecture does not require Gaussian extremality and hence is a weaker conjecture to that of McKean.

The following theorem presents an alternate characterization of completely monotone function.

Theorem 4.1 (Bernstein's theorem)**.** *Let $g(t) : [0, \infty) \to [0, \infty)$ be a continuous and infinitely differentiable function. The following are equivalent:*

- *g is completely monotone: $\forall n \in \mathbb{N}, \forall t > 0, (-1)^n g^{(n)}(t) \geq 0$;*
- *g is the Laplace transform of a finite Borel measure μ in $\mathbb{R}_+$:*

$$\forall x \in \mathbb{R}_+, g(x) = \int_0^\infty e^{-xt} d\mu(t).$$

It can be shown that any completely monotone function $g(t)$ is log-convex with respect to t, see [22]. Thus, if $I(\mu_t^X)$ is a completely monotone function with respect to t, then $\ln I(\mu_t^X)$ is convex with respect to t, which is also stated as Conjecture 2 in [15].

The main result of this chapter is establishing that $I(\mu_t^X)$ is log-convex in t, thus resolving affirmatively Conjecture 2 in [15]. We do this by extending the ideas developed in [15] and [71].

This chapter will first introduce the ideas and tools developed in [15] and [71], then present the results on the log-convexity of Fisher Information, and in the end, reveal its connection with the non-convex functional $H(Y_t) - \lambda H(X_t)$, where $X_t := X + \sqrt{t}Z$ is in the set of distributions along the heat flow and Y_t is obtained by passing X_t through an additive Gaussian noise channel.

4.1.2 Notations and Previous results

Given a random variable X on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}$, let the cumulative distribution function of X be $\tilde{F}(x) := Pr(X \leq x)$, $x \in \mathbb{R}$. For Z some independent standard Gaussian random variable with mean zero and variance one, consider $X_t := X + \sqrt{t}Z$, $t > 0$, with probability density function $\mu_t^X(x)$ with respect to the Lebesgue measure on $\mathbb{R}$. The density $\mu_t^X(x)$, $x \in \mathbb{R}$, can be written as

$$\mu_t^X(x) = \int_{\mathbb{R}} \frac{-z}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} \tilde{F}(x-z) dz.$$

It is well-known in literature, e.g., [27], that the probability density function $\mu_t^X(x)$ of X_t is always upper bounded by $1+t$, strictly positive and infinitely differentiable with respect to $x \in (-\infty, \infty)$ and $t \in (0, \infty)$, and satisfy that

$$\lim_{|x| \to \infty} \frac{\partial^n \mu_t^X(x)}{\partial x^n} = 0, \forall n \in \mathbb{Z}_+.$$

Besides, $\mu_t^X(x)$ also satisfies the heat equation, see, e.g., [59].

$$\frac{\partial}{\partial t} \mu_t^X(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \mu_t^X(x). \tag{4.2}$$

The differential entropy of X_t, $h(X_t)$, $t > 0$, is defined as

$$h(X_t) = -\int_{\mathbb{R}} \mu_t^X(x) \ln \mu_t^X(x) dx.$$

When X has a finite variance P, $h(X_t)$ exists and is maximized by X following a Gaussian distribution with variance P.

The Fisher information of X_t is defined as

$$I(\mu_t^X) := \int_{\mathbb{R}} \left(\frac{\partial}{\partial x} \ln \mu_t^X(x) \right)^2 \mu_t^X(x) dx.$$

One can verify that the Fisher information $I(\mu_t^X), t > 0$, always exists and is infinitively differentiable with respect to $t \in (0, \infty)$, see, e.g., [15].

The Fisher information $I(\mu_t^X)$ is closely related to the differential entropy of X_t via the de Bruijin's identity when X has a finite variance, see, e.g., [19]

$$\frac{\partial}{\partial t} h(X_t) = \frac{1}{2} I(\mu_t^X). \tag{4.3}$$

Conjecture 2 in [15] postulates that $\ln I(\mu_t^X)$ is convex in $t > 0$. In this chapter, a proof to this conjecture is presented along the lines of the arguments in [15] and [71].

For convenience of writing, we will suppress the dependence on t and write $v(x) := \ln \mu_t^X(x), t > 0$, and $v_k(x) := \frac{\partial^k \ln \mu_t^X(x)}{\partial x^k}, k \in \mathbb{Z}_+$, i.e., $v_k(x)$ is the k-th derivative of v as a function of $x \in \mathbb{R}$. Well-definedness of $v_k(x)$ for any $k \in \mathbb{Z}_+$ follows from the known properties of $\mu_t^X(x)$.

Proposition 4.1 (Proposition 2 in [15]). *For any $r, m_i, k_i \in \mathbb{Z}_+$,*

$$\int_{\mathbb{R}} \left| \prod_{i=1}^{r} v_{k_i}^{m_i}(x) \right| \mu_t^X(x) dx < \infty,$$

and

$$\lim_{|x| \to \infty} \left| \prod_{i=1}^{r} v_{k_i}^{m_i}(x) \right| \mu_t^X(x) = 0.$$

We define $\langle \varphi \rangle := \int_{\mathbb{R}} \varphi \mu_t^X(x) dx$ to denote the integration with respect to the probability measure $\mu_t^X(x)$. Under this notation

$$I(\mu_t^X) = \langle v_1^2 \rangle. \tag{4.4}$$

The following lemma is needed in our proof.

Lemma 4.1 (Lemma 3 in [71]). *For $k \geq 2$, let $\varphi(x)$ be some function continuously differentiable with respect to x satisfying that $\lim_{|x| \to \infty} \varphi v_{k-1} \mu_t^X = 0$, then*

$$\langle \varphi v_k + \varphi v_1 v_{k-1} + \frac{\partial \varphi}{\partial x} v_{k-1} \rangle = 0.$$

One can see that this lemma follows from the basic integration by parts property. We present the short proof here for being self-contained.

Proof.

$$\begin{aligned}
\langle \varphi v_k + \varphi v_1 v_{k-1} + \frac{\partial \varphi}{\partial x} v_{k-1} \rangle &= \int_{\mathbb{R}} \left(\varphi v_k \mu_t^X + \varphi v_{k-1} \frac{\partial \mu_t^X}{\partial x} + \frac{\partial \varphi}{\partial x} v_{k-1} \mu_t^X \right) dx \\
&\overset{(a)}{=} \int_{\mathbb{R}} \left(\frac{\partial}{\partial x} \varphi v_{k-1} \mu_t^X \right) dx \\
&= \varphi v_{k-1} \mu_t^X |_{-\infty}^{\infty} \\
&\overset{(b)}{=} 0.
\end{aligned}$$

Equality (a) follows from the integration by parts property, and equality (b) follows from the condition that $\lim_{|x| \to \infty} \varphi v_{k-1} \mu_t^X = 0$. $\square$

Notice that by Proposition 4.1 we could choose φ in Lemma 4.1 to be in the form of $\prod_{i=1}^{r} v_{k_i}^{m_i}(x)$, where $r, m_i, k_i \in \mathbb{Z}_+$.

Lemma 4.2 ([15], [71]). *Let $\varphi(x)$ be some function continuously differentiable with respect to x satisfying that $\lim_{|x|\to\infty}\varphi v_1\mu_t^X = 0$. For $k \geq 0$, the following hold:*

$$\frac{\partial}{\partial t}v_k = \frac{1}{2}\left(v_{k+2} + \sum_{i=0}^{k}\binom{k}{i}v_{i+1}v_{k-i+1}\right),$$

$$\frac{\partial}{\partial t}\langle\varphi\rangle = \langle\frac{\partial}{\partial t}\varphi - \frac{1}{2}\frac{\partial\varphi}{\partial x}v_1\rangle.$$

Proof. The proof idea is to interchange integral and derivatives by Proposition 4.1 and the Dominated Convergence Theorem, and the calculations follow from the following observations (for details, see Appendix A in [71]). We present the outline here for being rather self-contained.

$$\begin{aligned}
2\frac{\partial}{\partial t}v_k &= 2\frac{\partial}{\partial t}\left(\frac{\partial^k}{\partial x^k}\ln\mu_t^X(x)\right)\\
&= 2\frac{\partial^k}{\partial x^k}\left(\frac{\partial}{\partial t}\ln\mu_t^X(x)\right)\\
&\overset{(a)}{=} \frac{\partial^k}{\partial x^k}\left(\frac{\frac{\partial^2}{\partial x^2}\mu_t^X(x)}{\mu_t^X(x)}\right)\\
&= \frac{\partial^k}{\partial x^k}\left(v_2 + v_1^2\right)\\
&\overset{(b)}{=} v_{k+2} + \sum_{i=0}^{k}\binom{k}{i}v_{i+1}v_{k-i+1}.
\end{aligned}$$

Equality (a) is due to the heat equation (4.2) and (b) can be established by mathematical induction.

For the second part, observe that

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\varphi\rangle &= \langle\frac{\partial}{\partial t}\varphi\rangle + \int_{\mathbb{R}}\varphi\frac{\partial\mu_t^X}{\partial t}dx\\
&\overset{(a)}{=} \langle\frac{\partial}{\partial t}\varphi\rangle + \frac{1}{2}\int_{\mathbb{R}}\varphi\frac{\partial^2\mu_t^X}{\partial x^2}dx\\
&\overset{(b)}{=} \langle\frac{\partial}{\partial t}\varphi\rangle - \frac{1}{2}\int_{\mathbb{R}}\frac{\partial\varphi}{\partial x}\frac{\partial\mu_t^X}{\partial x}dx\\
&= \langle\frac{\partial}{\partial t}\varphi\rangle - \frac{1}{2}\langle\frac{\partial\varphi}{\partial x}v_1\rangle.
\end{aligned}$$

Equality (a) is again due to the heat equation (4.2) and (b) follows from integration by parts and the assumption that $\lim_{|x|\to\infty}\varphi v_1\mu_t^X = 0$. □

One can compute the derivatives of the Fisher information $I(\mu_t^X)$ with respect to t as following, see [38] and [71].

Lemma 4.3 ([15], [71])**.** *For $t > 0$, Fisher information $I(\mu_t^X)$ and its derivatives up to second order can be expressed as:*

$$\frac{d}{dt}I(\mu_t^X) = -\langle v_2^2\rangle,$$
$$\frac{d^2}{dt^2}I(\mu_t^X) = \langle v_3^2 + 2v_1^2v_2^2 + 4v_1v_2v_3\rangle.$$

Proof. In the interest of being self-contained, we outline the proof via applications of Lemmas 4.2 and 4.1. Observe that

$$\begin{aligned}\frac{d}{dt}I(\mu_t^X) &= \frac{d}{dt}\langle v_1^2\rangle \\ &\overset{(a)}{=} \langle 2v_1\frac{\partial v_1}{\partial t} - v_2v_1^2\rangle \\ &\overset{(b)}{=} \langle v_1(v_3 + 2v_1v_2) - v_2v_1^2\rangle \\ &\overset{(c)}{=} -\langle v_2^2\rangle.\end{aligned}$$

Here $(a), (b)$ follow from Lemma 4.2, and (c) follows from Lemma 4.1 by setting $\varphi = v_1$ and $k = 3$. Similarly, note that

$$\begin{aligned}\frac{d^2}{dt^2}I(\mu_t^X) &= -\frac{d}{dt}\langle v_2^2\rangle \\ &\overset{(a)}{=} \langle -2v_2\frac{\partial v_2}{\partial t} + v_2v_3v_1\rangle \\ &\overset{(b)}{=} \langle -v_2(v_4 + 2v_1v_3 + 2v_2^2) + v_2v_3v_1\rangle \\ &\overset{(c)}{=} \langle v_3^2 - 2v_2^3\rangle \\ &\overset{(d)}{=} \langle v_3^2 + 2v_1^2v_2^2 + 4v_1v_2v_3\rangle.\end{aligned}$$

Here $(a), (b)$ follow from Lemma 4.2, (c) follows from Lemma 4.1 by setting $\varphi = v_2$ and $k = 4$, and (d) follows from Lemma 4.1 by setting $\varphi = v_2^2$ and $k = 2$. $\square$

Remark 4.2. There are several equivalent ways of expressing $\frac{d^2}{dt^2}I(\mu_t^X)$ using Lemma 4.2. For instance, [71] expressed it as $\langle v_3^2 - 2v_2^3\rangle$. We choose this particular representation, $\langle v_3^2 + 2v_1^2v_2^2 + 4v_1v_2v_3\rangle$, as it turns out to be useful to prove the log-convexity of Fisher information.

Above set of tools and notations could make a short proof to Costa's Entropy Power Inequality (EPI) [16] in single dimension case. This proof comes from [62], which is in turn motivated by calculations of Bakry and Emery.

Lemma 4.4 (Costa's EPI, [16])*. Let X be a random variable on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}$, and Z some independent standard Gaussian random variable. $e^{2h(X+\sqrt{t}Z)}$ is concave in $t \geq 0$.*

Proof. By computing the second derivative of $e^{2h(X+\sqrt{t}Z)}$ with respect to t, we need to show :

$$2e^{2h(X+\sqrt{t}Z)}\frac{\partial^2 h(X+\sqrt{t}Z)}{\partial^2 t} + 4e^{2h(X+\sqrt{t}Z)}\left(\frac{\partial h(X+\sqrt{t}Z)}{\partial t}\right)^2 \leq 0$$

which can be rewritten in terms of Fisher information $I(\mu_t^X)$ and its derivatives:

$$e^{2h(X+\sqrt{t}Z)}\left[-\langle v_2^2\rangle + \langle v_1^2\rangle^2\right] \leq 0$$

In Lemma 4.1, choose $\varphi = 1$ and $k = 2$, we have $\langle v_2 + v_1^2\rangle = 0$, so above is equivalent to

$$-\langle v_2^2\rangle + \langle v_2\rangle^2 \leq 0,$$

which holds trivially by convexity of x^2 with respect to x. $\square$

The results of this chapter are new in this thesis. This is a joint work with Prof. Chandra Nair and Prof. Michel Ledoux from University of Toulouse – Paul-Sabatier.

4.2 Main Result

Theorem 4.2. *Let X be a random variable on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}$, and Z some independent standard Gaussian random variable. Consider $X_t := X + \sqrt{t}Z, t > 0$, with probability density function $\mu_t^X(x)$ with respect to the Lebesgue measure on $\mathbb{R}$.*

The Fisher information of X_t is log-convex in t, i.e.

$$\ln I(\mu_t^X) = \ln \int_{\mathbb{R}} \left(\frac{\partial}{\partial t}\ln \mu_t^X(x)\right)^2 \mu_t^X(x)dx$$

is convex in t.

Proof. Log-convexity of Fisher information is equivalent to showing

$$\left(\frac{d}{dt}I(\mu_t^X)\right)^2 \leq I(\mu_t^X)\frac{d^2}{dt^2}I(\mu_t^X).$$

Using Lemma 4.3, this is equivalent to showing

$$\langle v_2^2\rangle^2 \leq \langle v_1^2\rangle\langle v_3^2 + 2v_1^2 v_2^2 + 4v_1 v_2 v_3\rangle. \tag{4.5}$$

In Lemma 4.1, the choices that $k = 2, \varphi = v_2$ and that $k = 2, \varphi = v_1^2$ will lead to the following two equalities respectively

$$\langle v_2^2 + v_1^2 v_2 + v_1 v_3 \rangle = 0 \tag{4.6}$$

$$\langle v_1^4 + 3v_1^2 v_2 \rangle = 0. \tag{4.7}$$

Consequently, for any $\alpha \in \mathbb{R}$ we have

$$\langle v_2^2 \rangle = -\langle v_1 (v_3 + \alpha v_1 v_2 - \frac{1-\alpha}{3} v_1^3)\rangle.$$

The Cauchy-Schwarz inequality yields,

$$\langle v_2^2 \rangle^2 \leq \langle v_1^2 \rangle \langle (v_3 + \alpha v_1 v_2 - \frac{1-\alpha}{3} v_1^3)^2 \rangle.$$

Thus to show inequality (4.5), it suffices to show that

$$\langle (v_3 + \alpha v_1 v_2 - \frac{1-\alpha}{3} v_1^3)^2 \rangle \leq \langle v_3^2 + 2v_1^2 v_2^2 + 4 v_1 v_2 v_3 \rangle \tag{4.8}$$

holds for some $\alpha \in \mathbb{R}$. Expanding, (4.8) is equivalent to

$$\langle (2-\alpha^2) v_1^2 v_2^2 + (4 - 2\alpha) v_1 v_2 v_3 - \frac{1}{9}(1-\alpha)^2 v_1^6 + \frac{2}{3}(1-\alpha) v_1^3 v_3 + \frac{2}{3}\alpha(1-\alpha) v_1^4 v_2 \rangle \geq 0.$$

In Lemma 4.1, the choices that $k = 3, \varphi = v_1^3$ and that $k = 2, \varphi = v_1^4$ will lead to the following two equalities respectively.

$$\langle v_1^3 v_3 + v_2 v_1^4 + 3 v_1^2 v_2^2 \rangle = 0$$

$$\langle v_1^6 + 5 v_1^4 v_2 \rangle = 0.$$

Thus proving inequality (4.8) for some $\alpha \in \mathbb{R}$ is equivalent to proving the following inequality

$$\begin{aligned} \langle (2-\alpha^2) v_1^2 v_2^2 + (4 - 2\alpha) v_1 v_2 v_3 - \frac{1}{9}(1-\alpha)^2 v_1^6 + \frac{2}{3}(1-\alpha) v_1^3 v_3 + \frac{2}{3}\alpha(1-\alpha) v_1^4 v_2 \rangle \\ + \beta \langle v_1^3 v_3 + v_2 v_1^4 + 3 v_1^2 v_2^2 \rangle + \gamma \langle v_1^6 + 5 v_1^4 v_2 \rangle \geq 0 \end{aligned} \tag{4.9}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$.

We successively choose the values α, β, γ to eliminate the terms whose signs are not clear: first set $\alpha = 2$ to get rid of $\langle v_1 v_2 v_3 \rangle$, then $\beta = \frac{2}{3}$ to eliminate $\langle v_1^3 v_3 \rangle$, and finally $\gamma = \frac{2}{15}$ to handle $\langle v_1^4 v_2 \rangle$. With these choices, the above inequality (4.9) reduces to $\frac{1}{45}\langle v_1^6 \rangle \geq 0$, which holds trivially. $\square$

4.3 Discussion

4.3.1 Generalization of log-convexity to higher dimensions

One clear question that is definitely worth addressing is to determine whether the log-convexity of Fisher information along the heat flow also holds for random vectors. In particular we ask, whether

$$\left(\frac{d^3 h(\mathbf{X}+\sqrt{t}\mathbf{Z})}{dt^3}\right)\left(\frac{dh(\mathbf{X}+\sqrt{t}\mathbf{Z})}{dt}\right) \geq \left(\frac{d^2 h(\mathbf{X}+\sqrt{t}\mathbf{Z})}{dt^2}\right)^2$$

where $\mathbf{X}$ and $\mathbf{Z}(\sim \mathcal{N}(0, I_d))$ are independent random vectors taking values in $\mathbb{R}^d$. If $\mathbf{X}$ has independent components, then an application of the Cauchy-Schwartz inequality immediately implies affirmatively the inequality above.

While the techniques applied in the scalar case do have natural extensions to the vector case, preliminary investigations by the authors indicate that these extensions seem insufficient to establish the log-convexity for vector valued random variables.

4.3.2 Generalization of convexity of the output entropy

Let us consider a channel given by

$$\mathbf{Y} = A\mathbf{X} + \mathbf{Z}$$

where A is an $l \times d$ (channel-gain) matrix, $\mathbf{X}$ is the input, and $\mathbf{Z}(\sim \mathcal{N}(0, I_l))$ is the additive Gaussian noise. Then one can ask for flows in the space of input distributions, say characterized by X_t, where $h(X_t)$ is linear in t and $h(Y_t)$ is convex in t.

An interesting such flow exists in the space of Gaussian vectors. Let $\mathbf{X}_0 \sim \mathcal{N}(0, K_0)$ and $\mathbf{X}_1 \sim \mathcal{N}(0, K_1)$ be two Gaussian random vectors with $K_0, K_1 \succ 0$. Define

$$K_t = K_0^{\frac{1}{2}}\left(K_0^{-\frac{1}{2}} K_1 K_0^{-\frac{1}{2}}\right)^t K_0^{\frac{1}{2}},$$

and $X_t \sim \mathcal{N}(0, K_t)$. Note that this is a continuous path that connects the distribution of X_0 to that of X_1. Further, observe that $h(X_t)$ is linear in t. It follows from the seminal work in [37], and is well-known, that

$$h(Y_t) = \log|AK_tA^T + I|$$

is convex in t.

From the perspective of non-convex optimization problems that arise in the computation of achievable regions or outer bound in network information theory, it will be very helpful to find similar flows in a more general setting, i.e. outside the space of Gaussian vectors and more generally for larger class of channels. Such results may also be useful in showing the uniqueness of local maximizers in such settings as is observed in settings such as the MIMO Gaussian broadcast channels.

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