# On Tightness of Several Achievable Rate Regions in Network Information Theory

XIA, Lingxiao

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Doctor of Philosophy
in
Information Engineering

The Chinese University of Hong Kong
September 2016

This thesis is dedicated to my parents.

# Acknowledgements

I would like to thank my advisor Prof. Chandra Nair for guiding me through my PhD research. I would also like to take this chance to thank my colleague Mehdi Yazdanpanah, or Babak, for being a great friend. I will for sure miss the heated debates we have over lunch and I wish you all the best.

**Abstract of thesis entitled: On Tightness of Several Achievable Rate Regions in Network Information Theory**
**Submitted by XIA, Lingxiao**
**for the degree of Doctor of Philosophy**
**at The Chinese University of Hong Kong in July 2016**

Several popular achievable regions of two classic communication scenarios, discrete memoryless broadcast channel (DMBC) and discrete memoryless interference channel (DMIC), in network information theory are discussed in this thesis.

We show that superposition-coding achievable region is sub-optimal for general three-receiver more capable DMBCs whilst remains optimal for certain subclasses.

We define very weak interference for DMIC and we show that Han–Kobayashi (HK) achievable region is sum-rate optimal for some DMICs with very weak interference. The sum-capacity, for some channels with very weak interference, is established by developing a genie-based outer bound which turns out to be tight in some parameter instances. Of independent interest is the analysis of the genie based outer-bound where: ($i$) we develop novel techniques for establishing cardinality bounds on the genie variables; ($ii$) we show that there exist no genies for certain parameters that would reduce the outer bound to treating interference as noise.

We also show with a discrete memoryless Z-interference channel counter-example that HK achievable region is sub-optimal for general DMIC, solving a long standing open problem in network information theory.

**中文摘要**

本論文主要討論網絡信息論中數字廣播系統(DMBC) 和數字幹擾系統(DMIC) 的可傳輸速率範圍。數字廣播系統和數字幹擾系統是網絡信息論中最重要且基本的信息傳輸模型，而且兩個系統的最優化可傳輸速率範圍都是未知的。

作者證明對於一個有三個接收器的可排序廣播系統，疊加編碼可傳輸速率範圍只是特定信道的最優化可傳輸速率範圍。

另外，通過定義微弱信道幹擾的概念和提出速率外界精靈計算法，作者證明了漢-小林(Han–Kobayashi)編碼在某些有微弱信道幹擾的數字幹擾系統中的整體可傳輸速率最優性。在速率外界精靈計算法的推導過程中，作者還提出了精靈基數計算法以及證明了通過漢-小林編碼取得的內界和同於通過精靈計算法得到的外界在特定數字幹擾系統中的不等性。

最後，作者通過對Z型數字幹擾系統的研究證明了漢-小林編碼可傳輸速率範圍和數字幹擾系統最優化可傳輸速率範圍的不等性，解決了網絡信息論中的一個長久公開問題。

# Contents

# List of Figures

# Notations

We list the notation and terminology used throughout the thesis. It is only a general guideline. Cases where this guideline is not followed are pointed out on an *ad hoc* basis inside the thesis.

We use lowercase letters $x$, $y$, $\ldots$ to denote constants and values of random variables. We use $x_i^j = (x_i, x_{i+1}, \ldots, x_j)$ to denote an $(j-i+1)$-sequence/column vector for $1 \le i \le j$. When $i = 1$, we always drop the subscript, i.e., $x^j = (x_1, x_2, \ldots, x_j)$, unless there are multiple subscripts. Sometimes we write $\mathbf{x}$, $\mathbf{y}$, $\ldots$ for vectors with specified dimensions and $x_j$ for the $j$-th component of $\mathbf{x}$.

Let $\alpha, \beta \in [0,1]$. Then $\bar{\alpha} = 1 - \alpha$ and $\alpha * \beta = \alpha\bar{\beta} + \beta\bar{\alpha}$.

Let $x^n, y^n \in \{0,1\}^n$ be binary $n$-vectors. Then $x^n \oplus y^n$ is the componentwise modulo-2 sum of the two vectors.

$\mathbb{R}$ is the real line and $\mathbb{R}^d$ is the $d$-dimensional real Euclidean space.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of non-negative integers.

Calligraphic letters $\mathcal{X}$, $\mathcal{Y}$, $\ldots$ are used exclusively for finite sets and $|\mathcal{X}|$ denotes the cardinality of the set $\mathcal{X}$.

Script letters $\mathscr{C}$, $\mathscr{R}$, $\mathscr{H}$, $\ldots$ are used for subsets of $\mathbb{R}^d$.

For a pair of integers $i \le j$, we define the discrete interval $[i:j] = \{i, i+1, \ldots, j\}$. More generally, for $a \ge 0$ and integer $i \le 2^a$, we define

- $[i:2^a) = \{i, i+1, \ldots, 2^{\lfloor a \rfloor}\}$, where $\lfloor a \rfloor$ is the integer part of $a$, and
- $[i:2^a] = \{i, i+1, \ldots, 2^{\lceil a \rceil}\}$, where $\lceil a \rceil$ is the smallest integer larger or equal to $a$.

The probability of an event $\mathcal{A}$ is denoted by $P(\mathcal{A})$ and the conditional probability of $\mathcal{A}$ given $\mathcal{B}$ is denoted by $P(\mathcal{A}|\mathcal{B})$. We use uppercase letters $X$, $Y$, $\ldots$ to denote random variables. The random variables may take values from finite sets $\mathcal{X}$, $\mathcal{Y}$, $\ldots$ or from the real line $\mathbb{R}$. By convention, $X = \phi$ means that $X$ is a degenerate random variable (unspecified constant) regardless of its support. The probability of the event $\{X \in \mathcal{A}\}$ is denoted by $P\{X \in \mathcal{A}\}$.

In accordance with the notation for constant vectors, we use $X_i^j = (X_i, \ldots, X_j)$ to denote a $(j-i+1)$-sequence of random variables for $1 \le i \le j$. When

$i = 1$, we always drop the subscript and use $X^j = (X_1, \ldots, X_j)$ unless there are multiple subscripts. For example, we do not drop the subscripts of $X_{2,1}^j = (X_{2,1}, X_{2,2}, \ldots, X_{2,j})$ because $X_2^j = (X_2, X_3, \ldots, X_j)$ means something different.

The following notations are used to specify random variables and random vectors.

$X^n \sim p(x^n)$ means that $p(x^n)$ is the probability mass function (pmf) of the discrete random vector $X^n$. The function $p_{X^n}(\tilde{x}^n)$ denotes the probability mass of argument $\tilde{x}^n$, i.e., $p_{X^n}(\tilde{x}^n) = P\{X^n = \tilde{x}^n\}$ for all $\tilde{x}^n \in \mathcal{X}^n$. The functions $p(x^n)$ without subscript is understood to be the pmf of the random vector $X^n$ over $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$.

$(X^n, Y^n) \sim p(x^n, y^n)$ means that $p(x^n, y^n)$ is the joint pmf of $X^n$ and $Y^n$.

$p(y^n|x^n)$ is a collection of conditional pmfs on $\mathcal{Y}^n$, one for every $x^n \in \mathcal{X}^n$.

Given a random variable $X$, the expected value of its functions $g(X)$ is denoted by $\mathrm{E}_X(g(X))$, or $\mathrm{E}(g(X))$ in short. The conditional expectation of $X$ given $Y$ is denoted by $\mathrm{E}(X|Y)$. We use $\mathrm{Var}(X) = \mathrm{E}[(X - \mathrm{E}(X))^2]$ to denote the variance of $X$ and $\mathrm{Var}(X|Y) = \mathrm{E}[(X - \mathrm{E}(X))^2|Y]$ to denote the conditional variance of $X$ given $Y$.

We use the following notations for standard random variables and random vectors.

$X \sim \mathbf{Unif}(\mathcal{A})$: $X$ is a discrete uniform random variable over a finite set $\mathcal{A}$.

$X \sim \mathbf{Unif}[i : j]$ for integers $j > i$: $X$ is a discrete uniform random variable over $[i : j]$.

$X \sim \mathbf{Unif}[1 : 2^{nR}]$ for $n \in \mathbb{Z}$ and $R \in \mathbb{R}^+$ is a discrete uniform random variable over $[1 : 2^{nR}]$.

We say that $X \to Y \to Z$ forms a Markov chain if $p(x, y, z) = p(x)p(y|x)p(z|y)$. More generally, we say that $X_1 \to X_2 \to X_3 \to \ldots$ forms a Markov chain if $p(x_i|x^{i-1}) = p(x_i|x_{i-1})$ for $i \geq 2$.

The logarithm function log is assumed to be base 2 unless specified otherwise.

Binary entropy function: $H_b(p) = -p \log p - \bar{p} \log \bar{p}$ for $p \in [0, 1]$.

We use uppercase $A, B, \ldots$ to denote matrices. The entry in the $i$-th row and the $j$-th column of a matrix $A$ is denoted by $A(i, j)$ or $A_{ij}$. A transpose of a matrix $A$ is denoted by $A^T$. For a square matrix $A$, $|A| = \det(A)$ denotes the determinant of $A$ and $tr(A)$ denotes its trace.

The upper concave envelope of a function $f(x)$ over domain $\mathcal{D}$ is defined as

$$\mathfrak{C}[f(x)](x_0) := \inf\{g(x_0) : g(x) \text{ is concave in } x \in \mathcal{D},\ g(x) \geq f(x)\ \forall x \in \mathcal{D}\}.$$

Please refer to Appendix A for preliminary definitions and properties used throughout the thesis.

# Chapter 1

# Introduction

The field of network information theory as taught in [7] extends Shannon's information theory on point-to-point communication systems to multi-user settings with shared resources. It aims to solve for fundamental limits on information flow similar to those studied in point-to-point communication systems.

The purpose of this chapter is to review the communication models used in information theory and in particular, network information theory to establish the language used throughout this thesis.

The communication model which inspired this thesis makes several assumptions about the scenario.

It is assumed that the communication goes only one-way with no feedback. We assume that the channel takes in symbols from a finite alphabet $\mathcal{X}$ and each input symbol is transformed by the channel to an output symbol from another finite alphabet $\mathcal{Y}$ at the receiving end. For each input $x \in \mathcal{X}$, the output $Y$ follows a fixed (conditional) distribution denoted by $\mathfrak{q}(y|x)$. The channel is also assumed to be memoryless, *i.e.*, $p(y_i|x^i, y^{i-1}) = \mathfrak{q}(y_i|x_i)\ \forall i$. The resulting channel, $\mathfrak{q}_*(y^n|x^n)$, is called an $n$-product channel and in the absence of feedback implies that $\mathfrak{q}_*(y^n|x^n) = \prod_{i=1}^n \mathfrak{q}(y_i|x_i) := \mathfrak{q}^n(y^n|x^n)$. We call a point-to-point channel modeled thus as a $(\mathcal{X}, \mathfrak{q}(y|x), \mathcal{Y})$ channel, or sometimes $\mathfrak{q}(y|x)$ in short.

In the general point-to-point communication scenario depicted in Figure 1, the sender wishes to maximize the rate at which messages can be transmitted to the receiver, so that they can be decoded with a small probability of error.

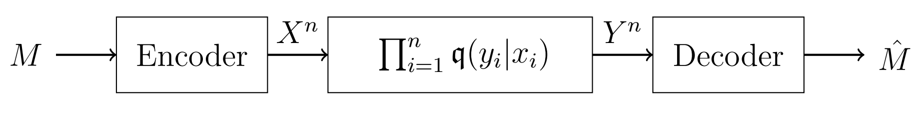


Figure 1.1: Discrete memoryless point-to-point channel

A $(2^{nR}, n)$ code (or a coding strategy) of size $2^{nR}$ that uses the channel $n$ times consists of:

(*i*) an encoder that maps a message, $M$, from the set $[1 : 2^{nR}]$ to $\mathcal{X}^n$,

(*ii*) a decoder that maps the received sequence in $\mathcal{Y}^n$ to an estimated message, $\hat{M}$, in $[1 : 2^{nR}]$.

The performance metric for a codebook is the average probability of error, *i.e.*, $P_e^{(n)} = P(M \neq \hat{M})$ when $M$ is uniformly distributed in the set $[1 : 2^{nR}]$.

A rate $R$ bits per channel use is said to be *achievable* if there is a sequence of $(2^{nR}, n)$ codes such that $P_e^{(n)} \to 0$ as $n$ tends to infinity. The supremum of all achievable rates for a given channel is called *channel capacity*, denoted by $\mathscr{C}$. The celebrated result of Shannon [23] established that $\mathscr{C}$ is given by $\max_{p(x)} I(X;Y)$ for a DMC.

The model extends easily to multi-user communication systems with similar discrete and memoryless assumptions. The study of such models constitutes the field of *network information theory* and the basic models and results can be found in [7]. However the capacity region, the maximal set of achievable rate tuples, is established only for very few settings. In this thesis we look at two of the most basic settings where capacity region is open, the broadcast channel and the interference channel; and we establish new results concerning the capacity region in various special settings. The results in this thesis completely resolves open Question 6.4 (a very well-known one) and partially resolves open Question 5.2 in [7].

## 1.1 Broadcast channel

Broadcast channel model extends the point-to-point channel model by adding more receivers. It models the communication of one sender and multiple receivers with a shared medium for transmission [4]. This models, for instance, the communication system from a base station to the receivers within its cellular range, commonly referred to as the downlink transmission. The capacity region for such a channel is largely unknown and the characterization of the capacity region is a classical and fundamental open problem in network information theory.

A general $k$-receiver discrete memoryless broadcast channel (DMBC) model $(\mathcal{X},\ \mathfrak{q}(y_1, y_2, \ldots, y_k|x),\ \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_k)$ consists of a finite alphabet $\mathcal{X}$ for input/transmitter, finite alphabets $\mathcal{Y}_1$, $\mathcal{Y}_1$, ..., $\mathcal{Y}_k$, one for each output/receiver and a collection of channel transition pmfs $\mathfrak{q}(y_1, y_2, \ldots, y_k|x)$.

This thesis focuses on broadcast channel with private message sets. A general $k$-receiver DMBC with private message sets is depicted in Figure 1.2, the sender wishes to communicate an independent, private message $M_r$ to each receiver $r$, $r = 1, 2, \ldots, k$. The sender wishes to maximize the rate at which messages can be transmitted to the receivers, so that they can be decoded with a small probability of error.

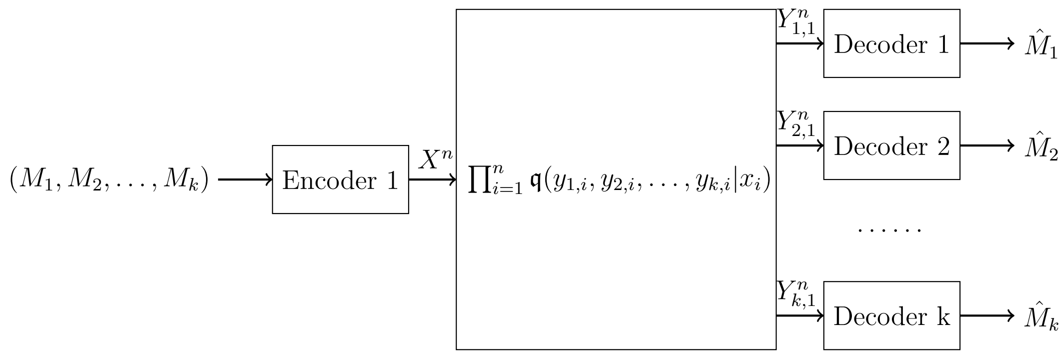


Figure 1.2: Discrete memoryless broadcast channel with private message sets

A $(2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_k}, n)$ code of size $(2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_k})$ that uses the channel $n$ times consists of:

($i$) an encoder that maps the messages $M_1$, $M_2$,..., $M_k$ from the set $[1, 2^{nR_1}] \times [1, 2^{nR_2}] \times \cdots \times [1, 2^{nR_k}]$ to $\mathcal{X}^n$,

($ii$) $k$ decoders, one for each receiver, that map the received sequence in $\mathcal{Y}^n$ to an estimated message $\hat{M}_r$ in $[1 : 2_r^{nR}]$, $r = 1, 2, \ldots, k$.

The performance metric for a codebook is the average probability of error, *i.e.*, $P_e^{(n)} = P((M_1, M_2, \ldots, M_k) \neq (\hat{M}_1, \hat{M}_2, \ldots, \hat{M}_k))$ when $(M_1, M_2, \ldots, M_k) \sim$ **Unif**$([1, 2^{nR_1}] \times [1, 2^{nR_2}] \times \cdots \times [1, 2^{nR_k}])$.

A rate tuple $(R_1, R_2, \ldots, R_k)$ is said to be achievable if there is a sequence of $(2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_k}, n)$ codes such that $P_e^{(n)} \to 0$ as $n$ tends to infinity. The closure of all achievable rate tuples for a given DMBC constitutes its *capacity region*, denoted by $\mathscr{C}$, which corresponds to the channel capacity concept of the DMC counter part. Although, unlike its counter part in DMC, the capacity region $\mathscr{C}$ of a DMBC is not known in general.

*Remark* 1. The marginal transition probabilities $\mathfrak{q}_r(y_r|x)$, $r = 1, 2, \ldots, k$ uniquely determines the channel capacity region of the DMBC and are referred to as the (marginal) channel from $X$ to $Y_r$.

Since the capacity region depends only on the marginal channels, it is natural to consider a partial order among the channels that capture the noise induced by a channel. Three prominent such partial orders between any two receivers are degraded [4], less-noisy and more-capable [13].

**Definition 1** (Degraded)**.** For a DMBC with one sender $X$ and two receivers $Y_1$, $Y_2$ with corresponding channels $\mathfrak{q}_1$ and $\mathfrak{q}_2$, we say that the channel to $Y_2$ is degraded with respect to the channel to $Y_1$, denoted as $\mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_2$, if there is $\mathfrak{q}'$ such that $\mathfrak{q}_2(y_2|x) = \mathfrak{q}_1(y_1|x)\mathfrak{q}'(y_2|y_1)$.

**Definition 2** (Less-noisy)**.** For a DMBC with one sender $X$ and two receivers $Y_1$, $Y_2$ with corresponding channels $\mathfrak{q}_1$ and $\mathfrak{q}_2$, we say that $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_2$ if $I(U;Y_1) \geq I(U;Y_2)$ for all distributions of $(U, X)$ where $U$ is an auxiliary random variable and $U \to X \to (Y_1, Y_2)$ forms a Markov chain.

**Definition 3** (More-capable)**.** For a DMBC with one sender $X$ and two receivers $Y_1$, $Y_2$ with corresponding channels $\mathfrak{q}_1$ and $\mathfrak{q}_2$, $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2$ if and only if $I(X;Y_1) \geq I(X;Y_2)$ for every distribution $p(x)$ of $X$.

It is a simple exercise to see that $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2$ implies that $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_2$, and $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_2$ implies that $\mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_2$. We say that a 2-receiver broadcast channel is degraded, less-noisy, more-capable if the corresponding marginal channels satisfy the degraded, less-noisy, more-capable partial orders respectively.

Consider a two-receiver DMBC model with private message sets depicted in Figure 1.3.

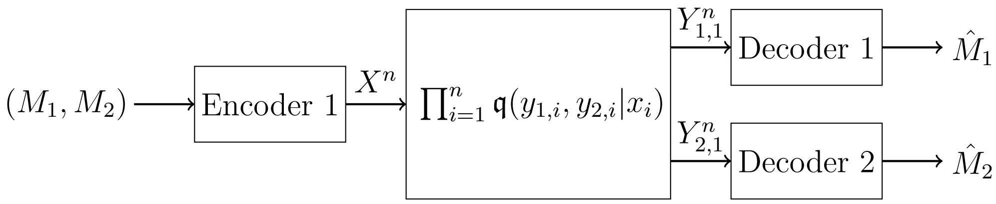


Figure 1.3: Two-receiver DMBC with private message sets

A simple achievable strategy for a broadcast channel, and one that is widely used in practice, is to employ time division; communicate to one receiver for a fraction of the slots and to the other receiver for the remaining fraction. A better strategy, in many cases, is the superposition-coding strategy introduced by Cover [4] to communicate over a degraded broadcast channel. This strategy was proven to be optimal for 2-receiver degraded broadcast channels (in the Gaussian case by Bergmans [2] and in the discrete memoryless setting by Gallager [8]), for 2-receiver less-noisy broadcast channels [13], and for the two-receiver more-capable broadcast channel [6].

**Theorem 1** (Superposition-coding inner bound)**.** *A rate pair* $(R_1, R_2)$ *is achievable for the DMBC depicted in Figure 1.3 if*

$$R_1 \leq I(X;Y_1|U)$$
$$R_2 \leq I(U;Y_2)$$

$$R_1 + R_2 \le I(X; Y_1)$$

*for some pmf $p(u, x)$ where $U \in \mathcal{U}$ is an auxiliary random variable and $|\mathcal{U}| \le |\mathcal{X}| + 1$.*

In the above region we assume that $Y_1$ is the "stronger" receiver.

While Gallager's proof of the optimality of superposition-coding region extends to a broadcast channel where the marginal channels follow the degraded partial order; the proofs for the less-noisy and more-capable do not extend. The optimality of superposition-coding region for a 3-receiver broadcast channel where the marginal channels follow the less-noisy partial order was recently established in [17]; and remains open (open problem 5.1 in [7]) for 4 or more receivers under a less-noisy ordering. Prior to the work included this thesis the optimality of superposition-coding region for a 3-receiver broadcast channel where the marginal channels follow the more-capable partial order was not known (open problem 5.2 in [7]). In this thesis we show that the capacity region for a 3-receiver broadcast channel can be strictly larger than the one given by the superposition-coding strategy. This results are first published in [20].

It is known that superposition-coding is not optimal for a general two-receiver DMBC. The best known achievable region is the Marton's inner bound [15]. It is not yet known whether the rate region given by the achievable region below differs from the true capacity region or not.

**Theorem 2** (Marton's inner bound)**.** *A rate pair $(R_1, R_2)$ is achievable for the DMBC depicted in Figure 1.3 if*

$$\begin{aligned} R_1 &\le I(W, U_1; Y_1) \\ R_2 &\le I(W, U_2; Y_2) \\ R_1 + R_2 &\le \min\{I(W; Y_1), I(W; Y_2)\} + I(U_1; Y_1|W) + I(U_2; Y_2|W) - I(U_1; U_2|W) \end{aligned}$$

*for some pmf $p(w, u_1, u_2, x)$, where $W$, $U_1$, $U_2$ are auxiliary random variables and $(W, U_1, U_2) \to X \to (Y_1, Y_2)$ forms a Markov chain.*

*Remark* 2. A recent result by Gohari and Anantharam [10] showed that Marton's region can be evaluated by restricting to auxiliary variable satisfying cardinality bounds $|\mathcal{W}| \le |\mathcal{X}| + 3$, $|\mathcal{U}_1| \le |\mathcal{X}|$, and $|\mathcal{U}_2| \le |\mathcal{X}|$.

## 1.2 Interference channel

Interference channel models the communication of two (or more) sender/receiver pairs with a shared medium for transmission. Like broadcast channel, characterization of the capacity region is a classical and fundamental open problem in network information theory. With the vast interests in wireless communications nowadays and the prominent presence of interference under such settings, characterization of the capacity region is becoming more urgent than ever.

A general discrete memoryless interference channel (DMIC) model $((\mathcal{X}_1 \times \mathcal{X}_2), \mathfrak{q}(y_1, y_2|x_1, x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ with two sender/receiver pairs consists of finite alphabets $\mathcal{X}_1$, $\mathcal{X}_2$, one for each input/transmitter; finite alphabets $\mathcal{Y}_1$, $\mathcal{Y}_2$, one for each output/receiver and a collection of channel transition pmfs $\mathfrak{q}(y_1, y_2|x_1, x_2)$.

The interference channel shown in Figure 1.4 depicts the primary model used in this thesis. Each sender wishes to communicate an independent, private message $M_r$ to its corresponding receiver $r$, $r = 1, 2$. The senders wish to maximize the rate at which messages can be transmitted to their corresponding receivers with a small probability of error.

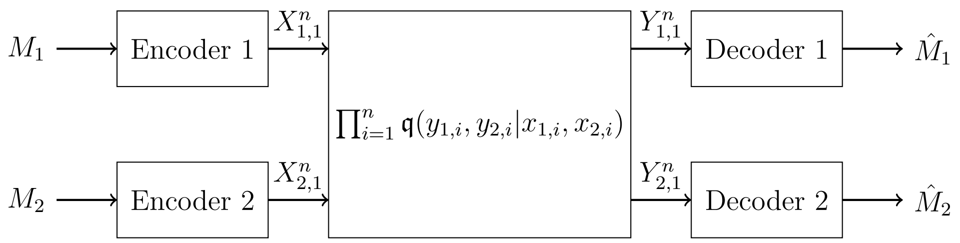


Figure 1.4: Discrete memoryless interference channel

A $(2^{nR_1}, 2^{nR_2}, n)$ code of size $(2^{nR_1}, 2^{nR_2})$ that uses the channel $n$ times consists of:

$(i)$ two encoders, one for each sender. Encoder $r$ maps the messages $M_r$ from the set $[1, 2^{nR_r}]$ to $\mathcal{X}_{r,1}^n$, $r = 1, 2$,

$(ii)$ two decoders, one for each receiver. Decoder $r$ maps the received sequence in $\mathcal{Y}_{r,1}^n$ to an estimated message $\hat{M}_r$ in $[1 : 2_r^{nR}]$, $r = 1, 2$.

The performance metric for a code is the average probability of error, *i.e.*, $P_e^{(n)} = Pr((M_1, M_2) \neq (\hat{M}_1, \hat{M}_2))$ when $(M_1, M_2) \sim \mathbf{Unif}([1, 2^{nR_1}] \times [1, 2^{nR_2}])$.

A rate tuple $(R_1, R_2)$ is said to be achievable if there is a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $P_e^{(n)} \to 0$ as $n$ tends to infinity. Similar to that of DMC and DMBC, the capacity region $\mathscr{C}$ of the DMIC is the closure of the set of all achievable rate pairs $(R_1, R_2)$. $\mathscr{C}$ is not known for a general DMIC.

*Remark* 3. The marginal transition probabilities $\mathfrak{q}_1(y_1|x_1, x_2)$ and $\mathfrak{q}_2(y_2|x_1, x_2)$ uniquely determines the channel capacity region of the DMIC for similar reasons as that of a DMBC.

The best known achievable region is described by Han–Kobayashi inner bound [11].

**Theorem 3** (Han–Kobayashi (HK) inner bound). *A rate-pair $(R_1, R_2)$ is achievable for the channel described in Figure 1.4 if*

$$R_1 < I(X_1; Y_1|U_2, Q), \tag{1.1}$$

$$R_2 < I(X_2; Y_2|U_1, Q), \tag{1.2}$$

$$R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q), \tag{1.3}$$

$$R_1 + R_2 < I(X_2, U_1; Y_2|Q) + I(X_1; Y_1|U_1, U_2, Q), \tag{1.4}$$

$$\begin{aligned} R_1 + R_2 < {} & I(X_1, U_2; Y_1|U_1, Q) \\ & + I(X_2, U_1; Y_2|U_2, Q), \end{aligned} \tag{1.5}$$

$$\begin{aligned} 2R_1 + R_2 < {} & I(X_1, U_2; Y_1|Q) + I(X_1; Y_1|U_1, U_2, Q) \\ & + I(X_2, U_1; Y_2|U_2, Q), \end{aligned} \tag{1.6}$$

$$\begin{aligned} R_1 + 2R_2 < {} & I(X_2, U_1; Y_2|Q) + I(X_2; Y_2`|U_1, U_2, Q) \\ & + I(X_1, U_2; Y_1|U_1, Q) \end{aligned} \tag{1.7}$$

*for some pmf $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$, where $U_1$, $U_2$ and $Q$ are auxiliary random variables. $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$ and $Q \in \mathcal{Q}$. $|\mathcal{U}_1| \leq |\mathcal{X}_1| + 4$, $|\mathcal{U}_2| \leq |\mathcal{X}_2| + 4$, and $|\mathcal{Q}| \leq 7$.*

The set of achievable rate pairs form the Han–Kobayashi achievable region, or HK region by short, and is denoted by $\mathscr{H}$. This region becomes the HK inner bound on capacity.

HK inner bound subsumes all other known inner bounds but the two auxiliary random variables $U_1$, $U_2$ and the presence of the various constraints makes the (numerical) evaluation of the bound impractical under most circumstances. The capacity region is known when the interference is *strong* [21, 3].

In this thesis we investigate the capacity region under the opposite end of the spectrum, *i.e.*, when the interference is very weak. A naive strategy (as well as HK strategy) for maximizing the sum-rate under this setting would be to treat interference as noise and we show that for some subset of channels in this class, the above strategy is indeed optimal (see Chapter 3). The results are first published in [14].

Further restricting to an even smaller subset, where computation of the entire HK region becomes numerically tractable, we show in Chapter 4 that by considering multi-letter extensions, there are channels where the HK strategy is strictly sub-optimal. The results are first published in [18].

# Chapter 2

# More-capable three-receiver discrete memoryless broadcast channel with private message sets

A three-receiver more-capable DMBC $(\mathcal{X}, \mathfrak{q}(y_1, y_2, y_3|x), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3)$ has one sender $X$, three receivers $Y_1$, $Y_2$, $Y_3$ with the corresponding marginal channels $\mathfrak{q}_1(y_1|x)$, $\mathfrak{q}_2(y_2|x)$ and $\mathfrak{q}_3(y_3|x)$ satisfy $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2 \overset{m.c.}{\succeq} \mathfrak{q}_3$.

In this chapter we show that superposition-coding is sum-rate optimal for general $k$-receiver DMBCs. However, we show by example that superposition-coding is sub-optimal when we consider the whole capacity region of the three-receiver more-capable DMBC. We further establish the true capacity region of the counter-example. Finally, we investigate superposition-coding's optimality when we impose stronger partial orders between some pairs of receivers.

## 2.1 Superposition-coding inner bound for three-receiver DMBC

The following theorem presents the superposition-coding inner bound for a three-receiver DMBC; where receiver $Y_1$ decodes the messages for $Y_2$ and $Y_3$, and receiver $Y_2$ decodes the message for receiver $Y_3$; implying an implicit order in their decoding capabilities.

**Theorem 4** (Superposition-coding inner bound for three-receiver DMBC)**.** *For a three-receiver DMBC $(\mathcal{X}, \mathfrak{q}(y_1, y_2, y_3|x), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3)$ with one sender $X$, three receivers $Y_1$, $Y_2$, $Y_3$ and private message sets $(M_1, M_2, M_3)$, a rate tuple $(R_1, R_2, R_3)$ is achievable if*

$$R_3 \leq I(U_3; Y_3) \tag{2.1}$$

$$R_2 + R_3 \leq I(U_2; Y_2|U_3) + I(U_3; Y_3) \tag{2.2}$$

$$R_2 + R_3 \leq I(U_2, U_3; Y_2) \tag{2.3}$$

$$R_1 + R_2 + R_3 \leq I(X; Y_1|U_2, U_3) + I(U_2; Y_2|U_3) + I(U_3; Y_3) \tag{2.4}$$

$$R_1 + R_2 + R_3 \leq I(X; Y_1|U_2, U_3) + I(U_2, U_3; Y_2) \tag{2.5}$$

$$R_1 + R_2 + R_3 \leq I(X; Y_1|U_3) + I(U_3; Y_3) \tag{2.6}$$

$$R_1 + R_2 + R_3 \leq I(X; Y_1) \tag{2.7}$$

*for some auxiliary random variables $U_2$, $U_3$ over $\mathcal{U}_2, \mathcal{U}_3$, respectively, where $|\mathcal{U}_2| \leq |\mathcal{X}| + 1$, $|\mathcal{U}_3| \leq |\mathcal{X}| + 4$ and $(U_3, U_2) \to X \to (Y_1, Y_2, Y_3)$ forms a Markov chain.*

*Remark* 4. The cardinality bounds on the auxiliary random variables are obtained from standard cardinality bounding techniques in Appendix B.

We show in the following sub-section that, if we restrict ourselves to sum-rate $R_1 + R_2 + R_3$, superposition-coding achieves the sum-rate capacity $\mathscr{C}_s$.

### 2.1.1 Optimality of superposition-coding for sum-rate

It has been shown in [9] that the more-capable ordering is a much weaker ordering than less-noisy ordering. In particular, it was shown that if one substitutes a receiver (in a two-receiver broadcast channel) with a more-capable receiver then the capacity region could strictly decrease (!). Further it was also shown that such a phenomenon would not occur for less-noisy ordering. Hence sub-optimality of superposition-coding for three-receiver more-capable DMBC, once proven, should not come as very surprising. However, based on the work in [9], a natural instinct for beating superposition-coding's achievable region would be to show that the maximum sum-rate achieved by superposition-coding is strictly smaller than the sum-rate capacity. This fails, however, as we show below that the sum-rate capacity of any $k$-receiver DMBC can be achieved by transmitting solely to the best receiver.

**Theorem 5.** *Any achievable rate tuple $(R_1, \ldots, R_k)$ for a $k$-receiver more-capable DMBC with private message sets $(M_1, M_2, \ldots, M_k)$ must satisfy*

$$R_1 + R_2 + \cdots + R_k \leq \max_{p(x)} I(X; Y_1). \tag{2.8}$$

*Proof.* We will prove the theorem for three-receiver more-capable channels, the proof for more receivers shall follow with similar steps. Note that

$$n(R_1 + R_2 + R_3) - n\epsilon_n$$
$$\stackrel{(a)}{\leq} I(M_1; Y_{1,1}^n) + I(M_2; Y_{2,1}^n|M_1) + I(M_3; Y_{3,1}^n|M_2, M_1)$$

$$
\begin{aligned}
&\le I(M_1;Y_{1,1}^n) + I(M_2;Y_{2,1}^n|M_1) + \sum_{i=1}^n I(X_i;Y_{3,i}|M_2,M_1,Y_{3,1}^{i-1}) && (2.9)\\
&= I(M_1;Y_{1,1}^n) + \sum_{i=1}^n \big(I(X_i;Y_{3,i}|M_2,M_1,Y_{2,i+1}^{\ n},Y_{3,1}^{i-1}) + I(Y_{2,i+1}^{\ n};Y_{3,i}|M_2,M_1,Y_{3,1}^{i-1})\\
&\qquad\qquad + I(M_2;Y_{2,i}|M_1,Y_{2,i+1}^{\ n})\big)\\
&\overset{(b)}{=} I(M_1;Y_{1,1}^n) + \sum_{i=1}^n \big(I(X_i;Y_{3,i}|M_2,M_1,Y_{2,i+1}^{\ n},Y_{3,1}^{i-1}) + I(Y_{3,1}^{i-1};Y_{2,i}|M_2,M_1,Y_{2,i+1}^{\ n})\\
&\qquad\qquad + I(M_2;Y_{2,i}|M_1,Y_{2,i+1}^{\ n})\big)\\
&= I(M_1;Y_{1,1}^n) + \sum_{i=1}^n \big(I(X_i;Y_{3,i}|M_2,M_1,Y_{2,i+1}^{\ n},Y_{3,1}^{i-1}) + I(M_2,Y_{3,1}^{i-1};Y_{2,i}|M_1,Y_{2,i+1}^{\ n})\big)\\
&\overset{(c)}{\le} I(M_1;Y_{1,1}^n) + \sum_{i=1}^n \big(I(X_i;Y_{2,i}|M_2,M_1,Y_{2,i+1}^{\ n},Y_{3,1}^{i-1}) + I(M_2,Y_{3,1}^{i-1};Y_{2,i}|M_1,Y_{2,i+1}^{\ n})\big)\\
&= I(M_1;Y_{1,1}^n) + \sum_{i=1}^n I(X_i;Y_{2,i}|M_1,Y_{2,i+1}^{\ n}) && (2.10)\\
&\overset{(d)}{=} \sum_{i=1}^n \big(I(X_i;Y_{2,i}|M_1,Y_{2,i+1}^{\ n},Y_{1,1}^{i-1}) + I(M_1,Y_{2,i+1}^{\ n};Y_{1,i}|Y_{1,1}^{i-1})\big)\\
&\overset{(e)}{\le} \sum_{i=1}^n \big(I(X_i;Y_{1,i}|M_1,Y_{2,i+1}^{\ n},Y_{1,1}^{i-1}) + I(M_1,Y_{2,i+1}^{\ n};Y_{1,i}|Y_{1,1}^{i-1})\big)\\
&= \sum_{i=1}^n I(X_i;Y_{1,i}|Y_{1,1}^{i-1})\\
&\le n \max_{p(x)} I(X;Y_1)
\end{aligned}
$$

As $n \to \infty$ we have $\epsilon_n \to 0$ and

$$R_1 + R_2 + R_3 \le \max_{p(x)} I(X;Y_1),$$

which, as we mentioned before, is achieved by transmitting solely to the best receiver with superposition-coding.

In the above chain of inequalities we have used Fano's inequality (inequality (a)), chain-rule for mutual information, data-processing inequality, Csiszár's sum lemma (equalities (b),(d)) and the more-capable ordering (inequalities (c),(e)). The data-processing inequalities used above come from the following Markov chain

$$(Y_{1,1}^{i-1}, Y_{2,i+1}^{\ n}, Y_{3,1}^{i-1}, M_1, M_2, M_3) \to X_i \to (Y_{1,i}, Y_{2,i}, Y_{3,i}).$$

Further, we can see from the similarities between step (2.9) and step (2.10) that the proof extends to the general $k$-receiver case by eliminating one receiver at a time. □

*Remark* 5. This is the first converse proof presented in this thesis. A converse proof proves that no higher rates can be achieved above a particular rate while a coding strategy gives achievable rates. All traditional converse proofs resemble this one in the sense that they invokes the same set of lemmas and inequalities (Fano's inequality, chain rule for mutual information, data processing inequality and Csiszár's sum lemma etc.). This thesis is about going beyond the traditional techniques used in converse proofs.

## 2.2 Sub-optimality of superposition-coding for three-receiver more-capable DMBC

Theorem 5 implies that it is not possible to beat the sum-rate. We prove the sub-optimality of superposition-coding by constructing a particular channel and beating the superposition-coding region along hyperplanes other than the sum-rate.

### 2.2.1 Channel construction

The particular channel we use is a three-receiver DMBC with $X \in \{0,1\}$, $Y_1 \in \{0,1,e\}$, $Y_2 \in \{0,1,e\}$ and $Y_3 \in \{0,1\}$, where the channel from $X$ to $Y_1$, $Y_2$ and $Y_3$ are BEC($\epsilon_1$), BEC($\epsilon_2$) and BSC($p$), respectively (see Figure 2.1). Let $p \in [0, \frac{1}{2}]$, $\epsilon_1 = 2p$ and $\epsilon_2 = H(p)$, then from [16] we know that this is a three-receiver more-capable DMBC.

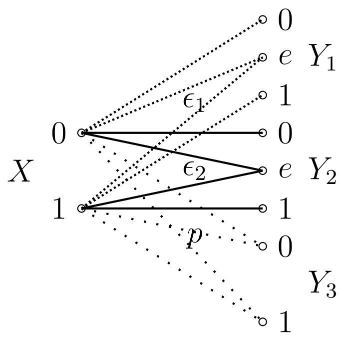


Figure 2.1: Three-receiver more-capable channel with $\epsilon_1 = 2p$ and $\epsilon_2 = H(p)$

### 2.2.2 Beating the superposition-coding region

Let $\mathscr{C}$ denote the true (as yet unknown) capacity region and $\mathscr{S}$ denote the superposition-coding region. Suppose the private message rates are $R_1$, $R_2$ and

$R_3$ for receivers $Y_1$, $Y_2$ and $Y_3$, respectively. We try to evaluate the following:

$$T = \max_{(R_1,R_2,R_3)\in\mathscr{C}} \quad \frac{R_1}{1-\epsilon_1} + \frac{R_2+R_3}{1-\epsilon_2}.$$

**Lemma 1.** *For any* $(R_1^*, R_2^*, R_3^*) \in \mathscr{S}$*, we have*

$$\frac{R_1^*}{1-\epsilon_1} + \frac{R_2^*+R_3^*}{1-\epsilon_2} \leq 1.$$

*Proof.* Suppose $(R_1^*, R_2^*, R_3^*) \in \mathscr{S}$, plugging $(R_1^*, R_2^* + R_3^*, 0)$ into the region in Theorem 4, we see that $(R_1^*, R_2^* + R_3^*, 0) \in \mathscr{S}$.

The channel $X \to (Y_1, Y_2)$ is a degraded DMBC and its capacity region is known to be the union of all non-negative rate pairs $(R_1, R_2)$ satisfying

$$\begin{aligned} R_1 &\leq I(X; Y_1|U), \\ R_2 &\leq I(U; Y_2), \end{aligned}$$

where $U$ is an auxiliary random variable and $U \to X \to (Y_1, Y_2)$ forms a Markov chain.

Therefore, we have

$$\frac{R_1^*}{1-\epsilon_1} + \frac{R_2^*+R_3^*}{1-\epsilon_2} \leq \frac{I(X;Y_1|U)}{1-\epsilon_1} + \frac{I(U;Y_2)}{1-\epsilon_2}. \tag{2.11}$$

Also, note that $X \to (Y_1, Y_2)$ consists of two BECs. Thus, for any $U \to X \to (Y_1, Y_2)$, we have $I(U; Y_2) = I(U; X) - I(U; X|Y_2) = (1 - \epsilon_2)I(U; X)$ and $I(X; Y_1|U) = H(X|U) - H(X|Y_1, U) = (1 - \epsilon_1)H(X|U)$. Putting them back into (2.11), we get

$$\frac{R_1^*}{1-\epsilon_1} + \frac{R_2^*+R_3^*}{1-\epsilon_2} \leq H(X|U) + I(U;X) = H(X) \leq 1.$$

□

Lemma 1 implies that if superposition-coding were optimal, we would have $T \leq 1$.

Next, we show that one can actually achieve $T > 1$. Instead of treating $Y_2$ as the second best receiver, we ignore $Y_2$ completely; *i.e.*, it does not need to decode any message. This way the channel is transformed into a two-receiver degraded DMBC with receivers $Y_1$ and $Y_3$. Using superposition-coding on this two-receiver channel, we can achieve $R_1 = I(X; Y_1|U), R_3 = I(U; Y_3)$ for any

$U \to X \to (Y_1, Y_3)$. Hence

$$T \geq \max_{U \to X \to (Y_1, Y_2, Y_3)} \frac{I(X;Y_1|U)}{(1-\epsilon_1)} + \frac{I(U;Y_3)}{1-\epsilon_2} = \max_{U \to X \to (Y_1, Y_2, Y_3)} \frac{I(X;Y_1|U)}{(1-\epsilon_1)} + \frac{I(U;Y_3)}{1-H(p)}.$$

Let $U \to X$ be a BSC with crossover probability $s$, $0 < s < \frac{1}{2}$. Further, let $P(U=0) = \frac{1}{2}$. We have,

$$T \geq \frac{I(X;Y_1|U)}{1-\epsilon_1} + \frac{I(U;Y_3)}{1-H(p)} = \frac{(1-\epsilon_1)H(s)}{1-\epsilon_1} + \frac{1-H(s*p)}{1-H(p)} = H(s) + \frac{1-H(s*p)}{1-H(p)}. \tag{2.12}$$

By setting $p$ and $s$ to $\frac{1}{10}$, we see that

$$T \geq H(s) + \frac{1-H(s*p)}{1-H(p)} = H(0.1) + \frac{1-H(0.18)}{1-H(0.1)} \geq 1.07.$$

Therefore, superposition-coding cannot be optimal.

### 2.2.3 Alternative achievable region for three-receiver more-capable DMBC

Since the sum-rate capacity is bounded by what we could transmit to receiver $Y_1$, as shown in Theorem 5, a natural guess would be to allow $Y_1$ to decode all the messages. On top of that, deploying Marton's binning scheme to transmit non-nested messages to receivers $Y_2$ and $Y_3$, we get an alternative achievable region.

**Theorem 6.** *Consider a three receiver more-capable broadcast channel with $Y_1$ being the most capable receiver and $Y_3$ the least, a non-negative rate tuple $(R_1, R_2, R_3)$ is achievable if*

$$\begin{aligned}
R_2 &\leq I(U_2, W; Y_2) \\
R_3 &\leq I(U_3, W; Y_3) \\
R_2 + R_3 &\leq \min\{I(W;Y_2), I(W;Y_3)\} + I(U_2;Y_2|W) + I(U_3;Y_3|W) \\
&\quad - I(U_2;U_3|W) \\
R_1 + R_2 + R_3 &\leq I(X;Y_1) \\
R_1 + R_2 + R_3 &\leq I(U_2, W; Y_2) + I(X;Y_1|U_2, W) \\
R_1 + R_2 + R_3 &\leq I(U_3, W; Y_3) + I(X;Y_1|U_3, W) \\
R_1 + R_2 + R_3 &\leq \min\{I(W;Y_2), I(W;Y_3)\} + I(U_2;Y_2|W) + I(U_3;Y_3|W) \\
&\quad + I(X;Y_1|U_2, U_3, W) - I(U_2;U_3|W)
\end{aligned}$$

$$R_1 + 2R_2 + 2R_3 \le I(U_2, W; Y_2) + I(U_3, W; Y_3) + I(X; Y_1|W) - I(U_2; U_3|W)$$
$$2R_1 + 2R_2 + 2R_3 \le I(U_2, W; Y_2) + I(U_3, W; Y_3) + I(X; Y_1|U_2, U_3, W) + I(X; Y_1|W) - I(U_2; U_3|W)$$

*where* $W$*,* $U_2$*,* $U_3$ *are auxiliary random variables and* $(W, U_2, U_3) \to X \to (Y_1, Y_2, Y_3)$ *forms a Markov chain.*

*Proof.* The proof follows standard techniques of random binning, superposition-coding, and jointly typical decoding. Receiver $Y_2$ decodes $(U_{2,1}^n, W^n)$, receiver $Y_3$ decodes $(U_{3,1}^n, W^n)$, and receiver $Y_1$ decodes $(W^n, U_{2,1}^n, U_{3,1}^n, X^n)$. The analysis is routine and straightforward (but messy) and hence is omitted. □

*Remark* 6. Fourier-Motzkin elimination gives $0 \le I(U_2; Y_2|W) + I(U_3; Y_3|W) - I(U_2; U_3|W)$ as one of the final conditions but it is easy to see that this is redundant to the computation of the region.

### 2.2.4 Capacity region of the particular channel construction

In the event where both $Y_2$ and $Y_3$ are stochastically degraded versions of $Y_1$(as in the counter example), the achievable region in Theorem 6 reduces to

$$\begin{aligned} R_1 + R_2 + R_3 &\le \min\big\{I(W; Y_2), I(W; Y_3)\big\} + I(U_2; Y_2|W) + I(U_3; Y_3|W) \\ &\quad + I(X; Y_1|U_2, U_3, W) - I(U_2; U_3|W) \\ R_2 + R_3 &\le \min\big\{I(W; Y_2), I(W; Y_3)\big\} + I(U_2; Y_2|W) + I(U_3; Y_3|W) \\ &\quad - I(U_2; U_3|W) \end{aligned} \tag{2.13}$$
$$\begin{aligned} R_2 &\le I(U_2, W; Y_2) \\ R_3 &\le I(U_3, W; Y_3) \end{aligned} \tag{2.14}$$

Further since $Y_3$ is *essentially less-noisy* [16] than $Y_2$ in the counter example, by symmetrization argument[1] we can assume $\mathrm{P}(X = 0) = \frac{1}{2}$, and hence $I(U_2, W; Y_3) \ge I(U_2, W; Y_2)$. Therefore we can set $\tilde{W} = (U_2, W)$, $\tilde{U}_2 = \emptyset$ and $\tilde{U}_3 = U_3$ to obtain the following achievable region:

**Theorem 7.** *For the channel depicted in 2.1, the union of rate triples* $(R_1, R_2, R_3)$ *satisfying*

$$R_1 + R_2 + R_3 \le I(\tilde{W}; Y_2) + I(\tilde{U}_3; Y_3|\tilde{W}) + I(X; Y_1|\tilde{U}_3, \tilde{W})$$

[1]Symmetrization argument can be found in [16, 19, 9] or in Chapter 5 of [7]. The main purpose of this argument is to show that points on the boundary for a binary input symmetric output channels can be computed using distributions that satisfy $\mathrm{P}(X = 0) = \frac{1}{2}$.

$$R_2 + R_3 \leq I(\tilde{W}; Y_2) + I(\tilde{U}_3; Y_3|\tilde{W})$$
$$R_2 \leq I(\tilde{W}; Y_2)$$

*over all* $(\tilde{W}, \tilde{U}_3) \to X \to (Y_1, Y_2, Y_3)$ *is achievable.*

This is just superposition-coding treating $Y_3$ as the second best receiver. We will prove that this is indeed the capacity region for the particular channel construction.

Note that it suffices to just show a converse to Theorem 7 to establish the capacity region.

The arguments are reasonably routine once the identifications of the auxiliaries have been made: $\tilde{U}_{3,i} = (M_3, Y_{1,1}^{i-1})$ and $\tilde{W}_i = (M_2, Y_{3,i+1}^{\,n}, Y_{2,1}^{i-1})$.

Observe that

$$\begin{aligned}
&n(R_1 + R_2 + R_3) - n\epsilon_n \\
&\quad \leq I(M_2; Y_{2,1}^n) + I(M_3; Y_{3,1}^n|M_2) + I(M_1; Y_{1,1}^n|M_2, M_3) \\
&\quad = \sum_{i=1}^n I(M_1; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}) + I(M_3; Y_{3,1}^n|M_2) + I(M_2; Y_{2,1}^n) \\
&\quad \leq I(M_2; Y_{2,1}^n) + \sum_{i=1}^n \big( I(M_1; Y_{1,i}|M_2, M_3, Y_{3,i+1}^{\,n}, Y_{1,1}^{i-1}) + I(Y_{3,i+1}^{\,n}; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}) \\
&\qquad\quad + I(M_3; Y_{3,i}|Y_{3,i+1}^{\,n}, M_2) \big) \\
&\quad \overset{(a)}{=} I(M_2; Y_{2,1}^n) + \sum_{i=1}^n \big( I(M_1; Y_{1,i}|M_2, M_3, Y_{3,i+1}^{\,n}, Y_{1,1}^{i-1}) + I(M_3, Y_{1,1}^{i-1}; Y_{3,i}|Y_{3,i+1}^{\,n}, M_2) \big) \\
&\quad \leq \sum_{i=1}^n I(M_2; Y_{2,i}|Y_{2,1}^{i-1}) + I(M_3, Y_{1,1}^{i-1}; Y_{3,i}|Y_{3,i+1}^{\,n}, M_2, Y_{2,1}^{i-1}) + I(Y_{2,1}^{i-1}; Y_{3,i}|Y_{3,i+1}^{\,n}, M_2) \\
&\qquad\qquad + I(M_1; Y_{1,i}|M_2, M_3, Y_{3,i+1}^{\,n}, Y_{1,1}^{i-1}) \\
&\quad \overset{(b)}{=} \sum_{i=1}^n I(Y_{3,i+1}^{\,n}, M_2; Y_{2,i}|Y_{2,1}^{i-1}) + I(M_1; Y_{1,i}|M_2, M_3, Y_{3,i+1}^{\,n}, Y_{1,1}^{i-1}) \\
&\qquad\qquad + I(M_3, Y_{1,1}^{i-1}; Y_{3,i}|Y_{3,i+1}^{\,n}, M_2, Y_{2,1}^{i-1}) \\
&\quad \leq \sum_{i=1}^n I(Y_{3,i+1}^{\,n}, Y_{2,1}^{i-1}, M_2; Y_{2,i}) + I(X_i; Y_{1,i}|M_2, M_3, Y_{3,i+1}^{\,n}, Y_{1,1}^{i-1}) \\
&\qquad\qquad + I(M_3, Y_{1,1}^{i-1}; Y_{3,i}|Y_{3,i+1}^{\,n}, M_2, Y_{2,1}^{i-1}) \\
&\quad \overset{(c)}{=} \sum_{i=1}^n I(Y_{3,i+1}^{\,n}, Y_{2,1}^{i-1}, M_2; Y_{2,i}) + I(M_3, Y_{1,1}^{i-1}; Y_{3,i}|Y_{3,i+1}^{\,n}, M_2, Y_{2,1}^{i-1}) \\
&\qquad\qquad + I(X_i; Y_{1,i}|M_2, M_3, Y_{3,i+1}^{\,n}, Y_{1,1}^{i-1}, Y_{2,1}^{i-1}) \\
&\quad = \sum_{i=1}^n I(X_i; Y_{1,i}|\tilde{U}_{3,i}, \tilde{W}_i) + I(\tilde{U}_{3,i}; Y_{3,i}|\tilde{W}_i) + I(\tilde{W}_i; Y_{2,i}).
\end{aligned}$$

We used Fano's inequality, Csiszar sum-lemma (equalities $(a)$, $(b)$), data-processing inequality, and chain rule of mutual information in the above analysis. All the data processing inequalities come from the following Markov chain:

$$(M_1, M_2, M_3, Y_{3,i+1}^{\;n}, Y_{1,1}^{i-1}, Y_{2,1}^{i-1}) \to X_i \to (Y_{1,i}, Y_{2,i}, Y_{3,i}).$$

Equality $(c)$ comes from the fact that $Y_2$ is a degraded version[2] of $Y_1$ and hence

$$Y_{2,1}^{i-1} \to Y_{1,1}^{i-1} \to (M_1, M_2, M_3, Y_{3,i+1}^{\;n}, X_i, Y_{1,i}, Y_{2,i}, Y_{3,i})$$

forms a Markov chain.

Finally, let $Q$ be an independent random variable distributed uniformly in $[1:n]$ and set $\tilde{W} = (\tilde{W}_Q, Q), \tilde{U}_3 = \tilde{U}_{3Q}, X = X_Q$.

The other inequalities follow a similar (but simpler) line of reasoning. Observe that

$$\begin{aligned}
&n(R_2 + R_3) - n\epsilon_n \\
&\quad\le I(M_2; Y_{2,1}^n) + I(M_3; Y_{3,1}^n | M_2) \\
&\quad= \sum_{i=1}^n I(M_2; Y_{2,i} | Y_{2,1}^{i-1}) + I(M_3; Y_{3,i} | M_2, Y_{3,i+1}^{\;n}) \\
&\quad\le \sum_{i=1}^n I(M_2; Y_{2,i} | Y_{2,1}^{i-1}) + I(Y_{2,1}^{i-1}; Y_{3,i} | M_2, Y_{3,i+1}^{\;n}) \\
&\qquad\qquad + I(M_3; Y_{3,i} | M_2, Y_{3,i+1}^{\;n}, Y_{2,1}^{i-1}) \\
&\quad= \sum_{i=1}^n I(M_2, Y_{3,i+1}^{\;n}; Y_{2,i} | Y_{2,1}^{i-1}) + I(M_3; Y_{3,i} | M_2, Y_{3,i+1}^{\;n}, Y_{2,1}^{i-1}) \\
&\quad\le \sum_{i=1}^n I(M_2, Y_{3,i+1}^{\;n}, Y_{2,1}^{i-1}; Y_{2,i}) + I(M_3, Y_{1,1}^{i-1}; Y_{3,i} | M_2, Y_{3,i+1}^{\;n}, Y_{2,1}^{i-1}) \\
&\quad= \sum_{i=1}^n I(\tilde{W}_i; Y_{2,i}) + I(\tilde{U}_{3,i}; Y_{3,i} | \tilde{W}_i).
\end{aligned}$$

The last inequality (on $R_2$) is very straightforward with this identification and is omitted. This completes the proof for the capacity region of the channel in Figure 2.1.

*Remark* 7. It may appear a bit strange to see that even though superposition-coding in the natural more-capable ordering (*i.e.*, $Y_1$ better than $Y_2$ better than $Y_3$) is sub-optimal, a re-ordering of the receivers, *i.e.*, $Y_1$ better than $Y_3$ better than $Y_2$, could make superposition-coding optimal again. But of course, this

[2] Since capacity region just depends on the marginals $\mathfrak{q}_1(y_1|x), \mathfrak{q}_2(y_2|x), \mathfrak{q}_3(y_3|x)$ we can without loss of generality assume that in the example in Figure 2.1 $Y_2$ is a physically degraded version of $Y_1$.

is a carefully chosen channel construction and hence the peculiar situation. It is natural to ask whether there exists a three-receiver more-capable broadcast channel where superposition-coding is not optimal with either ordering. We will show such an example (a minor perturbation of the example in Figure 2.1) in the next section.

### 2.2.5 A modified channel construction

Consider the same channel as in Figure 2.1. Set $\epsilon_1 = 2 * 0.1 = 0.2, \epsilon_2 = H(0.1)$. Slightly change the value of $p$ from 0.1 to 0.11. Clearly since the new receiver $Y_3$ is a degraded version of the old receiver $Y_3$ (which was $BSC(0.1)$), this setting is still a three-receiver more-capable channel. As before, we try to maximize

$$T = \max_{(R_1,R_2,R_3)\in\mathscr{C}} \frac{R_1}{1-\epsilon_1} + \frac{R_2+R_3}{1-\epsilon_2}.$$

If superposition-coding in the more-capable ordering were optimal, then again the same arguments would imply that $T \leq 1$. However if we again ignore $Y_2$ and use superposition-coding between receivers $Y_1$ and $Y_3$ we can obtain, taking $U \to X$to be $BSC(0.1)$ with uniform distribution,

$$\begin{aligned} T &\geq \frac{I(X;Y_1|U)}{1-\epsilon_1} + \frac{I(U;Y_3)}{1-\epsilon_2} \\ &= H(0.1) + \frac{1 - H(0.11 * 0.1)}{1 - H(0.1)} \\ &\geq 1.039. \end{aligned}$$

Hence, superposition-coding in the more-capable ordering is not optimal.

To show that superposition-coding in the $Y_1, Y_3, Y_2$ ordering is not optimal either, we maximize

$$T = \max_{(R_1,R_2,R_3)\in\mathscr{C}} R_2 + R_3.$$

If superposition-coding in $Y_1, Y_3, Y_2$ ordering were optimal, this would be the same as maximizing $R_3$, whose maximum is $1-H(0.11) \approx 0.501$. On the other hand, by just transmitting to receiver $Y_2$ we can obtain $R_2 = 1 - \epsilon_2 = 1 - H(0.1) \geq 0.531$. Thus, superposition-coding in the $Y_1, Y_3, Y_2$ order is also not optimal for this modified counter example.

*Remark* 8. The converse in the last section continues to hold for this modified setting. However since $Y_3$ is no longer an essentially less-noisy receiver than $Y_2$, the achievability of the region depicted by Theorem 7 fails to be true.

*Remark* 9. A natural conjecture for the capacity region in this modified counterexample would be given by the constraints in Equations (2.13) though a proof yet to be formulated.

## 2.3 Optimality of superposition-coding for three-receiver DMBC with enhanced partial orders

We used the more mathematically clean version of the definition for more-capable in previous parts. We revisit the original definition made by Körner and Marton, for both more-capable and less-noisy, here for drawing intuition regarding the optimality of superposition-coding with enhanced partial orders.

**Definition 4** (Equivalent definition of more-capable)**.** For a DMBC with one sender $X$ and two receivers $Y_1$, $Y_2$ with corresponding marginal channels $\mathfrak{q}_1$ and $\mathfrak{q}_2$, we say that $\mathfrak{q}_1$ is more-capable than $\mathfrak{q}_2$ and is denoted as $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2$ if for every $\epsilon$-error channel codebook[3] of size $2^{nR}$ from sender $X$ to $Y_2$, there exists an $\epsilon'$-error channel codebook of size $2^{n(R-\delta)}$ from sender $X$ to $Y_1$ where $\delta, \epsilon' \to 0$ as $\epsilon \to 0$.

In words, this definition implies that any good codebook for receiver $Y_2$ has a sub-codebook of essentially the same rate that can be decoded by receiver $Y_1$.

Consider a set $\mathcal{A} \subseteq \mathcal{X}^n$ and let $0 < \eta < 1$. Let $\mathfrak{q}$ be a channel that transforms $X$ to $Y$, where $Y$ is a random variable over alphabet $\mathcal{Y}$. Given a distribution $P(x)$, let $Q(y)$ be the induced distribution on $Y$ by the channel $\mathfrak{q}$. Let $Q^n$ denote the product distribution on $\mathcal{Y}^n$. We denote the size of the image of cluster $\mathcal{A}$ under (independent uses of) $\mathfrak{q}$, corresponding to input distribution $P$ as

$$G_{\mathfrak{q},P}(\mathcal{A},\eta) = \min\{Q^n(\mathcal{B}) : \mathcal{B} \subseteq \mathcal{Y}^n, \mathfrak{q}^n(Y^n \in \mathcal{B}|x^n) \geq \eta, \ \forall x^n \in \mathcal{A}\}.$$

**Definition 5** (Equivalent definition of less-noisy)**.** For a DMBC with one sender $X$ and two receivers $Y_1$, $Y_2$ with corresponding channels $\mathfrak{q}_1$ and $\mathfrak{q}_2$, we say that $Y_1$ is less-noisy than $Y_2$ and is denoted as $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_2$ if

$$\liminf_{n\to\infty} \min_{\mathcal{B}\subseteq\mathcal{X}^n} \frac{1}{n}\big(\log G_{\mathfrak{q}_1,p(x)}(\mathcal{B},\eta) - \log G_{\mathfrak{q}_2,p(x)}(\mathcal{B},\eta)\big) \geq 0$$

for every distribution $p(x)$ on $X$ and every $0 < \eta < 1$.

One can, using a bit of work, interpret this as the following: consider a set comprised of different clusters; if receiver $Y_2$ can distinguish between the clusters, then receiver $Y_1$ can also essentially distinguish between these clusters.

[3] An $\epsilon$-error codebook of size $2^{nR}$ for $Y_2$ consists of a set of codewords $x^n(m)$, $m \in [1 : 2^{nR}]$ and disjoint decoding regions $\mathcal{B}(m) \in \mathcal{Y}_{2,1}^n$ such that $\mathfrak{q}_{2,1}^n(Y_{2,1}^n \notin \mathcal{B}(m)|x^n(m) \text{ is transmitted}) < \epsilon$, $\forall m$.

*Remark* 10. Using the above definitions one can reconcile the reason for the sub-optimality of the superposition-coding strategy for three receiver more-capable broadcast channel. We claim that it may not be possible for receiver $Y_2$ to decode the message for receiver $Y_3$. Upon decoding $M_2 = m_2$, receiver $Y_2$ has a list consisting of codewords of the type $x^n(*, m_2, *)$ as the potential transmitted codewords. Receiver $Y_3$ is only guaranteed to distinguish between clusters (within this list) of the form $x^n(*, m_2, k)$ and $x^n(*, m_2, j)$ for $k \neq j$. However a more-capable ordering is too weak to guarantee that $Y_2$ can also distinguish between the clusters, while a less noisy ordering guarantees this. Using a cut-set bound like argument, one can see that receiver $Y_1$ should be able to decode all the messages.

To understand the ordering requirements among the three channels such that superposition-coding region remains optimal, we divide the collection of channels into various sub-classes defined by their pairwise ordering.

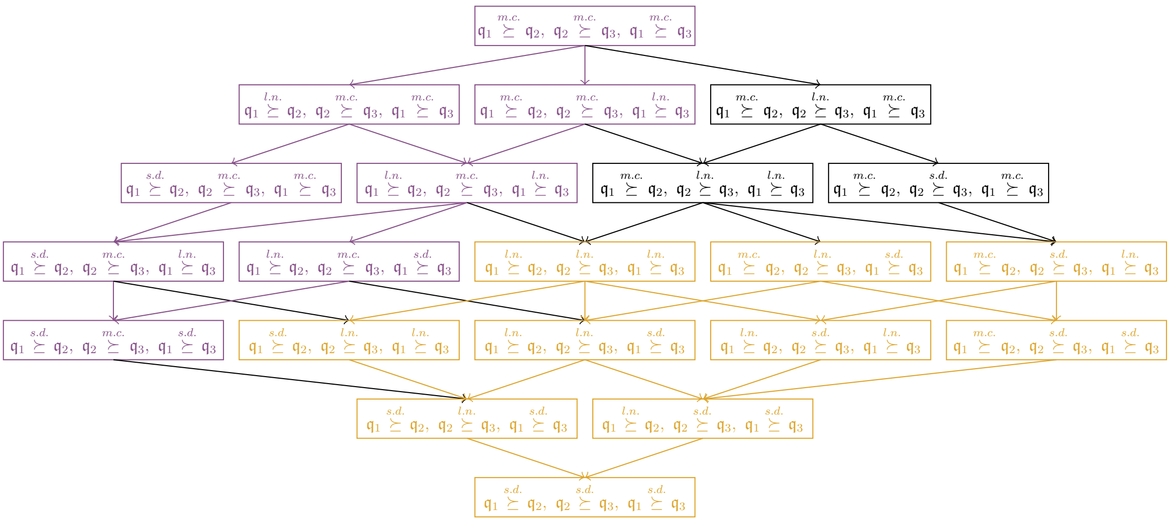


Figure 2.2: Relationship diagram of non-empty sub-classes of three-receiver more-capable DMBC and sub-optimality/optimality of superposition coding

Figure 2.2 depicts the relationship of all non-empty sub-classes of 3-receiver DMBCs as a tree diagram. Each node indicates a non-empty sub-class. Any parent node would be a larger sub-class and includes all its child nodes. For example, $\mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$ is a larger sub-class than its child $\mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$ because the former one only requires less-noisy ordering between $\mathfrak{q}_1$ and $\mathfrak{q}_3$ while the latter one requires the more restrictive degraded ordering. On top of this figure we have the most general 3-receiver more-capable DMBC while at the bottom we have the most restrictive 3-receiver degraded DMBC. We have shown using an example from sub-class $\mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{m.c.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$ that superposition-coding region is strictly sub-optimal, therefore it remains strictly sub-optimal for all its ancestor nodes, indicated in purple. On the other hand, it is previously known that superposition-coding is optimal for 3-receiver less-noisy DMBC indicated as node $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$, therefore we know superposition-coding remains optimal for all its descendants.

The intuition in Remark 10 suggests that superposition-coding should be optimal for all remaining sub-classes because they all satisfied $\mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3$. However, we have been unable to show it for all of them. In the rest of this chapter we show the optimality of superposition-coding region for node $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$ and node $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{s.d.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$, which then implies the optimality result for node $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{s.d.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$. All nodes where superposition-coding region is proven to be optimal are colored in gold.

Both sub-classes $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$ and $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2,\ \mathfrak{q}_2 \overset{s.d.}{\succeq} \mathfrak{q}_3,\ \mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$ satisfy $\mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3$ and $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$, therefore we have $I(U_2;Y_2|U_3) + I(U_3;Y_3) \leq I(U_2;Y_2|U_3) + I(U_3;Y_2) = I(U_2,U_3;Y_2)$ and $I(X;Y_1|U_3) + I(U_3;Y_3) \leq I(X;Y_1)$. Thus, the superposition-coding inner bound constraints stated in Theorem 4 could be simplified.

**Lemma 2** (Superposition-coding inner bound for a subset of three-receiver DMBC)**.** *For a three-receiver DMBC $(\mathcal{X}, \mathfrak{q}(y_1,y_2,y_3|x), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3)$ with one sender $X$, three receivers $Y_1$, $Y_2$, $Y_3$ and private message sets $(M_1,M_2,M_3)$, if $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$ and $\mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3$, then a rate tuple $(R_1,R_2,R_3)$ is achievable if*

$$
\begin{aligned}
R_3 &\leq I(U_3;Y_3)\\
R_2 + R_3 &\leq I(U_2;Y_2|U_3) + I(U_3;Y_3)\\
R_1 + R_2 + R_3 &\leq I(X;Y_1|U_2,U_3) + I(U_2;Y_2|U_3) + I(U_3;Y_3)\\
R_1 + R_2 + R_3 &\leq I(X;Y_1|U_3) + I(U_3;Y_3)
\end{aligned}
$$

*for some auxiliary random variables $U_2$, $U_3$ over $\mathcal{U}_2$, $\mathcal{U}_3$, respectively, where $|\mathcal{U}_2| \leq |\mathcal{X}| + 1$, $|\mathcal{U}_3| \leq |\mathcal{X}| + 4$ and $(U_3,U_2) \to X \to (Y_1,Y_2,Y_3)$ forms a Markov chain.*

We omit the proof here because it is standard and straightforward. We also need the following Lemma before proceeding to proving the optimality results.

**Lemma 3.** *For a two-receiver broadcast channel $X \to (Z_1,Z_2)$, suppose $M \to X^n \to (Z_{1,1}^n, Z_{2,1}^n)$ forms a Markov chain and $Z_1 \overset{l.n.}{\succeq} Z_2$, then we have for any positive integer $i \leq n$*

$$I(Z_{1,1}^{i-1}; Z_{1,i}|M) \geq I(Z_{2,1}^{i-1}; Z_{1,i}|M)$$

*Proof.*

$$I(Z_{1,1}^{i-1}; Z_{1,i}|M)$$

$$
\begin{aligned}
&= I(Z_{1,1}^{i-2}, Z_{1,i-1}; Z_{1,i}|M) \\
&= I(Z_{1,1}^{i-2}; Z_{1,i}|M) + I(Z_{1,i-1}; Z_{1,i}|M, Z_{1,1}^{i-2}) \\
&\overset{(a)}{\geq} I(Z_{1,1}^{i-2}; Z_{1,i}|M) + I(Z_{2,i-1}; Z_{1,i}|M, Z_{1,1}^{i-2}) \\
&= I(Z_{2,i-1}, Z_{1,1}^{i-2}; Z_{1,i}|M) \\
&= I(Z_{2,i-1}, Z_{1,1}^{i-3}, Z_{1,i-2}; Z_{1,i}|M) \\
&= I(Z_{2,i-1}, Z_{1,1}^{i-3}; Z_{1,i}|M) + I(Z_{1,i-2}; Z_{1,i}|M, Z_{2,i-1}, Z_{1,1}^{i-3}) \\
&\overset{(b)}{\geq} I(Z_{2,i-1}, Z_{1,1}^{i-3}; Z_{1,i}|M) + I(Z_{2,i-2}; Z_{1,i}|M, Z_{2,i-1}, Z_{1,1}^{i-3}) \\
&= I(Z_{2,i-2}^{i-1}, Z_{1,1}^{i-3}; Z_{1,i}|M) \\
&\geq \cdots \\
&\geq I(Z_{2,1}^{i-1}; Z_{1,i}|M),
\end{aligned}
$$

where $(a)$ comes from the Markov chain $(M, Z_{1,1}^{i-2}, Z_{1,i}) \to X_{1,i-1} \to (Z_{1,i-1}, Z_{2,i-1})$ and the less-noisy condition, $(b)$ comes from the Markov chain $(M, Z_{2,i-1}, Z_{1,1}^{i-3}, Z_{1,i})$ $\to X_{1,i-1} \to (Z_{1,i-2}, Z_{2,i-2})$ and the less-noisy condition. The chain of inequalities follow from similar arguments. □

We first prove the optimality of superposition-coding region for the sub-class where $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2, \mathfrak{q}_2 \overset{s.d.}{\succeq} \mathfrak{q}_3, \mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$.

**Theorem 8.** *For a three-receiver DMBC $(\mathcal{X}, \mathfrak{q}(y_1, y_2, y_3|x), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3)$ with one sender $X$, three receivers $Y_1$, $Y_2$, $Y_3$ and private message sets $(M_1, M_2, M_3)$, if $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2$, $\mathfrak{q}_2 \overset{s.d.}{\succeq} \mathfrak{q}_3$ and $\mathfrak{q}_1 \overset{l.n.}{\succeq} \mathfrak{q}_3$, then superposition-coding inner bound as stated in Lemma 2 is capacity region.*

*Proof.* Setting $U_{3,i} = (M_3, Y_{3,,i+1}^n)$ and $U_{2,i} = (M_2, Y_{2,,i+1}^n, Y_{1,1}^{i-1})$, we will prove the following

$$nR_3 \leq \sum_{i=1}^{n} I(U_{3,i}; Y_{3,i}) + n\epsilon_n, \tag{2.15}$$

$$nR_2 \leq \sum_{i=1}^{n} I(U_{2,1}; Y_{2,i}|U_{3,i}) + n\epsilon_n, \tag{2.16}$$

$$n(R_1 + R_2) \leq \sum_{i=1}^{n} I(X_i; Y_{1,i}|U_{3,i}) + n\epsilon_n, \tag{2.17}$$

$$n(R_1 + R_2) \leq \sum_{i=1}^{n} (I(X_i; Y_{1,i}|U_{2,i}, U_{3,i}) + I(U_{2,i}; Y_{2,i}|U_{3,i})) + n\epsilon_n, \tag{2.18}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Proof for (2.15):

$$\begin{aligned}
nR_3 - n\epsilon_n &\leq I(M_3; Y_{3,1}^n) && \{\text{Fano's inequality}\} \\
&= \sum_{i=1}^{n} I(M_3; Y_{3,i} | Y_{3,,i+1}^n) \\
&= \sum_{i=1}^{n} \big( I(M_3, Y_{3,,i+1}^n; Y_{3,i}) - I(Y_{3,,i+1}^n; Y_{3,i}) \big) \\
&\leq \sum_{i=1}^{n} I(M_3, Y_{3,,i+1}^n; Y_{3,i}) \\
&= \sum_{i=1}^{n} I(U_{3,i}; Y_{3,i})
\end{aligned}$$

Proof for (2.16):

$$\begin{aligned}
nR_2 - n\epsilon_n &\leq I(M_2; Y_{2,1}^n | M_3) && \{\text{Fano's inequality}\} \\
&= \sum_{i=1}^{n} I(M_2; Y_{2,i} | M_3, Y_{2,,i+1}^n) \\
&\leq \sum_{i=1}^{n} I(M_2, Y_{3,,i+1}^n; Y_{2,i} | M_3, Y_{2,,i+1}^n) \\
&= \sum_{i=1}^{n} \big( I(M_2, Y_{3,,i+1}^n, Y_{2,,i+1}^n; Y_{2,i} | M_3) - I(Y_{2,,i+1}^n; Y_{2,i} | M_3) \big) \\
&= \sum_{i=1}^{n} \big( I(M_2, Y_{2,,i+1}^n; Y_{2,i} | M_3, Y_{3,,i+1}^n) + I(Y_{3,,i+1}^n; Y_{2,i} | M_3) - I(Y_{2,,i+1}^n; Y_{2,i} | M_3) \big) \\
&\leq \sum_{i=1}^{n} I(M_2, Y_{2,,i+1}^n; Y_{2,i} | M_3, Y_{3,,i+1}^n) \\
& && \{\text{less-noisy condition and Lemma 3}\} \\
&\leq \sum_{i=1}^{n} I(M_2, Y_{2,,i+1}^n, Y_{1,1}^{i-1}; Y_{2,i} | M_3, Y_{3,,i+1}^n) \\
&= \sum_{i=1}^{n} I(U_{2,i}; Y_{2,i} | U_{3,i})
\end{aligned}$$

Proof for (2.17):

$$\begin{aligned}
& n(R_1 + R_2) - n\epsilon_n \\
&\quad \leq I(M_1; Y_{1,1}^n | M_3) + I(M_2; Y_{2,1}^n | M_1, M_3) && \{\text{Fano's inequality}\} \\
&\quad \leq I(M_1; Y_{1,1}^n | M_3) + I(X^n; Y_{2,1}^n | M_1, M_3) \\
&\quad \leq I(M_1; Y_{1,1}^n | M_3) + I(X^n; Y_{1,1}^n | M_1, M_3) && \{\text{more-capable condition}\} \\
&\quad = I(X^n; Y_{1,1}^n | M_3)
\end{aligned}$$

$$= \sum_{i=1}^{n} I(X_i; Y_{1,i}|M_3, Y_{1,,i+1}^n)$$

$$= \sum_{i=1}^{n} \big(I(X_i, Y_{1,,i+1}^n; Y_{1,i}|M_3) - I(Y_{1,,i+1}^n; Y_{1,i}|M_3)\big)$$

$$\leq \sum_{i=1}^{n} \big(I(X_i, Y_{1,,i+1}^n; Y_{1,i}|M_3) - I(Y_{3,,i+1}^n; Y_{1,i}|M_3)\big)$$

{less-noisy condition and Lemma 3}

$$= \sum_{i=1}^{n} \big(I(X_i; Y_{1,i}|M_3) - I(Y_{3,,i+1}^n; Y_{1,i}|M_3)\big)$$

$$= \sum_{i=1}^{n} \big(I(X_i, Y_{3,,i+1}^n; Y_{1,i}|M_3) - I(Y_{3,,i+1}^n; Y_{1,i}|M_3)\big)$$

$$= \sum_{i=1}^{n} I(X_i; Y_{1,i}|M_3, Y_{3,,i+1}^n)$$

$$= \sum_{i=1}^{n} I(X_i; Y_{1,i}|U_{3,i})$$

Proof for (2.18):

$$n(R_1 + R_2) - n\epsilon_n$$

$$\leq I(M_1; Y_{1,1}^n|M_2, M_3) + I(M_2; Y_{2,1}^n|M_3) \qquad \{\text{Fano's inequality}\}$$

$$= \sum_{i=1}^{n} \big(I(M_1; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}) + I(M_2; Y_{2,i}|M_3, Y_{2,,i+1}^n)\big)$$

$$= \sum_{i=1}^{n} \big(I(M_1; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^n) + I(Y_{2,,i+1}^n; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1})$$
$$- I(Y_{2,,i+1}^n; Y_{1,i}|M_1, M_2, M_3, Y_{1,1}^{i-1}) + I(M_2; Y_{2,i}|M_3, Y_{2,,i+1}^n)\big)$$

$$= \sum_{i=1}^{n} \big(I(M_1; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^n) + I(Y_{,1}^{i-1}; Y_{2,i}|M_2, M_3, Y_{2,,i+1}^n)$$
$$- I(Y_{2,,i+1}^n; Y_{1,i}|M_1, M_2, M_3, Y_{1,1}^{i-1}) + I(M_2; Y_{2,i}|M_3, Y_{2,,i+1}^n)\big)$$

{Csiszar's Sum Lemma}

$$= \sum_{i=1}^{n} \big(I(M_1; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^n) + I(M_2, Y_{,1}^{i-1}; Y_{2,i}|M_3, Y_{2,,i+1}^n)$$
$$- I(Y_{2,,i+1}^n; Y_{1,i}|M_1, M_2, M_3, Y_{1,1}^{i-1})\big)$$

$$\leq \sum_{i=1}^{n} \big(I(X_i; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^n, Y_{3,,i+1}^n) + I(Y_{3,,i+1}^n; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^n)$$
$$+ I(M_2, Y_{,1}^{i-1}; Y_{2,i}|M_3, Y_{2,,i+1}^n) - I(Y_{2,,i+1}^n; Y_{1,i}|M_1, M_2, M_3, Y_{1,1}^{i-1})\big)$$

$$= \sum_{i=1}^{n} \big(I(X_i; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^n, Y_{3,,i+1}^n) + I(Y_{3,,i+1}^n; Y_{1,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^n)$$

$$+ I(M_2, Y_{,1}^{i-1}, Y_{2,,i+1}^{n}; Y_{2,i}|M_3, Y_{3,,i+1}^{n}) - I(Y_{3,,i+1}^{n}; Y_{2,i}|M_2, M_3, Y_{1,1}^{i-1}, Y_{2,,i+1}^{n})$$
$$+ I(Y_{3,,i+1}^{n}; Y_{2,i}|M_3) - I(Y_{2,,i+1}^{n}; Y_{2,i}|M_3) - I(Y_{2,,i+1}^{n}; Y_{1,i}|M_1, M_2, M_3, Y_{1,1}^{i-1})\Big)$$
$$\leq \sum_{i=1}^{n} \Big(I(X_i; Y_{1,i}|U_{2,i}, U_{3,i}) + I(U_{2,i}; Y_{2,i}|U_{3,i})\Big),$$

where the last inequality goes through because $\mathfrak{q}_2 \overset{s.d.}{\succeq} \mathfrak{q}_3$.

Therefore, all constraints in the superposition-coding inner bound as stated in Lemma 2 get a converse proof. Superposition-coding achievable region is capacity region. □

Next, we prove the optimality of superposition-coding for the sub-class where $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2, \mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3, \mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$.

**Theorem 9.** *For a three-receiver DMBC $(\mathcal{X}, \mathfrak{q}(y_1, y_2, y_3|x), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3)$ with one sender $X$, three receivers $Y_1$, $Y_2$, $Y_3$ and private message sets $(M_1, M_2, M_3)$, if $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2$, $\mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3$ and $\mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$, then superposition-coding inner bound as stated in Lemma 2 is capacity region.*

We have seen from the proof of Theorem 8 the traditional way of proving converse. In the following proof, we use another technique which shows *two-letter tensorization*. The simplest form of this technique could be applied in the proof of the capacity of a point-to-point discrete memoryless channel. We know that $\max_{p(x)} I(X;Y)$ is achievable. We also know that $\frac{1}{n}\max_{p(x^n)} I(X^n;Y^n)$ is an outer bound. By showing

$$\begin{aligned}
\max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2) &= \max_{p(x_1,x_2)} H(Y_1, Y_2) - H(Y_1, Y_2|X_1, X_2) \\
&= \max_{p(x_1,x_2)} H(Y_1, Y_2) - H(Y_1|X_1) - H(Y_2|X_2) \\
&\leq \max_{p(x_1,x_2)} H(Y_1) + H(Y_2) - H(Y_1|X_1) - H(Y_2|X_2) \\
&= \max_{p(x_1,x_2)} I(X_1; Y_1) + I(X_2; Y_2) \\
&\leq \max_{p(x_1)} I(X_1; Y_1) + \max_{p(x_2)} I(X_2; Y_2),
\end{aligned}$$

we prove that the normalized two-letter rate does not exceed single letter rate, we say that the two-letter expression tensorizes. As a consequence, we have

$$\begin{aligned}
\max_{p(x^n)} I(X^n; Y^n) &\leq \max_{p(x_1)} I(X_1; Y_1) + \max_{p(x_{2,1}^n)} I(X_2^n; Y_2^n) \leq \dots \\
&\leq n \max_{p(x)} I(X; Y),
\end{aligned}$$

and therefore showing that $n$-letter tensorizes as well, thus establishing the converse.

*Proof.* In our particular setting, we know that superposition-coding region is achievable. We also know that the normalized $n$-letter form of the region is an outer bound. Therefore, if superposition-coding does not achieve a higher normalized rate for a product channel of any two three-receiver DMBCs both satisfying the partial orders listed in sub-class 12, superposition-coding inner bound is capacity region.

To show two-letter tensorization, we first get the $\lambda$ sum-rate ($\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3$) of superposition-coding region where $\lambda_1,\ \lambda_2,\ \lambda_3 \geq 0$. Suppose that max( $\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3$) is achieved by $(R_1^*, R_2^*, R_3^*)$ and $\max(\lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3) = \lambda_1 R_1^* + \lambda_2 R_2^* + \lambda_3 R_3^*$. From the characterization of superposition-coding, $(R_1^*, R_2^* + R_3^*, 0)$ is also achievable and $R_1^* + \lambda_2 R_2^* + \lambda_3 R_3^* = \max(R_1 + \lambda_2 R_2 + \lambda_3 R_3) \geq R_1 + \lambda_2 R_2 + \lambda_2 R_3 = R_1^* + \lambda_2 (R_2^* + R_3^*)$, implying that $\lambda_3 R_3^* \geq \lambda_2 R_3^*$. This further implies that either $\lambda_2 \leq \lambda_3$ or $R_3^* = 0$. $R_3^* = 0$ changes the DMBC to a 2-receiver more-capable channel which we already know the capacity of. Therefore, the non-trivial case is when $\lambda_2 \leq \lambda_3$. Similarly, we can assume $\lambda_1 \leq \lambda_2$. Also, without loss of generality, we can set $\lambda_1 = 1$.

Thus, we have, for $1 \leq \lambda_2 \leq \lambda_3$ and $\beta \in [0, 1]$, the $\lambda$ sum-rate of region stated in Lemma 2 satisfies

$$\begin{aligned}
&\max(R_1 + \lambda_2 R_2 + \lambda_3 R_3) \\
&\quad \leq \max_{p(u_2,u_3,x)} \min_{\beta} \beta\big(I(X;Y_1|U_2,U_3) + I(U_2;Y_2|U_3) + I(U_3;Y_3)\big) \\
&\qquad\qquad + (1-\beta)\big(I(X;Y_1|U_3) + I(U_3;Y_3)\big) \\
&\qquad\qquad\quad + (\lambda_2 - 1)\big(I(U_2;Y_2|U_3) + I(U_3;Y_3)\big) + (\lambda_3 - \lambda_2) I(U_3;Y_3) \\
&\quad = \max_{p(u_2,u_3,x)} \min_{\beta} \beta\big(I(X;Y_1|U_2,U_3) + I(U_2;Y_2|U_3) + I(U_3;Y_3)\big) \\
&\qquad\qquad + (1-\beta)\big(I(X;Y_1|U_3) + I(U_3;Y_3)\big) \\
&\qquad\qquad\quad + (\lambda_2 - 1) I(U_2;Y_2|U_3) + (\lambda_3 - 1) I(U_3;Y_3) \\
&\quad = \min_{\beta} \max_{p(x)} \lambda_3 I(X;Y_3) + \mathop{\mathfrak{C}}_{p(x)} \Big(\bar{\beta} I(X;Y_1) + (\lambda_2 - \bar{\beta}) I(X;Y_2) - \lambda_3 I(X;Y_3) \\
&\qquad\qquad\qquad + \mathop{\mathfrak{C}}_{p(x)} \big(\beta I(X;Y_1) - (\lambda_2 - \bar{\beta}) I(X;Y_2)\big)\Big),
\end{aligned} \tag{2.19}$$

where $\mathop{\mathfrak{C}}_{p(x)}(*)$ is the concave envelop function of $*$ over valid input distributions $p(x)$.

We show in the following that the two-letter expression of (2.19) tensorizes.

First, consider the inner concave envelop:

$$
\begin{aligned}
&\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1,X_2;Y_{1,1},Y_{1,2})-(\lambda_2-\bar{\beta})I(X_1,X_2;Y_{2,1},Y_{2,2})\big)\\
&\quad=\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1;Y_{1,1})+\beta I(X_2;Y_{1,2}|Y_{1,1})-(\lambda_2-\bar{\beta})I(X_2;Y_{2,2})\\
&\qquad\qquad-(\lambda_2-\bar{\beta})I(X_1;Y_{2,1}|Y_{2,2})\big)\\
&\quad=\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1;Y_{1,1}|Y_{2,2})+\beta I(X_2;Y_{1,2}|Y_{1,1})-(\lambda_2-\bar{\beta})I(X_2;Y_{2,2}|Y_{1,1})\\
&\qquad\qquad-(\lambda_2-\bar{\beta})I(X_1;Y_{2,1}|Y_{2,2})+(1-\lambda_2)I(Y_{1,1};Y_{2,2})\big)\\
&\quad\le\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1;Y_{1,1}|Y_{2,2})-(\lambda_2-\bar{\beta})I(X_1;Y_{2,1}|Y_{2,2})+\beta I(X_2;Y_{1,2}|Y_{1,1})\\
&\qquad\qquad-(\lambda_2-\bar{\beta})I(X_2;Y_{2,2}|Y_{1,1})\big)\\
&\quad\le\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1;Y_{1,1}|Y_{2,2})-(\lambda_2-\bar{\beta})I(X_1;Y_{2,1}|Y_{2,2})\big)\\
&\qquad\quad+\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_2;Y_{1,2}|Y_{1,1})-(\lambda_2-\bar{\beta})I(X_2;Y_{2,2}|Y_{1,1})\big)
\end{aligned}
$$

Next, for the first part of the outer concave envelop, note that

$$
\begin{aligned}
&\bar{\beta}I(X_1,X_2;Y_{1,1},Y_{1,2})+(\lambda_2-\bar{\beta})I(X_1,X_2;Y_{2,1},Y_{2,2})-\lambda_3 I(X_1,X_2;Y_{3,1},Y_{3,2})\\
&\quad\le\bar{\beta}I(X_1;Y_{1,1})+(\lambda_2-\bar{\beta})I(X_1;Y_{2,1})-\lambda_3 I(X_1;Y_{3,1})\\
&\qquad\quad+\bar{\beta}I(X_2;Y_{1,2}|Y_{3,1})+(\lambda_2-\bar{\beta})I(X_2;Y_{2,2}|Y_{3,1})-\lambda_3 I(X_2;Y_{3,2}|Y_{3,1}). \quad (2.20)
\end{aligned}
$$

This is because (2.20) is equivalent to

$$
\bar{\beta}\big(I(Y_{1,2};Y_{3,1})-I(Y_{1,2};Y_{1,1})\big)\le(\lambda_2-\bar{\beta})\big(I(Y_{2,2};Y_{2,1})-I(Y_{2,2};Y_{3,1})\big),
$$

which always holds as left hand side is less than or equal to 0 whilst right hand side is greater than or equal to 0.

Therefore, we have for the two-letter form of the expression inside the outer concave envelop:

$$
\begin{aligned}
&\bar{\beta}I(X_1,X_2;Y_{1,1},Y_{1,2})+(\lambda_2-\bar{\beta})I(X_1,X_2;Y_{2,1},Y_{2,2})-\lambda_3 I(X_1,X_2;Y_{3,1},Y_{3,2})\\
&\quad+\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1,X_2;Y_{1,1},Y_{1,2})-(\lambda_2-\bar{\beta})I(X_1,X_2;Y_{2,1},Y_{2,2})\big)\\
&\quad\le\bar{\beta}I(X_1;Y_{1,1})+(\lambda_2-\bar{\beta})I(X_1;Y_{2,1})-\lambda_3 I(X_1;Y_{3,1})\\
&\qquad\quad+\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1;Y_{1,1}|Y_{2,2})-(\lambda_2-\bar{\beta})I(X_1;Y_{2,1}|Y_{2,2})\big)\\
&\qquad\qquad+\bar{\beta}I(X_2;Y_{1,2}|Y_{3,1})+(\lambda_2-\bar{\beta})I(X_2;Y_{2,2}|Y_{3,1})-\lambda_3 I(X_2;Y_{3,2}|Y_{3,1})\\
&\qquad\qquad\quad+\underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_2;Y_{1,2}|Y_{1,1})-(\lambda_2-\bar{\beta})I(X_2;Y_{2,2}|Y_{1,1})\big)\\
&\quad=\bar{\beta}I(X_1;Y_{1,1})+(\lambda_2-\bar{\beta})I(X_1;Y_{2,1})-\lambda_3 I(X_1;Y_{3,1})
\end{aligned}
$$

$$
\begin{aligned}
&+ \underset{p(x_1,x_2)}{\mathfrak{C}}\Big( \sum_{y_2 \in \mathcal{Y}_2} P(Y_{2,2} = y_2)\big(\beta I(X_1; Y_{1,1}|Y_{2,2} = y_2) \\
&\qquad\qquad - (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1}|Y_{2,2} = y_2)\big)\Big) \\
&\quad + \bar{\beta} I(X_2; Y_{1,2}|Y_{3,1}) + (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{3,1}) - \lambda_3 I(X_2; Y_{3,2}|Y_{3,1}) \\
&\quad + \underset{p(x_1,x_2)}{\mathfrak{C}}\Big( \sum_{y_1 \in \mathcal{Y}_1} P(Y_{1,1} = y_1)\big(\beta I(X_2; Y_{1,2}|Y_{1,1} = y_1) \\
&\qquad\qquad - (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{1,1} = y_1)\big)\Big) \\
\overset{(a)}{\leq}\ & \bar{\beta} I(X_1; Y_{1,1}) + (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1}) - \lambda_3 I(X_1; Y_{3,1}) \\
&+ \sum_{y_2 \in \mathcal{Y}_2} P(Y_{2,2} = y_2) \underset{p(x_1,x_2)}{\mathfrak{C}} \big(\beta I(X_1; Y_{1,1}|Y_{2,2} = y_2) \\
&\qquad\qquad - (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1}|Y_{2,2} = y_2)\big) \\
&\quad + \bar{\beta} I(X_2; Y_{1,2}|Y_{3,1}) + (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{3,1}) - \lambda_3 I(X_2; Y_{3,2}|Y_{3,1}) \\
&\quad + \sum_{y_1 \in \mathcal{Y}_1} P(Y_{1,1} = y_1) \underset{p(x_1,x_2)}{\mathfrak{C}} \big(\beta I(X_2; Y_{1,2}|Y_{1,1} = y_1) \\
&\qquad\qquad - (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{1,1} = y_1)\big) \\
\overset{(b)}{=}\ & \bar{\beta} I(X_1; Y_{1,1}) + (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1}) - \lambda_3 I(X_1; Y_{3,1}) \\
&+ \sum_{y_2 \in \mathcal{Y}_2} P(Y_{2,2} = y_2) \underset{p(x_1,x_2)}{\mathfrak{C}} \big(\beta I(X_1; Y_{1,1}|Y_{2,2} = y_2) \\
&\qquad\qquad - (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1}|Y_{2,2} = y_2)\big) \\
&\quad + \bar{\beta} I(X_2; Y_{1,2}|Y_{3,1}) + (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{3,1}) - \lambda_3 I(X_2; Y_{3,2}|Y_{3,1}) \\
&\quad + \sum_{y_3 \in \mathcal{Y}_3} P(Y_{3,1} = y_3) \sum_{y_1 \in \mathcal{Y}_1} P(Y_{1,1} = y_1 | Y_{3,1} = y_3) \\
&\quad\quad \underset{p(x_1,x_2)}{\mathfrak{C}} \big(\beta I(X_2; Y_{1,2}|Y_{1,1} = y_1, Y_{3,1} = y_3) \\
&\qquad\qquad - (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{1,1} = y_1, Y_{3,1} = y_3)\big) \\
\overset{(c)}{\leq}\ & \bar{\beta} I(X_1; Y_{1,1}) + (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1}) - \lambda_3 I(X_1; Y_{3,1}) \\
&+ \underset{p(x_1,x_2)}{\mathfrak{C}} \big(\beta I(X_1; Y_{1,1}) - (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1})\big) \\
&\quad + \bar{\beta} I(X_2; Y_{1,2}|Y_{3,1}) + (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{3,1}) - \lambda_3 I(X_2; Y_{3,2}|Y_{3,1}) \\
&\quad + \sum_{y_3 \in \mathcal{Y}_3} P(Y_{3,1} = y_3) \underset{p(x_1,x_2)}{\mathfrak{C}} \big(\beta I(X_2; Y_{1,2}|Y_{3,1} = y_3) \\
&\qquad\qquad - (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}|Y_{3,1} = y_3)\big) \\
\overset{(d)}{\leq}\ & \underset{p(x_1)}{\mathfrak{C}} \Big(\bar{\beta} I(X_1; Y_{1,1}) + (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1}) - \lambda_3 I(X_1; Y_{3,1}) \\
&\qquad + \underset{p(x_1,x_2)}{\mathfrak{C}} \big(\beta I(X_1; Y_{1,1}) - (\lambda_2 - \bar{\beta}) I(X_1; Y_{2,1})\big)\Big) \\
&+ \underset{p(x_2)}{\mathfrak{C}} \Big(\bar{\beta} I(X_2; Y_{1,2}) + (\lambda_2 - \bar{\beta}) I(X_2; Y_{2,2}) - \lambda_3 I(X_2; Y_{3,2})
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_2;Y_{1,2}) - (\lambda_2-\bar\beta)I(X_2;Y_{2,2})\big)\Big) \\
&\overset{(e)}{=} \underset{p(x_1)}{\mathfrak{C}}\Big(\bar\beta I(X_1;Y_{1,1}) + (\lambda_2-\bar\beta)I(X_1;Y_{2,1}) - \lambda_3 I(X_1;Y_{3,1}) \\
&\qquad\qquad + \underset{p(x_1)}{\mathfrak{C}}\big(\beta I(X_1;Y_{1,1}) - (\lambda_2-\bar\beta)I(X_1;Y_{2,1})\big)\Big) \\
&\qquad + \underset{p(x_2)}{\mathfrak{C}}\Big(\bar\beta I(X_2;Y_{1,2}) + (\lambda_2-\bar\beta)I(X_2;Y_{2,2}) - \lambda_3 I(X_2;Y_{3,2}) \\
&\qquad\qquad + \underset{p(x_2)}{\mathfrak{C}}\big(\beta I(X_2;Y_{1,2}) - (\lambda_2-\bar\beta)I(X_2;Y_{2,2})\big)\Big),
\end{aligned}
$$

where (a) holds because $\underset{x}{\mathfrak{C}}\big(\sum_i a_i g_i(x)\big) \le \sum_i a_i \underset{x}{\mathfrak{C}}\big(g_i(x)\big)$, (b) holds because $Y_{1,1} \overset{s.d.}{\succeq} Y_{3,1}$, (c) holds from Jensen's inequality, (d) holds because the outer concave envelop makes the function value larger and (e) holds because the two parts are only functions of $p(x_1)$ and $p(x_2)$, respectively.

We have on the right hand side of the last equation a concave function in $p(x_1,x_2)$. Therefore, by definition of concave envelop, we have

$$
\begin{aligned}
&\underset{p(x_1,x_2)}{\mathfrak{C}}\Big(\bar\beta I(X_1,X_2;Y_{1,1},Y_{1,2}) + (\lambda_2-\bar\beta)I(X_1,X_2;Y_{2,1},Y_{2,2}) - \lambda_3 I(X_1,X_2;Y_{3,1},Y_{3,2}) \\
&\qquad\qquad + \underset{p(x_1,x_2)}{\mathfrak{C}}\big(\beta I(X_1,X_2;Y_{1,1},Y_{1,2}) - (\lambda_2-\bar\beta)I(X_1,X_2;Y_{2,1},Y_{2,2})\big)\Big) \\
&\le \underset{p(x_1)}{\mathfrak{C}}\Big(\bar\beta I(X_1;Y_{1,1}) + (\lambda_2-\bar\beta)I(X_1;Y_{2,1}) - \lambda_3 I(X_1;Y_{3,1}) \\
&\qquad\qquad + \underset{p(x_1)}{\mathfrak{C}}\big(\beta I(X_1;Y_{1,1}) - (\lambda_2-\bar\beta)I(X_1;Y_{2,1})\big)\Big) \\
&\qquad + \underset{p(x_2)}{\mathfrak{C}}\Big(\bar\beta I(X_2;Y_{1,2}) + (\lambda_2-\bar\beta)I(X_2;Y_{2,2}) - \lambda_3 I(X_2;Y_{3,2}) \\
&\qquad\qquad + \underset{p(x_2)}{\mathfrak{C}}\big(\beta I(X_2;Y_{1,2}) - (\lambda_2-\bar\beta)I(X_2;Y_{2,2})\big)\Big).
\end{aligned}
$$

Finally, because $I(X;Y_3)$ tensorizes by itself the same way point-to-point channel tensorizes, the whole expression in (2.19) tensorizes which concludes the proof. □

*Remark* 11. This proof of tensorization uses the concavity of concave envelopes as a new technique instead of the traditional way of looking for particular candidates for the auxiliary random variables. It should be noted though that the traditional way of proving still goes through.

As mentioned before, both Theorem 8 and Theorem 9 would imply the optimality of superposition-coding region for the sub-class where $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2$, $\mathfrak{q}_2 \overset{s.d.}{\succeq} \mathfrak{q}_3$, $\mathfrak{q}_1 \overset{s.d.}{\succeq} \mathfrak{q}_3$.

At this point, we are left with three sub-classes in Figure 2.2 colored in black where the optimality/sub-optimality of superposition-coding region remains unknown. We make the following conjecture which would imply optimality result for all three of them.

**Conjecture 1.** *For a three-receiver DMBC* $(\mathcal{X}, \mathfrak{q}(y_1, y_2, y_3|x), \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3)$ *with one sender* $X$*, three receivers* $Y_1$*,* $Y_2$*,* $Y_3$ *and private message sets* $(M_1, M_2, M_3)$*, if* $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_2$*,* $\mathfrak{q}_2 \overset{l.n.}{\succeq} \mathfrak{q}_3$ *and* $\mathfrak{q}_1 \overset{m.c.}{\succeq} \mathfrak{q}_3$*, then superposition-coding inner bound as stated in Theorem 4 is capacity region.*

## 2.4 Conclusion

The optimality/sub-optimality of superposition-coding region for three-receiver DMBC with ordered receivers is a rather delicate topic and it was studied in extensive detail.

Revisiting the sub-classes listed in Figure 2.2, we showed in this chapter that superposition-coding is sub-optimal for 8 of the sub-classes. Also, we showed that superposition-coding is optimal for 3 of the sub-classes.

Based on the intuition in Remark 10, we suspect that superposition-coding region remains optimal for the three unsolved sub-classes. However standard converse techniques seem to run into issues; thus these classes are perhaps some of the easier cases based on which new converse techniques may be developed.

# Chapter 3

# Very weak interference channel

Interference channel models the communication of two(or more) sender/receiver pairs with a shared medium for transmission. The characterization of the capacity region for interference channel is a classical and fundamental open problem in multi-terminal information theory. With the vast interests in wireless communications nowadays and the prominent presence of interference under such settings, characterization of the capacity region is becoming more pressing than ever.

In this chapter, we define the notion of *very weak interference* for the DMIC model described in 1.2. We restrict ourselves to the analysis of the sum-rate capacity $\mathscr{C}_s = \max_{R_1, R_2 \in \mathscr{C}}(R_1 + R_2)$, where $\mathscr{C}$ denotes the capacity of the DMIC in Figure 1.4. We use Han-Kobayashi(HK) inner bound to calculate the best known achievable sum-rate which reduces to treating interference as noise under very weak interference. We develop a genie based sum-rate outer bound. With the help of HK sum-rate inner bound and genie based sum-rate outer bound, we identify the sum-rate capacity under certain conditions for a new class of channels that we call binary skewed Z interference channel(BSZIC) with very weak interference.

## 3.1 Very weak interference

Capacity region of a general interference channel is unknown. Before we go into details of *very weak interference*, consider the following case when capacity is known.

**Definition 6** (Very strong interference)**.** An interference channel as described in Figure 1.4 is said to have *very strong interference* if

$$I(X_1; Y_1|X_2) \leq I(X_1; Y_2),$$
$$I(X_2; Y_2|X_1) \leq I(X_2; Y_1),$$

for all $p_1(x_1)p_2(x_2)$.

In layman's terms, a phrasing of the definition is the following: If the unintended receiver could decode the interfering signal treating its own as noise at a higher rate than the intended receiver, being (magically) given the interference, could, then the channel is said to have very strong interference.

Under very strong interference, HK inner bound reduces to an inner bound without auxiliary random variables and it turns out to be capacity. The optimal strategy turns out to be decoding the entire interfering signal before its intended signal.

We want to find a regime in contrast to very strong interference such that HK inner bound on the sum-rate reduces to treating interference as noise. Motivated by this intuition, we make the following definition.

**Definition 7** (Very weak interference)**.** An interference channel $\mathfrak{q}(y_1, y_2|x_1, x_2)$ is said to have *very weak interference* if

$$\begin{aligned} I(U_1; Y_1) &\geq I(U_1; Y_2|X_2), \\ I(U_2; Y_2) &\geq I(U_2; Y_1|X_1). \end{aligned} \tag{3.1}$$

for all auxiliaries $(U_1, U_2)$ such that the joint probability distribution satisfies $p(u_1, u_2, x_1, x_2, y_1, y_2) = p_1(u_1, x_1)p_2(u_2, x_2)\mathfrak{q}(y_1, y_2|x_1, x_2)$.

The following observation captures some intuitions: $I(U_1; Y_1)$ captures the rate of information from $U_1$ (a part of $X_1$ or a cloud center among $X_{1,1}^n$ sequences) to $Y_1$ while $Y_1$ treats the rest (including interference) as noise. For $Y_2$ to decode the same $U_1$, the rate is at most $I(U_1; Y_2|X_2)$, which is achieved in the best situation where $Y_2$ is fully aware of its intended message $X_2$. The first inequality indicates that the rate at which $Y_2$ decodes any part of $X_1$ is *less* than the rate at which $Y_1$ could. Thus, to maximize $(R_1 + R_2)$, $Y_2$ should not attempt to decode any part of $X_1$ (*i.e.*, $U_1$) at all. The second inequality gives similar conclusion. Therefore, it intuitively suggests that treating interference as noise would optimize the HK inner bound. The following proposition helps in proving this.

**Proposition 1.** *The conditions given in* (3.1) *are equivalent to the following:*

$I(X_1; Y_1) - I(X_1; Y_2|X_2)$ *is concave in* $p_1(x_1)$ *for a fixed* $p_2(x_2)$ *, and* $I(X_2; Y_2) - I(X_2; Y_1|X_1)$ *is concave in* $p_2(x_2)$ *for a fixed* $p_1(x_1)$.

*Proof.* Since $U_1 \to X_1 \to (X_2, Y_1, Y_2)$ forms a Markov chain, observe that

$$\begin{aligned} &I(U_1; Y_1) \geq I(U_1; Y_2|X_2) \\ &\qquad \iff I(X_1; Y_1) - I(X_1; Y_2|X_2) \geq I(X_1; Y_1|U_1) - I(X_1; Y_2|U_1, X_2), \end{aligned}$$

which is equivalent to concavity w.r.t. $p_1(x_1)$. Similar holds for the second equation w.r.t. $p_2(x_2)$. □

Below we state the HK sum-rate inner bound. This could be obtained by performing Fourier-Motzkin elimination on the original region stated in Theorem 3.

**Theorem 10** (Han-Kobayashi sum-rate inner bound)**.** *Any non-negative $R_1+R_2$ satisfying*

$$R_1 + R_2 \leq I(X_1; Y_1|U_2, Q) + I(X_2; Y_2|U_1, Q), \tag{3.2}$$

$$R_1 + R_2 \leq I(U_2, X_1; Y_1|Q) + I(X_2; Y_2|U_2, U_1, Q), \tag{3.3}$$

$$R_1 + R_2 \leq I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_2, U_1, Q), \tag{3.4}$$

$$R_1 + R_2 \leq I(U_2, X_1; Y_1|U_1, Q) + I(U_1 X_2; Y_2|U_2, Q), \tag{3.5}$$

*for some $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$ is achievable.*

Now we are ready to prove that HK sum-rate inner bound reduces to treating interference as noise under very weak interference.

**Proposition 2.** *The maximum achievable sum-rate of Han-Kobayashi inner bound, denoted as $\mathscr{H}_s$, reduces to*

$$\mathscr{H}_s = \max_{p_1(x_1)p_2(x_2)} I(X_1; Y_1) + I(X_2; Y_2)$$

*under very weak interference as defined in* (3.1)*.*

*Proof.* Treating interference as noise, or in particular, setting $Q = U_1 = U_2 = 0$ (*i.e.*, the trivial random variable) gives that $\max_{p_1(x_1)p_2(x_2)} I(X_1; Y_1) + I(X_2; Y_2)$ is achievable. This indicates that

$$\mathscr{H}_s \geq \max_{p_1(x_1)p_2(x_2)} I(X_1; Y_1) + I(X_2; Y_2). \tag{3.6}$$

Next, note that equation (3.5) satisfies:

$$\begin{aligned}
&I(U_2, X_1; Y_1|U_1, Q) + I(U_1, X_2; Y_2|U_2, Q)\\
&\quad \stackrel{(a)}{=} I(U_2, X_1; Y_1|Q) - I(U_1; Y_1|Q) + I(U_1, X_2; Y_2|Q) - I(U_2; Y_2|Q)\\
&\quad = I(X_1; Y_1|Q) + I(U_2; Y_1|X_1, Q) - I(U_2; Y_2|Q) + I(X_2; Y_2|Q) + I(U_1; Y_2|X_2, Q)\\
&\qquad\quad - I(U_1; Y_1|Q)\\
&\quad \stackrel{(b)}{\leq} I(X_1; Y_1|Q) + I(X_2; Y_2|Q),
\end{aligned}$$

where $(a)$ holds because $U_1 \to X_1 \to (U_2, X_2, Y_1, Y_2)$, $U_2 \to X_2 \to (U_1, X_1, Y_1, Y_2)$ form Markov chains conditioning on $Q = q$. $(b)$ is immediate consequence of very weak interference. Since $\mathscr{H}_s$ has to be smaller than the maximum of any of the four expressions, and that the average over $Q$ is dominated by the maximum value, we have $\mathscr{H}_s \le \max_{p_1(x_1)p_2(x_2)} I(X_1; Y_1) + I(X_2; Y_2)$. Combining this with (3.6), the proposition is established. □

*Remark* 12. To characterize the entire HK region, one needs to maximize $\lambda R_1 + R_2$. Treating interference as noise, or in this case, $\max_{p_1(x_1)p_2(x_2)} \lambda I(X_1; Y_1) + I(X_2; Y_2)$, might not be optimal under very weak interference. Thus the definition of very weak interference is tailored for sum-rate (*i.e.*, $\lambda = 1$).

Two classes of channels with very weak interference are provided as examples.

### 3.1.1 Gaussian Z interference channel

We show that our definition for very weak interference on DMIC could be extended to Gaussian interference channel.

Consider a Gaussian Z interference channel,

$$Y_1 = X_1 + Z_1$$
$$Y_2 = X_2 + aX_1 + Z_2$$

where $X_1$, $X_2$ are independent continues random variables with $E[X_1^2] \le P_1$ and $E[X_2^2] \le P_2$. $Z_1$, $Z_2$ are independent Gaussian noise $\mathcal{N}(0, 1)$. $0 \le a \le 1$.

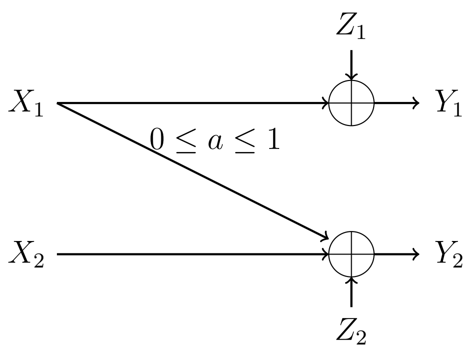


Figure 3.1: Gaussian Z interference channel

**Proposition 3.** *A Gaussian Z interference channel as described in Figure 3.1 with $a \le 1$ has very weak interference.*

*Proof.* Let $U_1 \to X_1 \to (Y_1, Y_2)$, $U_2 \to X_2 \to (Y_1, Y_2)$. Then

$$I(U_2; Y_1|X_1) = I(U_2; X_1 + Z_1|X_1) = I(U_2; Z_1) = 0$$

Hence $I(U_2;Y_1|X_1) \leq I(U_2;Y_2)$.

The second inequality is established as follow.

$$\begin{aligned} I(U_1;Y_2|X_2) &= I(U_1;X_2+aX_1+Z_2|X_2) = I(U_1;aX_1+Z_2) = I(U_1;X_1+\frac{1}{a}Z_2) \\ &\leq I(U_1;X_1+Z_1) = I(U_1;Y_1) \end{aligned}$$

where the inequality holds because $U_1 \to X_1+Z_1 \to X_1+\frac{1}{a}Z_2$ is stochastically-degraded when $a \leq 1$. □

### 3.1.2 Binary skewed-Z interference channel (BSZIC)

Going back to the discrete memoryless case, we introduce a class of binary interference channels that satisfy the very weak interference conditions.

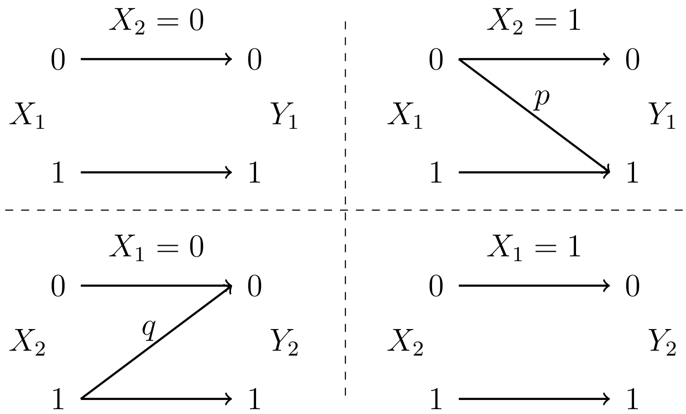


Figure 3.2: Binary skewed-Z interference channel (BSZIC)

Figure 3.2 depicts the transition probabilities $\mathfrak{q}_1(y_1|x_1,x_2)$, $\mathfrak{q}_2(y_2|x_1,x_2)$ of an interference channel where $p,q \in [0,1]$ are constants. $\mathfrak{q}_1$ and $\mathfrak{q}_2$ or in this case in particular, $p,q$ uniquely determines the capacity region of this channel. We call it a $(p,q)$ *binary skewed-Z interference channel (BSZIC)*, or BSZIC$(p,q)$.

**Proposition 4.** *The BSZIC$(p,q)$ shown in Figure 3.2 has very weak interference if and only if* $0 \leq p+q \leq 1$.

*Proof.* From Proposition 1, we know that in order for the channel to have very weak interference, $I(X_1;Y_1) - I(X_1;Y_2|X_2)$ has to be concave in $p_1(x_1)$ and $I(X_2;Y_2) - I(X_2;Y_1|X_1)$ has to be concave in $p_2(x_2)$.

Let $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ denote the binary entropy function. Let $\mathrm{P}(X_2=0)=a$ and $\mathrm{P}(X_1=0)=x$. We first determine the set of values of $p,q \in [0,1]$ within which $I(X_1;Y_1) - I(X_1;Y_2|X_2)$ is concave in $x$ for all $a \in [0,1]$.

$$I(X_1;Y_1) - I(X_1;Y_2|X_2) = H(x(1-\bar{a}p)) - xH(1-\bar{a}p) - \bar{a}H(xq) + \bar{a}xH(q),$$

where $\bar{a} = 1 - a$.

Concavity is equivalent to second derivative with respect to $x$ being non-positive. Note that the second and the last terms are linear in $x$ so they do not effect the second derivative. Concavity is equivalent to the following,

$$\frac{1 - \bar{a}p}{1 - x(1 - \bar{a}p)} \geq \frac{\bar{a}q}{1 - xq},$$
$$i.e., \quad (1 - \bar{a}p)(1 - xq) \geq \bar{a}q(1 - x(1 - \bar{a}p)).$$

The above condition must hold for all $x \in [0, 1]$. Since both sides of the inequality are linear in $x$, it suffices to verify at just $x = 0$ and $x = 1$. Substituting them in, we obtain the following two conditions.

$$\begin{cases} 1 - \bar{a}p \geq \bar{a}q, \\ (1 - \bar{a}p)(1 - q) \geq pq\bar{a}^2. \end{cases}$$

Both conditions have to be satisfied for all $a \in [0, 1]$. Keeping in mind that $p, q \in [0, 1]$, it is easy to check that this is equivalent to $0 \leq p + q \leq 1$.

Similarly, $I(X_2; Y_2) - I(X_2; Y_1|X_1)$ becomes concave in $p_2(x_2)$ when $0 \leq p+q \leq 1$. Therefore, the binary skewed-Z interference channel shown in Figure 3.2 has very weak interference if and only if $0 \leq p + q \leq 1$. □

## 3.2 Genie-based sum-rate outer bound

Binary input/output simplifies expressions and gives intuition for more general DMIC. Our goal is to find sum-rate capacity but before we continue with BSZIC, we develop a genie based outer bound which helps us with our converse proof later and could be of independent interest, too.

We strongly believe that existing outer bounds are not tight beyond where capacity is known. Hence we try to develop a new outer bound on sum-rate for general interference channels.

The sum-rate capacity of a subset of scalar Gaussian interference channels was established in [22], [1] where optimality (or converse) was shown using "genie-aided" receivers. Inspired by this technique, we develop the following sum-rate outer bound for general interference channels.

**Theorem 11.** *Let $T_1, T_2$ be any pair of random variables such that $p(y_1, t_1|x_1, x_2) = p(t_1|x_1)p(y_1|t_1, x_1, x_2)$, $p(y_2, t_2|x_1, x_2) = p(t_2|x_2)p(y_2|t_2, x_1, x_2)$, and the marginals are consistent with the given channel transition probabilities, i.e., $p(y_1|x_1, x_2) = \mathfrak{q}_1(y_1|x_1, x_2)$ and $p(y_2|x_1, x_2) = \mathfrak{q}_2(y_2|x_1, x_2)$. The achievable sum-rate of the* *discrete memoryless interference channel characterized by $\mathfrak{q}(y_1, y_2|x_1, x_2)$ can be upper bounded as follows:*

$$\begin{aligned}R_1 + R_2 \leq \max_{p_1(x_1)p_2(x_2)} & I(X_1; T_1, Y_1) + I(X_2; T_2, Y_2) \\ & + \mathfrak{C}\big(I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)\big) \\ & - I(X_2; T_2|X_1, T_1) + I(X_2; Y_1|T_1, X_1) \\ & + \mathfrak{C}\big(I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)\big) \\ & - I(X_1; T_1|X_2, T_2) + I(X_1; Y_2|T_2, X_2), \end{aligned} \tag{3.7}$$

*where $\mathfrak{C}[I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)]$ denotes the upper concave envelope of the function $I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)$ evaluated with respect to the space of product distributions $p_1(x_1)p_2(x_2)$. Similarly, $\mathfrak{C}[I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)]$ denotes the upper concave envelope of the function $I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)$ evaluated with respect to the same space of product distribution $p_1(x_1)p_2(x_2)$.*

*Proof.* See section 3.5. □

This genie-based sum-rate outer bound provides an upper bound on sum-capacity for *every* valid pair $(T_1, T_2)$. One could minimize over all feasible choice of genies to get a tighter upper bound. In particular, if there exists a pair of $(T_1, T_2)$ such that $I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)$ and $I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)$ become concave in $p_2(x_2)$ and $p_1(x_1)$, respectively, the outer bound reduces to

$$R_1 + R_2 \leq \max_{p_1(x_1)p_2(x_2)} I(X_1; T_1, Y_1) + I(X_2; T_2, Y_2).$$

Moreover, for $X_1^*, X_2^*$ maximizing $I(X_1; Y_1, T_1) + I(X_2; Y_2, T_2)$, if the genie pair $(T_1, T_2)$ satisfies $X_r^* \to Y_r^* \to T_r, r = 1, 2$, the outer bound becomes

$$R_1 + R_2 \leq I(X_1^*; Y_1^*) + I(X_2^*; Y_2^*),$$

which can be achieved exactly by treating interference as noise with $X_1^*, X_2^*$. Hence sum-capacity would be established.

Optimality result in Gaussian interference channel can be derived from genie-aided outer bound. In [1], the "useful" genies are choices that make $I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)$ and $I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)$ concave. The "smart" genies are those satisfying $X_r \to Y_r \to T_r, r = 1, 2$. We will use similar intuitions to show sum-capacity for binary skewed Z interference channel in the next chapter.

## 3.3 Sum-rate capacity analysis for BSZIC

Consider a BSZIC$(p, q)$, Proposition 4 states that the channel has very weak interference when $p + q \leq 1$. In this range, HK sum-rate inner bound is achieved by treating interference as noise. The following theorem shows that this is actually sum-rate capacity for a wide range of parameters.

**Theorem 12.** *Treating interference as noise is sum-rate optimal for BSZIC when channel parameters $(p, q)$ satisfy*

$$p + q + 3pq \leq 1.$$

The regime of parameters (as a subset of the very weak interference regime) is shown in Figure 3.3.

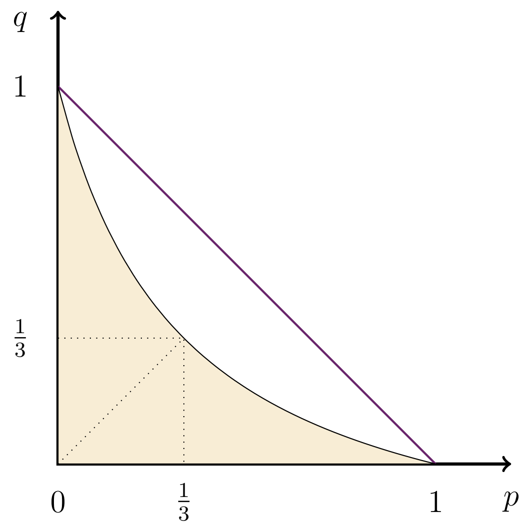


Figure 3.3: Regime of parameters where sum-rate capacity is established for very weak BSZIC(p,q)

The following proposition aids in our proof of the theorem.

**Proposition 5.** *Let $\mathfrak{C}[f](x, y)$ denote the upper concave envelope of $f(x, y)$ over the space of product distributions where $\mathrm{P}(X_1 = 0) = x$ and $\mathrm{P}(X_2 = 1) = y$. Suppose $f(x, y)$ is linear in $x$. Let $g_0(y) = f(0, y)$ and $g_1(y) = f(1, y)$, then $f(x, y) = (1 - x)g_0(y) + xg_1(y)$ and*

$$\mathfrak{C}[f](x, y) = (1 - x)\mathfrak{C}[g_0](y) + x\mathfrak{C}[g_1](y),$$

*where $\mathfrak{C}[g_0](y)$, $\mathfrak{C}[g_1](y)$ denotes the upper concave envelope of $g_0(y)$, $g_1(y)$, respectively, w.r.t. $y \in [0, 1]$.*

*Proof.* For a generic random variable $x \in [0,1]$, let $\bar{x} = 1-x$. Now consider a maximizing convex combination at $(x\bar{y}, xy, \bar{x}\bar{y}, \bar{x}y)$, *i.e.*, a probability vector $\{\alpha_i\}$ and product distributions $(x_i\bar{y}_i, x_iy_i, \bar{x}_i\bar{y}_i, \bar{x}_iy_i)$ such that $\sum_i \alpha_i(x_i\bar{y}_i, x_iy_i, \bar{x}_i\bar{y}_i, \bar{x}_iy_i) = (x\bar{y}, xy, \bar{x}\bar{y}, \bar{x}y)$ and $\sum_i \alpha_i f(x_i, y_i) = \mathfrak{C}[f](x,y)$. Note that $\sum_i \alpha_i\bar{x}_i = \sum_i \alpha_i\bar{x}_i(\bar{y}_i + y_i) = \bar{x}\bar{y} + \bar{x}y = \bar{x}$, $\sum_i \alpha_i x_i = x$, $\sum_i \alpha_i\bar{x}_iy_i = \bar{x}y$ and $\sum_i \alpha_i x_iy_i = xy$. Therefore,

$$
\begin{aligned}
\mathfrak{C}[f](x,y) &= \sum_i \alpha_i f(x_i, y_i) \\
&= \sum_i (\alpha_i\bar{x}_i f(0, y_i) + \alpha_i x_i f(1, y_i)) \\
&= \bar{x}\left(\sum_i \frac{\alpha_i\bar{x}_i}{\bar{x}} f(0, y_i)\right) + x\left(\sum_i \frac{\alpha_i x_i}{x} f(1, y_i)\right) \\
&\leq \bar{x}\mathfrak{C}[g_0](\sum_i \frac{\alpha_i\bar{x}_i}{\bar{x}} y_i) + x\mathfrak{C}[g_1](\sum_i \frac{\alpha_i x_i}{x} y_i) \\
&= \bar{x}\mathfrak{C}[g_0](y) + x\mathfrak{C}[g_1](y).
\end{aligned}
$$

The other direction is immediate as one can always mix the convex combination that achieves $\mathfrak{C}[g_0](y)$ and the convex combination that achieves $\mathfrak{C}[g_1](y)$ to obtain $(1-x)\mathfrak{C}[g_0](y) + x\mathfrak{C}[g_1](y)$. □

*Proof of Theorem 12.* Let $p_1^*(x_1)p_2^*(x_2)$ be the maximizing input for equation (3.7) and $Pr(X_1 = 0) = x^*$, $Pr(X_2 = 1) = y^*$ at $p_1^*(x_1)p_2^*(x_2)$. We will show the existence of a valid pair of genies $(T_1, T_2)$ corresponds to any point of the green region of Figure 3.3 such that the following two conditions hold:

1. $X_r \to Y_r \to T_r$, at $p_1^*(x_1)p_2^*(x_2)$, $r = 1, 2$.
2. $I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)$ and $I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)$ are concave w.r.t. product distributions $p_1(x_1)p_2(x_2)$.

The above conditions immediately imply that (3.7) reduces to

$$
\begin{aligned}
R_1 + R_2 &\leq I(X_1^*; Y_1) + I(X_2^*; Y_2) \\
&\leq \max_{p_1(x_1)p_2(x_2)} I(X_1; Y_1) + I(X_2; Y_2),
\end{aligned}
$$

which is achievable by treating interference as noise. Hence establishing sum-rate capacity.

One should note that the above conditions, though sufficient, are not necessary for genie-based sum-rate outer bound to match HK sum-rate inner bound. The second condition could be relaxed to that the functions match their corresponding concave envelopes at $p_1^*(x_1)p_2^*(x_2)$. Requiring the functions to be concave everywhere simplifies the calculations.

For the first condition to hold, given that the valid genies should also satisfy $T_2 \to X_2 \to X_1 \to T_1$ and channel transition probabilities $\mathfrak{q}(y_1|x_1, x_2)$, $\mathfrak{q}(y_2|x_1, x_2)$, one could verify that distributions $p_1(x_1, x_2, y_1, t_1)$ and $p_2(x_1, x_2, y_2, t_2)$ must be of the form given in Table 3.1, where $\{a_i\}, \{b_i\}$ are two generic probability vectors of size $|T_1|$ and $\{c_i\}, \{d_i\}$ are two generic probability vectors of size $|T_2|$. $Pr(X_1 = 0) = x$, $Pr(X_2 = 1) = y$.

Table 3.1: Generic probability distribution for genies that satisfy the Markov conditions

| $X_1$ | $X_2$ | $Y_1$ | $T_1$ | Probability |
|---|---|---|---|---|
| 0 | 0 | 0 | $i$ | $x(1-y)((1-p)a_i + pb_i))$ |
| 1 | 0 | 1 | $i$ | $(1-x)(1-y)b_i$ |
| 0 | 1 | 0 | $i$ | $xy(1-p)a_i$ |
| 0 | 1 | 1 | $i$ | $xypb_i$ |
| 1 | 1 | 1 | $i$ | $(1-x)yb_i,$ |
| $X_1$ | $X_2$ | $Y_2$ | $T_2$ | Probability |
| 1 | 1 | 1 | $i$ | $(1-x)y((1-q)c_i + qd_i))$ |
| 0 | 1 | 0 | $i$ | $xyqd_i$ |
| 1 | 0 | 0 | $i$ | $(1-x)(1-y)d_i$ |
| 0 | 1 | 1 | $i$ | $xy(1-q)c_i$ |
| 0 | 0 | 0 | $i$ | $x(1-y)d_i,$ |

*Remark* 13. Suppose the Markov chains hold for $Pr(X_1 = 0) = x_*$, $Pr(X_2 = 1) = y_*$, note that our final joint distributions are independent of $(x_*, y_*)$. This is because if the Markov chains hold for some $(x_*, y_*)$, they continue to hold for any other product distribution. This is a chance observation (peculiar to the Binary skewed-Z interference channel) that greatly simplified our analysis.

Next, we will discuss the concavity condition for genies. Define $f(x, y)$, $\tilde{f}(x, y)$ as

$$f(x,y) := \left(I(X_2;T_2|X_1,T_1) - I(X_2;Y_1|X_1,T_1)\right)|_{\mathrm{P}(X_1=0)=x,\mathrm{P}(X_2=1)=y},$$
$$\tilde{f}(x,y) := \left(I(X_1;T_1|X_2,T_2) - I(X_1;Y_2|X_2,T_2)\right)|_{\mathrm{P}(X_1=0)=x,\mathrm{P}(X_2=1)=y}.$$

For a generic variable $x \in [0, 1]$, let $\bar{x} = 1 - x$ and $L(x) = -x \log_2 x$. Then

$$f(x,y) = \sum_i \Bigg( L(\bar{y}d_i + y(\bar{q}c_i + qd_i)) - \bar{y}L(d_i) - yL(\bar{q}c_i + qd_i) \\ - (xpb_i + x\bar{p}a_i)L\left(\frac{ypb_i}{pb_i + \bar{p}a_i}\right) - x(pb_i + \bar{p}a_i)L\left(\frac{\bar{y}pb_i + \bar{p}a_i}{pb_i + \bar{p}a_i}\right)$$

$$+ xy(pb_i + \bar{p}a_i)L\left(\frac{pb_i}{pb_i + \bar{p}a_i}\right) + xy(pb_i + \bar{p}a_i)L\left(\frac{\bar{p}a_i}{pb_i + \bar{p}a_i}\right)\Bigg).$$

Note that $f(x, y)$ is linear in $x$. Therefore we could write it as the linear combination of two functions $g_0(y) = f(0, y)$ and $g_1(y) = f(1, y)$ as in Proposition 5.

$$g_0(y) := \sum_i \left(L(\bar{y}d_i + y(\bar{q}c_i + qd_i)) - \bar{y}L(d_i) - yL(\bar{q}c_i + qd_i)\right),$$

$$\begin{aligned} g_1(y) := \sum_i \Bigg( & L(\bar{y}d_i + y(\bar{q}c_i + qd_i)) - \bar{y}L(d_i) - yL(\bar{q}c_i + qd_i) \\ & - (pb_i + \bar{p}a_i)L\left(\frac{ypb_i}{pb_i + \bar{p}a_i}\right) - (pb_i + \bar{p}a_i)L\left(\frac{\bar{y}pb_i + \bar{p}a_i}{pb_i + \bar{p}a_i}\right) \\ & + y(pb_i + \bar{p}a_i)L\left(\frac{pb_i}{pb_i + \bar{p}a_i}\right) + y(pb_i + \bar{p}a_i)L\left(\frac{\bar{p}a_i}{pb_i + \bar{p}a_i}\right)\Bigg). \end{aligned}$$

Similar for $\tilde{f}(x, y)$, define $\tilde{g}_0(x)$, $\tilde{g}_1(x)$ such that $\tilde{f}(x, y) = (1 - y)\tilde{g}_0(x) + y\tilde{g}_1(x)$.

Based on Lemma 4 from Appendix 3.5, it is safe to consider only binary genies. *i.e.*, $T_1, T_2 \in \{0, 1\}$.

Then, by Proposition 5, the concavity condition is equivalent to that $g_0(y)$, $g_1(y)$ be concave for all $y \in (0, 1)$ and $\tilde{g}_0(x)$, $\tilde{g}_1(x)$ be concave for all $x \in (0, 1)$. Since $g_0(y)$, $\tilde{g}_0(x)$ are already concave w.r.t. $y$, $x$, respectively. The condition is further reduced to $g_1(y)$ and $\tilde{g}_1(x)$ be concave. *i.e.*, their second derivatives be non-positive:

$$\sum_{i=0}^{1} -\frac{\bar{q}^2(c_i - d_i)^2}{\bar{y}d_i + y(\bar{q}c_i + pd_i)} + \frac{pb_i}{y} + \frac{p^2b_i^2}{\bar{y}pb_i + \bar{p}a_i} \le 0 \tag{3.8}$$

$$\sum_{i=0}^{1} -\frac{\bar{p}^2(a_i - b_i)^2}{\bar{x}b_i + y(\bar{p}a_i + pb_i)} + \frac{qd_i}{x} + \frac{q^2d_i^2}{\bar{x}qd_i + \bar{q}c_i} \le 0 \tag{3.9}$$

Note that in (3.8), either $d_0$ or $d_1$ has to be 0 in order to cancel $\frac{pb_i}{y}$ while $y \to 0^+$. Similarly, either $b_0$ or $b_1$ has to be zero because of (3.9). Without loss of generality, we assume that $d_0 = d = 0$ and $b_0 = b = 0$. Setting $a_0 = a$, $a_1 = \bar{a}$, $c_0 = c$ and $c_1 = \bar{c}$, (3.8) becomes equivalent to, for all $y \in (0, 1)$,

$$\begin{aligned} & -\frac{\bar{q}c}{y} + \frac{p}{y} - \frac{\bar{q}^2(\bar{c} - 1)^2}{\bar{y} + y(\bar{q}\bar{c} + q)} + \frac{p^2}{\bar{y}p + \bar{p}\bar{a}} \le 0 \\ \Leftrightarrow\ & \frac{p - \bar{p}c}{y} - \frac{\bar{p}^2c^2}{1 - y\bar{p}c} + \frac{p^2}{\bar{y}p + \bar{p}\bar{a}} \le 0 \\ \Leftrightarrow\ & \frac{p}{y} + \frac{p^2}{\bar{y}p + \bar{p}\bar{a}} \le \frac{\bar{p}c}{y} + \frac{\bar{p}^2c^2}{1 - y\bar{p}c} \end{aligned}$$

$$\Leftrightarrow \frac{p^2 + p\bar{p}\bar{a}}{\bar{y}p + \bar{p}\bar{a}} \le \frac{\bar{p}c}{1 - y\bar{p}c}$$

$$\Leftrightarrow (p^2 + p\bar{p}\bar{a})(1 - y\bar{p}c) \le (\bar{p}c)(\bar{y}p + \bar{p}\bar{a}), \forall y \in (0,1) \tag{3.10}$$

As the expression is linear in $y$ on both sides, it suffices to check the validity of (3.10) for when $y = 0$ and $y = 1$, *i.e.*, (3.10) is equivalent to

$$\begin{cases} p \le \bar{q}c, \\ p + \dfrac{p^2}{\bar{p}\bar{a}} \le \dfrac{\bar{q}c}{1 - \bar{q}c}. \end{cases}$$

Rearranging the first inequality we get

$$\begin{cases} \dfrac{p}{\bar{p}} \le \dfrac{\bar{q}c}{1 - \bar{q}c}, \\ p + \dfrac{p^2}{\bar{p}\bar{a}} \le \dfrac{\bar{q}c}{1 - \bar{q}c}. \end{cases}$$

Note that $p + \frac{p^2}{\bar{p}\bar{a}} = p(1 + \frac{p/\bar{a}}{\bar{p}}) \ge p(1 + \frac{p}{\bar{p}}) = \frac{p}{\bar{p}}$. Therefore, the first inequality is redundant and we are left with a single constraint

$$p + \frac{p^2}{\bar{p}\bar{a}} \le \frac{\bar{q}c}{1 - \bar{q}c}.$$

Similarly, inequality (3.9) is equivalent to the following,

$$q + \frac{q^2}{\bar{q}\bar{c}} \le \frac{\bar{p}a}{1 - \bar{p}a}.$$

Further, without loss of generality, we assume $p \le q$. Putting all the conditions together, we get

$$0 \le a \le 1 \tag{3.11}$$

$$0 \le c \le 1 \tag{3.12}$$

$$0 \le p \le q \le 1 \tag{3.13}$$

$$0 \le p + q \le 1 \tag{3.14}$$

$$p + \frac{p^2}{\bar{p}\bar{a}} \le \frac{\bar{q}c}{1 - \bar{q}c} \tag{3.15}$$

$$q + \frac{q^2}{\bar{q}\bar{c}} \le \frac{\bar{p}a}{1 - \bar{p}a} \tag{3.16}$$

Rearranging (3.15), we have

$$\bar{p}a \le \frac{\bar{p}\bar{q}c - p\bar{p}}{\bar{q}c - p^2\bar{q}c - p\bar{p}}$$

$$\frac{\bar{p}a}{1-\bar{p}a} \le \frac{\bar{q}c-p}{p\bar{q}c}$$

Note

$$\frac{\bar{q}c-p}{p\bar{q}c} = \frac{1-p/\bar{q}c}{p} \le \frac{\bar{p}}{1-\bar{p}}$$

This means (3.11) is redundant.

Combining with (3.16) we have the condition

$$\frac{q\bar{q}\bar{c}+q^2}{\bar{c}} \le \frac{\bar{q}c-p}{pc}$$

$$(1-pq)\bar{q}c^2 - (1+p)\bar{q}c + p \le 0 \tag{3.17}$$

This inequality must holds for some $c \in [0,1]$.

When $c = \frac{1+p}{2(1-pq)}$. $0 \le c \le 1$ is given by the following

$$0 \le \frac{1+p}{2(1-pq)} = \frac{1+p}{1+(1-2pq)} \le \frac{1+p}{1+(1-q)} \le \frac{1+p}{1+(1-\bar{p})} = 1$$

where first inequality is due to $p \le \frac{1}{2}$ and the second one is due to $q \le \bar{p}$. So we can let $c = \frac{1+p}{2(1-pq)}$.

Then inequality (3.17) gives

$$p - \frac{(1+p)^2\bar{q}}{4(1-pq)} \le 0$$

$$q \le \frac{1-p}{1+3p}$$

To satisfy (3.13), we need $p \le \frac{1-p}{1+3p}$. That is $0 \le p \le \frac{1}{3}$.

Same analysis can be applied to the case $q \le p$.

Hence we derive the conditions for the existence of smart and useful genie,

$$\begin{aligned} 0 \le p \le \frac{1}{3}, \\ p \le q \le \frac{1-p}{1+3p}, \end{aligned} \qquad \text{or} \qquad \begin{aligned} 0 \le q \le \frac{1}{3}, \\ q \le p \le \frac{1-q}{1+3q}. \end{aligned}$$

It is easy to verify that this region is equivalent to requiring $p+q+3pq \le 1$ and $p, q \ge 0$. □

In Theorem 12, we obtain sum-rate capacity for a certain range of $(p,q)$ for BSZIC by imposing a Markov condition and a concavity condition. No point outside this region would satisfy both conditions simultaneously. But as mentioned in the proof, these two conditions are not necessary for the genie-based sum-rate outer bound to match the HK sum-rate inner bound. The necessary and sufficient version is stated as below.

For a valid pair of genies $(T_1, T_2)$, let $p_1^*(x_1)p_2^*(x_2)$ be the maximizing input for equation (3.7) and $Pr(X_1 = 0) = x^*$, $Pr(X_2 = 1) = y^*$ at $p_1^*(x_1)p_2^*(x_2)$. For the genie-based sum-rate outer bound (3.7) to reduce to $I(X_1; Y_1)+I(X_2; Y_2)|_{p_1^*(x_1)p_2^*(x_2)}$ is equivalent to saying:

1. $X_r \to Y_r \to T_r$, at $p_1^*(x_1)p_2^*(x_2)$, $r = 1, 2$.
2. $\mathfrak{C}[I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)] = I(X_2; T_2|X_1, T_1) + I(X_2; Y_1|T_1, X_1)$ and $\mathfrak{C}[I(X_1; T_1|X_2, T_2)-I(X_1; Y_2|T_2, X_2)] = I(X_1; T_1|X_2, T_2)+I(X_1; Y_2|T_2, X_2)$ at $p_1^*(x_1)p_2^*(x_2)$

We could parameterize the joint distribution in exactly the same way as before because the Markov condition remains the same. Among the class of genies that satisfy the Markov chains, one is further interested in a subclass such that the two upper concave envelopes coincide with the two functions at $p_1^*(x_1)p_2^*(x_2)$. Define $f(x, y)$ as before,

$$f(x, y) := (I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|X_1, T_1))|_{\mathrm{P}(X_1=0)=x, \mathrm{P}(X_2=1)=y}\,.$$

Expending the expression, we have the same linearity in $x$ and $f(x, y) = (1 - x)g_0(y) + xg_1(y)$, where $g_0(y) = f(0, y)$ and $g_1(y) = f(1, y)$. By Proposition 5, we have $\mathfrak{C}[f](x, y) = (1 - x)\mathfrak{C}[g_0](y) + x\mathfrak{C}[g_1](y)$. $g_0(y)$ is concave as before and $g_1(y)$ could be either convex or concave in $y \in (0, 1)$ when $p + q + 3pq \geq 1$ and $p + q \leq 1$. Therefore, $\mathfrak{C}[f](x, y) = (1 - x)g_0(y) + x\mathfrak{C}[g_1](y)$.

For the genie based outer bound to reduce to treating interference as noise, it is necessary that we find, among the pairs of genies that satisfy the Markov chains, one such that $\mathfrak{C}[g_1(y)] = g_1(y)$ at $y^*$ and $\mathfrak{C}[\tilde{g}_1(x)] = \tilde{g}_1(x)$ at $x^*$, the maximizing point.

We shall see that this is not always possible, which indicates either our genie based sum-rate outer bound is not always tight or treating interference as noise is not always sum-rate optimal for interference channels with very weak interference.

**Proposition 6.** *For the binary skewed-Z interference channel when* $p = q = \frac{1}{2}$, *the genie based outer bound is strictly greater than treating-interference-as-noise inner bound.*

*Proof.* Define $f(x, y)$, $g_0(y)$ and $g_1(y)$ in the same way as before. The joint distribution is the same as defined in Table 3.1.

Setting $p = q = \frac{1}{2}$ and taking second derivative of $g_1(y)$, we get

$$\frac{d^2 g_1(y)}{dy^2} = \sum_i \left( -\frac{(c_i - d_i)^2}{2y(c_i - d_i) + 4d_i} + \frac{b_i}{2y} + \frac{b_i^2}{2\bar{y}b_i + 2a_i} \right)$$

$$
\begin{aligned}
&= -\sum_i \frac{(c_i - d_i)^2}{2y(c_i - d_i) + 4d_i} + \sum_i \frac{b_i}{2y} + \sum_i \frac{yb_i^2}{2y(\bar{y}b_i + a_i)} \\
&\geq -\sum_i \frac{c_i^2 + d_i^2}{2y(c_i - d_i) + 4d_i} + \sum_i \frac{b_i}{2y} + \sum_i \frac{yb_i^2}{2y(\bar{y}b_i + a_i)} \\
&= -\sum_i \frac{c_i^2}{2yc_i - 2yd_i + 4d_i} - \sum_i \frac{d_i^2}{2yc_i - 2yd_i + 4d_i} + \frac{1}{2y} + \sum_i \frac{yb_i^2}{2y(\bar{y}b_i + a_i)} \\
&\geq -\sum_i \frac{c_i^2}{2yc_i} - \sum_i \frac{d_i^2}{-2yd_i + 4d_i} + \frac{1}{2y} + \sum_i \frac{yb_i^2}{2y(\bar{y}b_i + a_i)} \\
&= -\frac{1}{2y} - \frac{1}{-2y+4} + \frac{1}{2y} + \frac{\bar{y}+1}{2}\left(\sum_i \frac{\bar{y}b_i + a_i}{\bar{y}+1} \frac{b_i^2}{(\bar{y}b_i + a_i)^2}\right) \\
&\overset{(a)}{\geq} -\frac{1}{-2y+4} + \frac{\bar{y}+1}{2}\left(\sum_i \frac{\bar{y}b_i + a_i}{\bar{y}+1} \frac{b_i}{\bar{y}b_i + a_i}\right)^2 \\
&= -\frac{1}{-2y+4} + \frac{1}{2(\bar{y}+1)} \\
&= 0,
\end{aligned}
$$

where $(a)$ holds because $\mathrm{E}(X^2) \geq \mathrm{E}(X)^2$. Thus $g_1(y)$ is convex in general. The only hope for the outer bound to work would be if $g_1(y)$ were a straight line. Next we analyze if this is possible.

Note $\frac{d^2 g_1(y)}{dy^2} = 0$ would imply that $c_i d_i = 0$ (for the first inequality to be equality) and $a_i = b_i$ (for the inequality labeled $(a)$ to be an equality).

For the symmetric condition to hold, define $\tilde{f}(x,y)$ as

$$
I(X_1; T_1 | X_2 T_2) - I(X_1; Y_2 | T_2 X_2)|_{\mathrm{P}(X_1=0)=x, \mathrm{P}(X_2=1)=y}
$$

Split $\tilde{f}(x,y)$ in same way as for $f(x,y)$,

$$
\tilde{f}(x,y) = (1-y)\tilde{g}_0(x) + y\tilde{g}_1(x)
$$

Computing derivative of $\tilde{g}_1(x)$, we have

$$
\frac{d^2 \tilde{g}_1(x)}{dx^2} \geq 0
$$

with equality holding only *iff* $a_i b_i = 0$ and $c_i = d_i$.

Clearly, both equalities cannot hold at the same time. At least one of $g_1$ and $\tilde{g}_1$ is strictly convex. Therefore, for any $(x,y) \in (0,1)^2$,

$$
\begin{aligned}
&\mathfrak{C}[f](x,y) + \mathfrak{C}[\tilde{f}](x,y) \\
&\quad = x\mathfrak{C}[g_0](y) + (1-x)\mathfrak{C}[g_1](y) + y\mathfrak{C}[\tilde{g}_0](x) + (1-y)\mathfrak{C}[\tilde{g}_1](x) \\
&\quad > xg_0(y) + (1-x)g_1(y) + y\tilde{g}_0(x) + (1-y)\tilde{g}_1(x)
\end{aligned}
$$

$$= f(x, y) + f_{c_i,d_i,a_i,b_i}(y, x)$$

□

If we restrict our attention to the symmetric case where $p = q$. We have shown already that there are genies for $g_1(y)$ to be concave throughout $y \in [0, 1]$ as long as $0 \leq p = q \leq \frac{1}{3}$.

Now we consider the range $\frac{1}{3} \leq p = q \leq \frac{1}{2}$. Consider genies with binary alphabets, $g_1(y)$ displays an interesting behaviour. The function is concave in some interval $[0, \hat{y}]$ and convex in the remainder. Hence the concave envelope of $g_1(y)$ matches the function in the interval $[0, y^\dagger]$ ($y^\dagger \leq \hat{y}$) and follows the tangent to the curve $g_1(y)$ (at $y^\dagger$) in the interval $[y^\dagger, 1]$. Here $y^\dagger$ is the unique point in $[0, 1]$ such the tangent to the curve $g_1(y)$ at $y^\dagger$ passes through $g_1(1)$ when $y = 1$. See Figure 3.4.

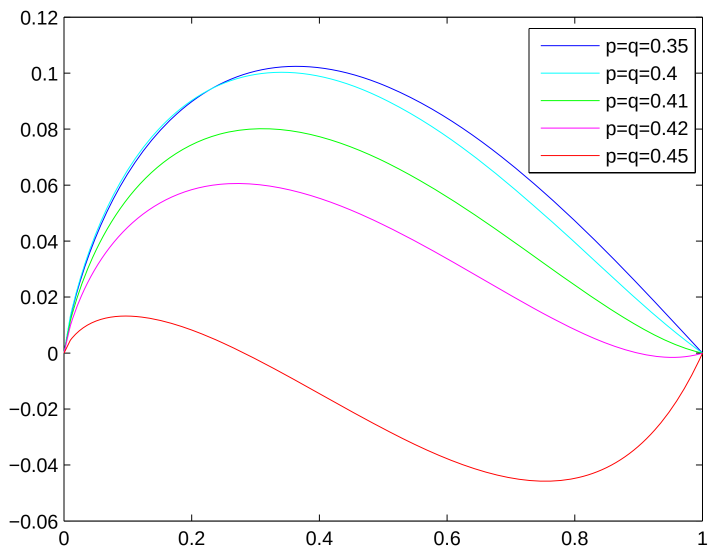


Figure 3.4: Graph of $g_1(x)$ for different values $(p, q)$ of BSZIC(p,q).[1]

Numerical simulations indicate that there are such genies when $0 \leq p = q \leq 0.39$. The updated graph is drawn below.

## 3.4 Generalized genie-based sum-rate outer bound

In genie-based sum-rate outer bound, genie random variable $T_1$ is a distorted version of $X_1$. It carries some information of $X_1$ which helps in decoding $Y_1$. Note that $Y_1$ also contains interference from $X_2$, it is a natural generalization to

[1]Set $p = q$, $a = \frac{1+q}{2(1-pq)}$ and $c = \frac{p(1+p)}{1+p^2-(1+2p-p^2)q} \wedge 1$ in function $g_1(y)$. When $p = 0.35$, the convexity occurs slightly to the left of $y = 1$.

create another genie $S_2$ carrying some information from $X_2$ to help counteract such interference. Similar for $Y_2$, a natural generalization is to create a genie $S_1$ carrying some information from $X_1$. Theorem 13 provides a generalized genie-based sum-rate outer bound in which $(T_1, S_2)$ helps in decoding $Y_1$ and $(T_2, S_1)$ helps in decoding $Y_2$.

**Theorem 13.** *Let $T_1, S_1, T_2, S_2$ be any random variables satisfying:*

- $p(y_1, t_1, s_2|x_1, x_2) = p(t_1|x_1)p(s_2|x_2)p(y_1|t_1, s_2, x_1, x_2)$,
  $p(y_2, t_2, s_1|x_1, x_2) = p(t_2|x_2)p(s_1|x_1)p(y_2|t_2, s_1, x_1, x_2)$.
- *The marginals are consistent with the given channel transition probabilities, that is,*
  $p(y_1|x_1, x_2) = \mathfrak{q}(y_1|x_1, x_2)$ *and* $p(y_2|x_1, x_2) = \mathfrak{q}(y_2|x_1, x_2)$.
- *For each $r = 1, 2$, $T_r, S_r$ has degraded order, i.e., either $X_r \to T_r \to S_r$ or $X_r \to S_r \to T_ir$ must form a Markov chain.*

*The achievable sum-rate of the discrete memoryless interference channel characterized by $\mathfrak{q}(y_1, y_2|x_1, x_2)$ can be upper bounded as follows:*

$$
\begin{aligned}
R_1 + R_2 \leq \max_{p_1(x_1)p_2(x_2)} & I(X_1; T_1, Y_1|S_2) + I(X_2; T_2, Y_2|S_1) \\
& + \mathfrak{C}\big(I(X_1; T_1|X_2, T_2, S_1) - I(X_1; Y_2|X_2, T_2, S_1)\big) \\
& - I(X_1; T_1|X_2, T_2, S_1) + I(X_1; Y_2|X_2, T_2, S_1) \\
& + \mathfrak{C}\big(I(X_2; T_2|X_1, T_1, S_2) - I(X_2; Y_1|X_1, T_1, S_2)\big) \\
& - I(X_2; T_2|X_1, T_1, S_2) + I(X_2; Y_1|X_1, T_1, S_2),
\end{aligned}
\tag{3.18}
$$

*where $\mathfrak{C}[\cdot]$ denotes the upper concave envelope of a function as before.*

*Proof.* See appendix 3.5. □

Note that when $S_1 = S_2 = \emptyset$, generalized genie-based sum-rate outer bound reduces to the bound in (3.7) as it should. The additional genies provide more freedom in searching for *good* genies but computation becomes more complicated.

Another thing to note is that the two genies originated from each sender need to form a degraded order. This degradation condition is easy to establish in Gaussian interference channels. And the generalized genie-based sum-rate outer bound, as it turns out, is actually tight for where sum-rate capacity is previously known.

## 3.5 Conclusion

This paper defines *very weak interference* for interference channels and also proposes a *genie-based sum-rate outer bound* for general interference channels. More importantly, we also discover particular continuous and discrete channels which have very weak interference. For the discrete example, we find a non-trivial set of parameters for which sum-rate capacity can be established because of the genie-based sum-rate outer bound. A generalized version of the genie-based sum-rate outer bound is then proposed. We prove that for continues channels with independent Gaussian noises, the generalized genie-based sum-rate outer bound is tight for all cases where sum-rate capacity is previously known. Additional advantages of genie-based outer bounds will be explored in the future.

# Appendices

## Proof of Theorem 11

*Proof of Theorem 11.* Consider a sequence of codebooks with growing block length $n$ such that their decoding error probabilities tend to zero as $n$ goes to infinity. The distribution on the $n$-tuples is given by

$$\begin{aligned} & p(m_1, m_2, x_{1,1}^n, x_{2,1}^n, y_{1,1}^n, t_{1,1}^n, y_{2,1}^n, t_{2,1}^n) \\ &\quad = p(m_1, x_{1,1}^n) p(m_2, x_{2,1}^n) \\ &\qquad \prod_{i=1}^{n} p(t_{1,i}|x_{1,i}) p(y_{1,i}|x_{1,i}, x_{2,i}, t_{1,i}) p(t_{2,i}|x_{2,i}) p(y_{2,i}|x_{1,i}, x_{2,i}, t_{2,i}). \end{aligned}$$

Keep in mind that the channel capacity of an interference channel depends only on the marginals $\mathfrak{q}(y_1|x_1, x_2)$ and $\mathfrak{q}(y_2|x_1, x_2)$ and that the distribution above is consistent with the marginal distributions by assumption. One can get an upper bound on the sum-rate by following manipulations. The initial part mimics the manipulations in the Gaussian argument as presented in the Appendix of Chapter 6 in [7].

$$\begin{aligned} & n(R_1 + R_2) - n\epsilon_n \\ &\quad = H(M_1) + H(M_2) \\ &\quad \le I(M_1; Y_{1,1}^n) + I(M_2; Y_{2,1}^n) && \{\text{by Fano's inequality}\} \\ &\quad \le I(X_{1,1}^n; Y_{1,1}^n) + I(X_{2,1}^n; Y_{2,1}^n) \\ &\quad \le I(X_{1,1}^n; Y_{1,1}^n, T_{1,1}^n) + I(X_{2,1}^n; Y_{2,1}^n, T_{2,1}^n) \\ &\quad = I(X_{1,1}^n; T_{1,1}^n) + I(X_{1,1}^n; Y_{1,1}^n | T_{1,1}^n) \end{aligned}$$

$$
\begin{aligned}
&\quad + I(X_{2,1}^n; T_{2,1}^n) + I(X_{2,1}^n; Y_{2,1}^n | T_{2,1}^n) \\
&= H(T_{1,1}^n) - H(T_{1,1}^n | X_{1,1}^n) + H(Y_{1,1}^n | T_{1,1}^n) - H(Y_{1,1}^n | T_{1,1}^n, X_{1,1}^n) \\
&\quad + H(T_{2,1}^n) - H(T_{2,1}^n | X_{2,1}^n) + H(Y_{2,1}^n | T_{2,1}^n) - H(Y_{2,1}^n | T_{2,1}^n, X_{2,1}^n).
\end{aligned}
$$

Firstly, consider the term $H(T_{1,1}^n) - H(Y_{2,1}^n | X_{2,1}^n, T_{2,1}^n)$, note that

$$
\begin{aligned}
&H(T_{1,1}^n) - H(Y_{2,1}^n | X_{2,1}^n, T_{2,1}^n) \\
&\quad = H(T_{1,1}^n | T_{2,1}^n, X_{2,1}^n) - H(Y_{2,1}^n | X_{2,1}^n, T_{2,1}^n) \\
&\qquad\qquad \{\text{since } T_{1,1}^n \text{ is independent of } (T_{2,1}^n, X_{2,1}^n)\} \\
&\quad = \sum_i H(T_{1,i} | T_{1,1}^{i-1}, T_{2,1}^n, X_{2,1}^n) - H(Y_{2,i} | Y_{2,i+1}^n, X_{2,1}^n, T_{2,1}^n) \\
&\quad = \sum_i H(T_{1,i} | Y_{2,i+1}^n, T_{1,1}^{i-1}, T_{2,1}^n, X_{2,1}^n) - H(Y_{2,i} | T_{1,1}^{i-1}, Y_{2,i+1}^n, X_{2,1}^n, T_{2,1}^n) \\
&\qquad\qquad \{\text{Csiszar-sum lemma}\} \\
&\quad = \sum_i H(T_{1,i} | U_i, X_{2,i}, T_{2,i}) - H(Y_{2,i} | U_i, X_{2,i}, T_{2,i}). \\
&\qquad\qquad \{U_i := (Y_{2,i+1}^n, T_{1,1}^{i-1}, T_{2,1}^{n \backslash i}, X_{2,1}^{n \backslash i})\}
\end{aligned}
$$

Consider a Bayesian network representation in Figure 3.5 of the variables. It is clear that any path from $X_{1,i}$ to $X_{2,i}$ is d-separated. Indeed the variable

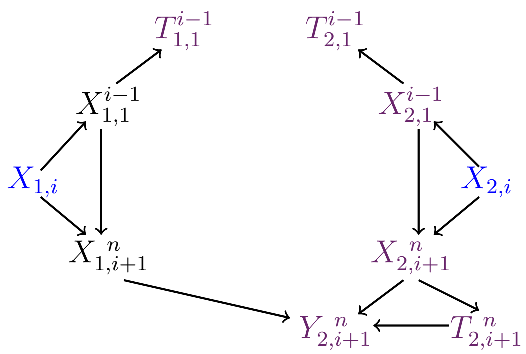


Figure 3.5: Bayesian network of dependence

$X_{2,i+1}^n$ d-separates the variables into two sets. Hence we have Markov chain $X_{1,i} \to U_i \to X_{2,i}$.

Similarly

$$
\begin{aligned}
&H(T_{2,1}^n) - H(Y_{1,1}^n | X_{1,1}^n, T_{1,1}^n) \\
&\quad = \sum_i H(T_{2,i} | V_i, X_{1,i}, T_{1,i}) - H(Y_{1,i} | V_i, X_{1,i}, T_{1,i})
\end{aligned}
$$

where $V_i = (Y_{1,i+1}^n, T_{2,1}^{i-1}, T_{1,1}^{n \backslash i}, X_{1,1}^{n \backslash i})$ and $X_{1,i} \to V_i \to X_{2,i}$.

Secondly, from the $n$-tuple distribution we get that

$$H(T_{1,1}^n|X_{1,1}^n) = \sum_{i=1}^{n} H(T_{1,i}|X_{1,i}, X_{1,1}^{n\setminus i}, T_{1,1}^{i-1}) = \sum_{i=1}^{n} H(T_{1,i}|X_{1,i}),$$
$$H(T_{2,1}^n|X_{2,1}^n) = \sum_{i=1}^{n} H(T_{2,i}|X_{2,i}, X_{2,1}^{n\setminus i}, T_{2,1}^{i-1}) = \sum_{i=1}^{n} H(T_{2,i}|X_{2,i}).$$

Following chain rule and that conditioning reduces entropy,

$$H(Y_{1,1}^n|T_{1,1}^n) \le \sum_{i=1}^{n} H(Y_{1,i}|T_{1,i}),$$
$$H(Y_{2,1}^n|T_{2,1}^n) \le \sum_{i=1}^{n} H(Y_{2,i}|T_{2,i}).$$

Combining the above arguments, using routine manipulations, we obtain that

$$\begin{aligned}
&n(R_1 + R_2) - n\epsilon_n \\
&\quad \le H(T_{1,1}^n) - H(T_{1,1}^n|X_{1,1}^n) + H(Y_{1,1}^n|T_{1,1}^n) - H(Y_{1,1}^n|T_{1,1}^n, X_{1,1}^n) \\
&\qquad + H(T_{2,1}^n) - H(T_{2,1}^n|X_{2,1}^n) + H(Y_{2,1}^n|T_{2,1}^n) - H(Y_{2,1}^n|T_{2,1}^n, X_{2,1}^n) \\
&\quad \le \sum_i H(T_{2,i}|V_i, X_{1,i}, T_{1,i}) - H(Y_{1,i}|V_i, X_{1,i}, T_{1,i}) \\
&\qquad - H(T_{1,i}|X_{1,i}) + H(Y_{1,i}|T_{1,i}) \\
&\qquad\quad + H(T_{1,i}|U_i, X_{2,i}, T_{2,i}) - H(Y_{2,i}|U_i, X_{2,i}, T_{2,i}) \\
&\qquad\qquad - H(T_{2,i}|X_{2,i}) + H(Y_{2,i}|T_{2,i}) \\
&\quad = \sum_i I(X_{2,i}; T_{2,i}|V_i, X_{1,i}, T_{1,i}) + I(V_i, X_{1,i}; Y_{1,i}|T_{1,i}) \\
&\qquad + I(X_{1,i}; T_{1,i}|U_i, X_{2,i}, T_{2,i}) + I(U_i, X_{2,i}; Y_{2,i}|T_{2,i}) \\
&\quad = \sum_i I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) - I(V_i; T_{2,i}|X_{1,i}, T_{1,i}) \\
&\qquad\qquad\qquad \{\text{since } I(V_i, X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) = I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i})\ \} \\
&\qquad + I(X_{1,i}; Y_{1,i}|T_{1,i}) + I(V_i; Y_{1,i}|T_{1,i}, X_{1,i}) \\
&\qquad\quad + I(X_{1,i}; T_{1,i}|X_{2,i}, T_{2,i}) - I(U_i; T_{1,i}|X_{2,i}, T_{2,i}) \\
&\qquad\qquad\qquad \{\text{since } I(U_i, X_{1,i}; T_{1,i}|X_{2,i}, T_{2,i}) = I(X_{1,i}; T_{1,i}|X_{2,i}, T_{2,i})\ \} \\
&\qquad\qquad + I(X_{2,i}; Y_{2,i}|T_{2,i}) + I(U_i; Y_{2,i}|T_{2,i}, X_{2,i}) \\
&\quad = \sum_i I(X_{2,i}; T_{2,i}) - I(V_i; T_{2,i}|X_{1,i}, T_{1,i}) \\
&\qquad + I(X_{1,i}; Y_{1,i}|T_{1,i}) + I(V_i; Y_{1,i}|T_{1,i}, X_{1,i}) \\
&\qquad\quad + I(X_{1,i}; T_{1,i}) - I(U_i; T_{1,i}|X_{2,i}, T_{2,i})
\end{aligned}$$

$$
\begin{aligned}
&+ I(X_{2,i}; Y_{2,i}|T_{2,i}) + I(U_i; Y_{2,i}|T_{2,i}, X_{2,i}) \\
&\qquad\qquad \{\text{since } (X_1, T_1) \text{ and } (X_2, T_2) \text{ are independent}\} \\
&= \sum_i I(X_{1,i}; T_{1,i}, Y_{1,i}) + I(X_{2,i}; T_{2,i}, Y_{2,i}) \\
&\quad - I(V_i; T_{2,i}|X_{1,i}, T_{1,i}) + I(V_i; Y_{1,i}|T_{1,i}, X_{1,i}) \\
&\quad - I(U_i; T_{1,i}|X_{2,i}, T_{2,i}) + I(U_i; Y_{2,i}|T_{2,i}, X_{2,i})
\end{aligned}
$$

Now since $V_i \to (X_{1,i}, T_{1,i}, X_{2,i}) \to (Y_{1,i}, T_{2,i})$ and $U_i \to (X_{1,i}, X_{2,i}, T_{2,i}) \to (Y_{2,i}, T_{1,i})$, one can rewrite the above as

$$
\begin{aligned}
&n(R_1 + R_2) - n\epsilon_n \\
&\le \sum_i I(X_{1,i}; T_{1,i}, Y_{1,i}) + I(X_{2,i}; T_{2,i}, Y_{2,i}) \\
&\quad - I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) + I(X_{2,i}; Y_{1,i}|T_{1,i}, X_{1,i}) \\
&\quad + I(X_{2,i}; T_{2,i}|V_i, X_{1,i}, T_{1,i}) - I(X_{2,i}; Y_{1,i}|V_i, T_{1,i}, X_{1,i}) \\
&\quad - I(X_{1,i}; T_{1,i}|X_{2,i}, T_{2,i}) + I(X_{1,i}; Y_{2,i}|T_{2,i}, X_{2,i}) \\
&\quad + I(X_{1,i}; T_{1,i}|U_i, X_{2,i}, T_{2,i}) - I(X_{1,i}; Y_{2,i}|U_i, T_{2,i}, X_{2,i}) \\
&\le \sum_i I(X_{1,i}; T_{1,i} Y_{1,i}) + I(X_{2,i}; T_{2,i} Y_{2,i}) \\
&\quad - I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) + I(X_{2,i}; Y_{1,i}|T_{1,i}, X_{1,i}) \\
&\quad + \mathfrak{C}\big(I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) - I(X_{2,i}; Y_{1,i}|T_{1,i}, X_{1,i})\big) \\
&\quad - I(X_{1,i}; T_{1,i}|X_{2,i}, T_{2,i}) + I(X_{1,i}; Y_{2,i}|T_{2,i}, X_{2,i}) \\
&\quad + \mathfrak{C}\big(I(X_{1,i}; T_{1,i}|X_{2,i}, T_{2,i}) - I(X_{1,i}; Y_{2,i}|T_{2,i}, X_{2,i})\big),
\end{aligned}
$$

where $\mathfrak{C}[I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) - I(X_{2,i}; Y_{1,i}|T_{1,i}, X_{1,i})]$ is the upper concave envelope of the function $I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) - I(X_{2,i}; Y_{1,i}|T_{1,i}, X_{1,i})$ defined on the space of distributions $p_1(x_1)p_2(x_2)$. It is easy to see from the definition of the upper concave envelope that

$$
\begin{aligned}
&\mathfrak{C}[I(X_{2,i}; T_{2,i}|X_{1,i}, T_{1,i}) - I(X_{2,i}; Y_{1,i}|T_{1,i}, X_{1,i})] \\
&\quad = \sup_{\substack{U: X_{1,i} \to U \to X_{2,i} \\ U \to (X_{1,i}, X_{2,i}) \to (Y_{1,i}, T_{2,i}, T_{1,i})}} I(X_{1,i}; T_{1,i}|U, X_{2,i}, T_{2,i}) - I(X_{1,i}; Y_{2,i}|U, T_{2,i}, X_{2,i}).
\end{aligned}
$$

By Fenchel-Bunt's extension [12] of the Caratheodory's theorem, it suffices to consider $U$ with cardinality $|\mathcal{U}| \le |\mathcal{X}_1||\mathcal{X}_2|$ in computing the upper concave envelope.

Thus for any valid choice of genies $T_1, T_2$, we obtain an outer bound to the sum-rate given by

$$
\begin{aligned}
&R_1 + R_2 \\
&\quad \le \max_{p_1(x_1)p_2(x_2)} I(X_1;T_1,Y_1) + I(X_2;T_2,Y_2) \\
&\qquad + \mathfrak{C}\big(I(X_2;T_2|X_1,T_1) - I(X_2;Y_1|T_1,X_1)\big) - I(X_2;T_2|X_1,T_1) + I(X_2;Y_1|T_1,X_1) \\
&\qquad\quad + \mathfrak{C}\big(I(X_1;T_1|X_2,T_2) - I(X_1;Y_2|T_2,X_2)\big) - I(X_1;T_1|X_2,T_2) + I(X_1;Y_2|T_2,X_2)
\end{aligned} \tag{3.19}
$$

□

## Cardinality bound on genies

For an outer bound with auxiliaries, we need to find some cardinality bounds for the auxiliaries because the outer bound is obtained by taking *union* of every possible joint distribution. Without cardinality, the union is over infinite dimensional space and thus is non-evaluable. However for the genie case, any valid genie pair yields a valid outer bound. Cardinality bound on genie is not necessary. Nevertheless, to find the best genie which can yield tight upper bound, we need a cardinality bound beyond which there are no benefit to tighten genie-aided outer bound. Unfortunately, traditional methods of bounding cardinalities using Caratheodory theorem does not go through as the cardinality bounds for $T_1$ and $T_2$ would end up depending on each other's. We will deploy a tailored method for our case.

By Proposition 5, $g_1(y)$ is concave for $y \in [0,1]$ if genies satisfy concavity condition. Taking second derivative of $g_1(y)$ with respect to $y$,

$$
\frac{d^2 g_1(y)}{dy^2} = \sum_i \left( -\frac{\bar{q}^2(c_i-d_i)^2}{y\bar{q}(c_i-d_i)+d_i} + \frac{pb_i}{y} + \frac{p^2 b_i^2}{\bar{y}pb_i + \bar{p}a_i} \right)
$$

$T_2$ is characterized by $\{c_i\}$ and $\{d_i\}$. The following lemma provides cardinality bound for $T_2$.

**Lemma 4.** *Let $n \ge 3$ and $T_{2n}$ be the set of all genies with cardinality $n$. If $T_{2n}(\mathbf{c},\mathbf{d})$ is a genie defined by $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ such that $\frac{d^2 g_1(y)}{dy^2} \le 0$, then there is always another set of coefficients $\hat{\mathbf{c}}$, $\hat{\mathbf{d}}$ with $(n-1)$ coordinates each such that $T_{2(n-1)}(\hat{\mathbf{c}},\hat{\mathbf{d}})$ defines a genie such that $\frac{d^2 g_1(y)}{dy^2} \le 0$.*

*Proof.* For $1 \le i \le n$, let $\epsilon \ge 0$ and $c'_i = c_i(1+\epsilon l_i)$, $d'_i = d_i(1+\epsilon l_i)$. $\mathbf{c}' = (c'_1, \ldots, c'_n)$ and $\mathbf{d}' = (d'_1, \ldots, d'_n)$ form a valid $T_{2n}(\mathbf{c}',\mathbf{d}')$ with some $\mathbf{l} = (l_1, l_2, \ldots, l_n)$ if $\sum_i c_i l_i = 0$, $\sum_i d_i l_i = 0$ and $\epsilon$ small enough. Note that as long as there exists of a non-zero $\mathbf{l}$ independent of $\epsilon$ such that $T_{2n}(\mathbf{c}', \mathbf{d}')$ forcing $\frac{d^2 g_1(y)}{dy^2} \leq 0$ for $0 \leq \epsilon \leq \epsilon_0$, we could increase $\epsilon$ from 0 gradually until for some $i$, $1+\epsilon l_i$ becomes 0. Dropping the 0 coefficients, we get an equivalent genie in $T_{2(n-1)}$. Therefore, it suffices to show the existence of one such $\mathbf{l}$ for $n \geq 3$.

Note that one of the $d_i$'s has to be 0 and the corresponding $c_i$ has to satisfy $\bar{q}c_i \geq p$ in order for $\frac{d^2 g_1(y)}{dy^2}$ to be non-positive when $y \to 0$. In cases where more than one of the $d_i$'s are 0, we could sum over the corresponding $c_i$'s and form a new smart and useful genie with smaller cardinality. Therefore, without loss of generality, we assume that $d_1 = 0$, $\bar{q}c_1 \geq p$ and $d_i > 0, \forall i \geq 2$. All assumptions about $\mathbf{c}$ and $\mathbf{d}$ are as below.

$$\begin{cases} \mathbf{c} \geq \mathbf{0}, \\ \sum_{i=1}^{n} c_i = 1, \\ \bar{p}c_1 \geq p, \\ d_1 = 0, \\ (d_2, d_3, \ldots, d_n) > \mathbf{0}, \\ \sum_{i=2}^{n} d_i = 1, \\ -\frac{\bar{q}c_1}{y} + \frac{pb_1}{y} + \frac{p^2 b_1^2}{\bar{y}pb_1 + \bar{p}a_1} \\ \quad + \sum_{i=2}^{n} \left( -\frac{\bar{q}^2 (c_i - d_i)^2}{y\bar{q}(c_i - d_i) + d_i} + \frac{pb_i}{y} + \frac{p^2 b_i^2}{\bar{y}pb_i + \bar{p}a_i} \right) \\ \quad \leq 0, \forall y \in [0,1]. \end{cases}$$

We need to find $\mathbf{l}$ such that

$$\begin{cases} \mathbf{l} \neq 0, \\ c_1 l_1 + \sum_{i=2}^{n} l_i c_i = 0, \\ \sum_{i=2}^{n} l_i d_i = 0, \\ -\frac{\bar{q}c_1(1+\epsilon l_1)}{y} + \frac{pb_1}{y} + \frac{p^2 b_1^2}{\bar{y}pb_1 + \bar{p}a_1} \\ \quad + \sum_{i=2}^{n} \left( -\frac{\bar{q}^2 (c_i - d_i)^2 (1+\epsilon l_i)}{y\bar{q}(c_i - d_i) + d_i} + \frac{pb_i}{y} + \frac{p^2 b_i^2}{\bar{y}pb_i + \bar{p}a_i} \right) \\ \quad \leq 0, \forall y \in [0,1], \epsilon \in [0, \epsilon_0] \end{cases}$$

Combining above two sets of conditions, and given $\epsilon \geq 0$

$$\begin{cases} \mathbf{l} \neq 0, \\ c_1 l_1 + \sum_{i=2}^{n} l_i c_i = 0, \\ \sum_{i=2}^{n} l_i d_i = 0, \\ -\frac{\bar{q} c_1 l_1}{y} - \sum_{i=2}^{n} \frac{\bar{q}^2 (c_i - d_i)^2 l_i}{y\bar{q}(c_i - d_i) + d_i} \leq 0, \forall y \in [0,1]. \end{cases}$$

Since $c_1 > 0$, set $l_1 = -\frac{\sum_{i=2}^{n} l_i c_i}{c_1}$. We get the new set of conditions for $l_2, \ldots, l_n$.

$$\begin{cases} \sum_{i=2}^{n} l_i d_i = 0, \\ \sum_{i=2}^{n} \frac{l_i d_i (y\bar{q}(c_i - d_i) + c_i)}{y(y\bar{q}(c_i - d_i) + d_i)} \leq 0, \forall y \in [0,1]. \end{cases}$$

Setting $l_i = 0, \forall i \geq 4$, we get

$$\begin{cases} l_2 d_2 + l_3 d_3 = 0, \\ \frac{l_2 d_2 (y\bar{q}(c_2 - d_2) + c_2)}{y\bar{q}(c_2 - d_2) + d_2} + \frac{l_3 d_3 (y\bar{q}(c_3 - d_3) + c_3)}{y\bar{q}(c_3 - d_3) + d_3} \\ \quad \leq 0, \forall y \in [0,1]. \end{cases}$$

Let $l_3 = -\frac{l_2 d_2}{d_3}$. It reduces to show the existence of $(c_2, c_3)$, $(d_2, d_3)$ and $l_2$ such that

$$l_2 d_2 \left( \frac{y\bar{q}(c_2 - d_2) + c_2}{y\bar{q}(c_2 - d_2) + d_2} - \frac{y\bar{q}(c_3 - d_3) + c_3}{y\bar{q}(c_3 - d_3) + d_3} \right) \leq 0, \forall y \in [0,1].$$

This is equivalent to

$$\frac{l_2 d_2 (c_2 d_3 - c_3 d_2)}{(y\bar{q} c_2 + (1 - y\bar{q}) d_2)(y\bar{q} c_3 + (1 - y\bar{q}) d_3)} \leq 0, \forall y \in [0,1].$$

Therefore, by setting $l_2 = \frac{1}{d_2}$ when $c_2 d_3 \leq c_3 d_2$ and setting $l_2 = -\frac{1}{d_2}$ when $c_2 d_3 > c_3 d_2$, we get a particular non-zero $\mathbf{l}$.

$$\mathbf{l} = \begin{cases} \left( \frac{-c_2 d_3 + d_2 c_3}{c_1 d_2 d_3}, \frac{1}{d_2}, -\frac{1}{d_3}, 0, \ldots, 0 \right), \text{ if } c_2 d_3 \leq c_3 d_2 \\ \left( \frac{c_2 d_3 - d_2 c_3}{c_1 d_2 d_3}, -\frac{1}{d_2}, \frac{1}{d_3}, 0, \ldots, 0 \right), \text{ if } c_2 d_3 > c_3 d_2 \end{cases}$$

□

The above lemma means that for a particular $(p, q)$, the existence of a smart and useful genie with cardinality greater or equal to 3 implies the existence of such a genie within smaller cardinalities. In other words, we could stop searching if we do not find any smart and useful genie within binary choices.

Similar argument can be applied to $T_1$ .

## Proof of Theorem 13

*Proof of Theorem 13.* The proof is basically following Csiszar sum lemma and manipulation of mutual information to reduce n-letter expression to 1-letter expression.

$$
\begin{aligned}
& n(R_1 + R_2) - n\epsilon_n \\
& \quad \leq H(M_1) + H(M_2) \\
& \quad \leq I(X_{1,1}^n; Y_{1,1}^n) + I(X_{1,1}^n; Y_{1,1}^n) \\
& \quad \leq I(X_{1,1}^n; Y_{1,1}^n, T_{1,1}^n, S_{2,1}^n) + I(X_{2,1}^n; Y_{2,1}^n, T_{2,1}^n, S_{1,1}^n) \\
& \quad = I(X_{1,1}^n; T_{1,1}^n) + I(X_{1,1}^n; Y_{1,1}^n | T_{1,1}^n, S_{2,1}^n) + I(X_{2,1}^n; T_{2,1}^n) + I(X_{2,1}^n; Y_{2,1}^n | T_{2,1}^n, S_{1,1}^n) \\
& \quad = \underline{H(T_{1,1}^n)} - H(T_{1,1}^n | X_{1,1}^n) + H(Y_{1,1}^n | T_{1,1}^n, S_{2,1}^n) \underline{- H(Y_{1,1}^n | T_{1,1}^n, S_{2,1}^n, X_{1,1}^n)} \\
& \qquad \underline{+ H(T_{2,1}^n)} - H(T_{2,1}^n | X_{2,1}^n) + H(Y_{2,1}^n | T_{2,1}^n, S_{1,1}^n) \underline{- H(Y_{2,1}^n | T_{2,1}^n, S_{1,1}^n, X_{2,1}^n)}
\end{aligned}
$$

Note that for underlined expressions, we have

$$
\begin{aligned}
& H(T_{1,1}^n) - H(Y_{2,1}^n | T_{2,1}^n, S_{1,1}^n X_{2,1}^n) \\
& \quad = H(T_{1,1}^n | S_{1,1}^n) + I(T_{1,1}^n; S_{1,1}^n) - H(Y_{2,1}^n | T_{2,1}^n, S_{1,1}^n, X_{2,1}^n) \\
& \quad = H(T_{1,1}^n | T_{2,1}^n, S_{1,1}^n, X_{2,1}^n) + I(T_{1,1}^n; S_{1,1}^n) - H(Y_{2,1}^n | T_{2,1}^n, S_{1,1}^n, X_{2,1}^n) \\
& \quad = \sum_i \Big( H(T_{1,i} | T_{1,1}^{i-1}, Y_{2,i+1}^n, T_{2,1}^n, S_{1,1}^n, X_{2,1}^n) \\
& \qquad\qquad - H(Y_{2,i} | T_{1,1}^{i-1}, Y_{2,i+1}^n, T_{2,1}^n, S_{1,1}^n, X_{2,1}^n) \Big) + I(T_{1,1}^n; S_{1,1}^n)
\end{aligned}
$$

The last equality is due to Csiszar sum identity. We have

$$
\begin{aligned}
& n(R_1 + R_2) - n\epsilon_n \\
& \quad \leq \sum_i \Big( H(T_{1,i} | T_{1,1}^{i-1}, Y_{2,i+1}^n, T_{2,1}^n, S_{1,1}^n, X_{2,1}^n) - H(Y_{2,i} | T_{1,1}^{i-1}, Y_{2,i+1}^n, T_{2,1}^n, S_{1,1}^n, X_{2,1}^n) \\
& \qquad\qquad - H(T_{1,i} | X_{1,i}) + H(Y_{1,i} | T_{1,i}, S_{2,i}) \\
& \qquad\qquad\quad + H(T_{2,i} | T_{2,1}^{i-1}, Y_{1,i+1}^n, T_{1,1}^n, S_{2,1}^n, X_{1,1}^n) - H(Y_{1,i} | T_{2,1}^{i-1}, Y_{1,i+1}^n, T_{1,1}^n, S_{2,1}^n, X_{1,1}^n) \\
& \qquad\qquad\qquad - H(T_{2,i} | X_{2,i}) + H(Y_{2,i} | T_{2,i}, S_{1,i}) \Big) \\
& \qquad + I(T_{1,1}^n; S_{1,1}^n) + I(T_{2,1}^n; S_{2,1}^n)
\end{aligned}
$$

Use substitution $U_{1,i} = (T_{1,1}^{i-1}, S_{1,1}^{n\backslash i})$, $V_{1,i} = (X_{2,1}^{n\backslash i}, T_{2,1}^{n\backslash i}, Y_{2,i+1}^{n})$,$U_{2,i} = (T_{2,1}^{i-1}, S_{2,1}^{n\backslash i})$, $V_{2,i} = (X_{1,1}^{n\backslash i}, T_{1,1}^{n\backslash i}, Y_{1,i+1}^{n})$,

$$
\begin{aligned}
&n(R_1 + R_2) - n\epsilon_n \\
&\quad= \sum_i \Big( H(T_{1,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) - H(Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) \\
&\qquad\qquad - \underline{H(T_{1,i}|X_{1,i}, S_{1,i})} - I(T_{1,i}; S_{1,i}|X_{1,i}) + H(Y_{1,i}|T_{1,i}, S_{2,i}) \\
&\qquad\qquad + H(T_{2,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) - H(Y_{1,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) \\
&\qquad\qquad\quad - \underline{H(T_{2,i}|X_{2,i}, S_{2,i})} - I(T_{2,i}; S_{2,i}|X_{2,i}) + H(Y_{2,i}|T_{2,i}, S_{1,i}) \Big) \\
&\qquad + I(T_{1,1}^n; S_{1,1}^n) + I(T_{2,1}^n; S_{2,1}^n) \\
&\quad\le \sum_i \Big( \underline{H(T_{1,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i})} - H(Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) \\
&\qquad\qquad - \underline{H(T_{1,i}|X_{1,i}, U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i})} + H(Y_{1,i}|T_{1,i}, S_{2,i}) \\
&\qquad\qquad + \underline{H(T_{2,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i})} - H(Y_{1,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) \\
&\qquad\qquad\quad - \underline{H(T_{2,i}|X_{2,i}, U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i})} + H(Y_{2,i}|T_{2,i}, S_{1,i}) \\
&\qquad\qquad\quad - I(T_{1,i}; S_{1,i}|X_{1,i}) - I(T_{2,i}; S_{2,i}|X_{2,i}) \Big) + I(T_{1,1}^n; S_{1,1}^n) + I(T_{2,1}^n; S_{2,1}^n) \\
&\quad= \sum_i \Big( I(X_{1,i}; T_{1,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) - I(X_{1,i}; Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) \\
&\qquad\qquad - \underline{H(Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}, X_{1,i})} + H(Y_{1,i}|T_{1,i}, S_{2,i}) \\
&\qquad\qquad + I(X_{2,i}; T_{2,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) - I(X_{2,i}; Y_{1,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) \\
&\qquad\qquad\quad - \underline{H(Y_{1,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}, X_{2,i})} + H(Y_{2,i}|T_{2,i}, S_{1,i}) \\
&\qquad\qquad\quad - I(T_{1,i}; S_{1,i}|X_{1,i}) - I(T_{2,i}; S_{2,i}|X_{2,i}) \Big) + I(T_{1,1}^n; S_{1,1}^n) + I(T_{2,1}^n; S_{2,1}^n)
\end{aligned}
$$

Memoryless property indicates that $(U_{1,i}, V_{1,i}) \to (X_{2,i}, X_{1,i}) \to (T_{2,i}, S_{1,i})$ and $(U_{2,i}, V_{2,i}) \to (X_{1,i}, X_{2,i}) \to (T_{1,i}, S_{2,i})$, then

$$
\begin{aligned}
&n(R_1 + R_2) - n\epsilon_n \\
&\quad= \sum_i \Big( I(X_{1,i}; T_{1,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) - I(X_{1,i}; Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) \\
&\qquad\qquad - \underline{H(Y_{2,i}|T_{2,i}, S_{1,i}, X_{2,i}, X_{1,i})} + \underline{H(Y_{1,i}|X_{1,i}, T_{1,i}, S_{2,i})} \\
&\qquad\qquad + I(X_{2,i}; T_{2,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) - I(X_{2,i}; Y_{1,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) \\
&\qquad\qquad\quad - \underline{H(Y_{1,i}|T_{1,i}, S_{2,i}, X_{1,i}, X_{2,i})} + \underline{H(Y_{2,i}|X_{2,i}, T_{2,i}, S_{1,i})} \\
&\qquad\qquad\quad + I(X_{1,i}; Y_{1,i}|T_{1,i}, S_{2,i}) + I(X_{2,i}; Y_{2,i}|T_{2,i}, S_{1,i}) \\
&\qquad\qquad\qquad - I(T_{1,i}; S_{1,i}|X_{1,i}) - I(T_{2,i}; S_{2,i}|X_{2,i}) \Big) + I(T_{1,1}^n; S_{1,1}^n) + I(T_{2,1}^n; S_{2,1}^n) \\
&\quad= \sum_i \Big( I(X_{1,i}; T_{1,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) - I(X_{1,i}; Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i})
\end{aligned}
$$

$$
\begin{aligned}
&\quad + I(X_{2,i};Y_{1,i}|X_{1,i},T_{1,i},S_{2,i}) + I(X_{2,i};T_{2,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) \\
&\qquad - I(X_{2,i};Y_{1,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) + I(X_{1,i};Y_{2,i}|X_{2,i},T_{2,i},S_{1,i}) \\
&\qquad\quad + I(X_{1,i};Y_{1,i}|T_{1,i},S_{2,i}) + I(X_{2,i};Y_{2,i}|T_{2,i},S_{1,i}) \\
&\qquad\qquad - I(T_{1,i};S_{1,i}|X_{1,i}) - I(T_{2,i};S_{2,i}|X_{2,i})\Big) + I(T_{1,1}^n;S_{1,1}^n) + I(T_{2,1}^n;S_{2,1}^n) \\
&= \sum_i \Big( I(X_{1,i};T_{1,i}|U_{1,i},V_{1,i},T_{2,i},S_{1,i},X_{2,i}) - I(X_{1,i};Y_{2,i}|U_{1,i},V_{1,i},T_{2,i},S_{1,i},X_{2,i}) \\
&\quad - I(X_{1,i};T_{1,i}|T_{2,i},S_{1,i},X_{2,i}) + I(X_{1,i};Y_{2,i}|X_{2,i},T_{2,i},S_{1,i}) \\
&\qquad + I(X_{2,i};T_{2,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) - I(X_{2,i};Y_{1,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) \\
&\qquad\quad - I(X_{2,i};T_{2,i}|T_{1,i},S_{2,i},X_{1,i}) + I(X_{2,i};Y_{1,i}|X_{1,i},T_{1,i},S_{2,i}) \\
&\qquad\qquad + \underline{I(X_{1,i};T_{1,i}|T_{2,i},S_{1,i},X_{2,i})} + \underline{I(X_{2,i};T_{2,i}|T_{1,i},S_{2,i},X_{1,i})} \\
&\qquad\qquad\quad + \underline{I(X_{1,i};Y_{1,i}|T_{1,i},S_{2,i})} + \underline{I(X_{2,i};Y_{2,i}|T_{2,i},S_{1,i})} \\
&\qquad\qquad\quad - I(T_{1,i};S_{1,i}|X_{1,i}) - I(T_{2,i};S_{2,i}|X_{2,i})\Big) + I(T_{1,1}^n;S_{1,1}^n) + I(T_{2,1}^n;S_{2,1}^n) \\
&= \sum_i \Big( I(X_{1,i};T_{1,i}|U_{1,i},V_{1,i},T_{2,i},S_{1,i},X_{2,i}) - I(X_{1,i};Y_{2,i}|U_{1,i},V_{1,i},T_{2,i},S_{1,i},X_{2,i}) \\
&\quad - I(X_{1,i};T_{1,i}|T_{2,i},S_{1,i},X_{2,i}) + I(X_{1,i};Y_{2,i}|X_{2,i},T_{2,i},S_{1,i}) \\
&\qquad + I(X_{2,i};T_{2,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) - I(X_{2,i};Y_{1,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) \\
&\qquad\quad - I(X_{2,i};T_{2,i}|T_{1,i},S_{2,i},X_{1,i}) + I(X_{2,i};Y_{1,i}|X_{1,i},T_{1,i},S_{2,i}) \\
&\qquad\qquad + I(X_{1,i};T_{1,i}|S_{1,i}) + I(X_{2,i};T_{2,i}|S_{2,i}) + I(X_{1,i};Y_{1,i}T_{1,i}|S_{2,i}) \\
&\qquad\qquad\quad - \underline{I(X_{1,i};T_{1,i}|S_{2,i})} + I(X_{2,i};Y_{2,i},T_{2,i}|S_{1,i}) - \underline{I(X_{2,i};T_{2,i}|S_{1,i})} \\
&\qquad\qquad\quad - \underline{I(T_{1,i};S_{1,i}|X_{1,i})} - \underline{I(T_{2,i};S_{2,i}|X_{2,i})}\Big) + I(T_{1,1}^n;S_{1,1}^n) + I(T_{2,1}^n;S_{2,1}^n) \\
&= \sum_i \Big( I(X_{1,i};T_{1,i}|U_{1,i},V_{1,i},T_{2,i},S_{1,i},X_{2,i}) - I(X_{1,i};Y_{2,i}|U_{1,i},V_{1,i},T_{2,i},S_{1,i},X_{2,i}) \\
&\quad - I(X_{1,i};T_{1,i}|T_{2,i},S_{1,i},X_{2,i}) + I(X_{1,i};Y_{2,i}|X_{2,i},T_{2,i},S_{1,i}) \\
&\qquad + I(X_{2,i};T_{2,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) - I(X_{2,i};Y_{1,i}|U_{2,i},V_{2,i},T_{1,i},S_{2,i},X_{1,i}) \\
&\qquad\quad - I(X_{2,i};T_{2,i}|T_{1,i},S_{2,i},X_{1,i}) + I(X_{2,i};Y_{1,i}|X_{1,i},T_{1,i},S_{2,i}) + I(X_{1,i};T_{1,i}|S_{1,i}) \\
&\qquad\qquad + I(X_{2,i};T_{2,i}|S_{2,i}) + I(X_{1,i};Y_{1,i},T_{1,i}|S_{2,i}) + I(X_{2,i};Y_{2,i},T_{2,i}|S_{1,i}) \\
&\qquad\qquad\quad - I(T_{1,i};S_{1,i},X_{1,i}) - I(T_{2,i};S_{2,i},X_{2,i})\Big) + I(T_{1,1}^n;S_{1,1}^n) + I(T_{2,1}^n;S_{2,1}^n)
\end{aligned}
$$

When genies has degraded order, say $X_1 \to T_1 \to S_1$, we have

$$
\begin{aligned}
I(T_{1,1}^n;S_{1,1}^n) &= H(S_{1,1}^n) - H(S_{1,1}^n|T_{1,1}^n) \\
&\leq \sum_i H(S_{1,i}) - H(S_{1,i}|S_{1,1}^{i-1},T_{1,1}^n) \\
&= \sum_i H(S_{1,i}) - H(S_{1,i}|T_{1,i})
\end{aligned}
$$

$$= \sum_i I(T_{1,i}; S_{1,i})$$

since $(S_{1,1}^{i-1}, T_{1,1}^{n\setminus i}) \to X_{1,i} \to T_{1,i} \to S_{1,i}$.

$$\begin{aligned}
&n(R_1 + R_2) - n\epsilon_n \\
&\quad \le \sum_i I(X_{1,i}; T_{1,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) - I(X_{1,i}; Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) \\
&\qquad - I(X_{1,i}; T_{1,i}|T_{2,i}, S_{1,i}, X_{2,i}) + I(X_{1,i}; Y_{2,i}|X_{2,i}, T_{2,i}, S_{1,i}) \\
&\qquad + I(X_{2,i}; T_{2,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) - I(X_{2,i}; Y_{1,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) \\
&\qquad - I(X_{2,i}; T_{2,i}|T_{1,i}, S_{2,i}, X_{1,i}) + I(X_{2,i}; Y_{1,i}|X_{1,i}, T_{1,i}, S_{2,i}) \\
&\qquad + \underline{I(X_{1,i}; T_{1,i}|S_{1,i})} + \underline{I(X_{2,i}; T_{2,i}|S_{2,i})} + I(X_{1,i}; Y_{1,i}, T_{1,i}|S_{2,i}) \\
&\qquad + I(X_{2,i}; Y_{2,i}, T_{2,i}|S_{1,i}) - \underline{I(T_{1,i}; S_{1,i}, X_{1,i})} - \underline{I(T_{2,i}; S_{2,i}, X_{2,i})} \\
&\qquad + \underline{I(T_{1,i}; S_{1,i})} + \underline{I(T_{2,i}; S_{2,i})} \\
&\quad = \sum_i I(X_{1,i}; T_{1,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) - I(X_{1,i}; Y_{2,i}|U_{1,i}, V_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) \\
&\qquad - I(X_{1,i}; T_{1,i}|T_{2,i}, S_{1,i}, X_{2,i}) + I(X_{1,i}; Y_{2,i}|X_{2,i}, T_{2,i}, S_{1,i}) \\
&\qquad + I(X_{2,i}; T_{2,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) - I(X_{2,i}; Y_{1,i}|U_{2,i}, V_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) \\
&\qquad - I(X_{2,i}; T_{2,i}|T_{1,i}, S_{2,i}, X_{1,i}) + I(X_{2,i}; Y_{1,i}|X_{1,i}, T_{1,i}, S_{2,i}) \\
&\qquad + I(X_{1,i}; Y_{1,i}, T_{1,i}|S_{2,i}) + I(X_{2,i}; Y_{2,i}, T_{2,i}|S_{1,i}) \\
&\quad = \sum_i I(X_{1,i}; T_{1,i}|U_{1,i}, S_{1,i}) - I(X_{1,i}; Y_{2,i}|U_{1,i}, T_{2,i}, S_{1,i}, X_{2,i}) \\
&\qquad - I(X_{1,i}; T_{1,i}|S_{1,i}) + I(X_{1,i}; Y_{2,i}|X_{2,i}, T_{2,i}, S_{1,i}) \\
&\qquad + I(X_{2,i}; T_{2,i}|U_{2,i}, S_{2,i}) - I(X_{2,i}; Y_{1,i}|U_{2,i}, T_{1,i}, S_{2,i}, X_{1,i}) \\
&\qquad - I(X_{2,i}; T_{2,i}|S_{2,i}) + I(X_{2,i}; Y_{1,i}|X_{1,i}, T_{1,i}, S_{2,i}) \\
&\qquad + I(X_{1,i}; Y_{1,i}, T_{1,i}|S_{2,i}) + I(X_{2,i}; Y_{2,i}, T_{2,i}|S_{1,i})
\end{aligned}$$

The last equality is due to the fact that $(X_1, T_1, S_1, U_1, V_2)$ is independent of $(X_2, T_2, S_2, U_2, V_1)$. More over, memoryless property suggests that channel structure remains the same when conditioned on $U = u$. Hence we have

$$\begin{aligned}
\mathscr{C}_s \le \max_{p_1(x_1)p_2(x_2)} & I(X_1; Y_1, T_1|S_2) + I(X_2; Y_2, T_2|S_1) \\
&+ \mathfrak{C}\left(I(X_1; T_1|T_2, S_1, X_2) - I(X_1; Y_2|T_2, S_1, X_2)\right) \\
&- I(X_1; T_1|T_2, S_1, X_2) + I(X_1; Y_2|X_2, T_2, S_1) \\
&+ \mathfrak{C}\left(I(X_2; T_2|T_1, S_2, X_1) - I(X_2; Y_1|T_1, S_2, X_1)\right) \\
&- I(X_2; T_2|T_1, S_2, X_1) + I(X_2; Y_1|X_1, T_1, S_2) \qquad \square
\end{aligned}$$

# Chapter 4

# Sub-optimality of the Han–Kobayashi inner bound for the capacity region of the interference channel

We show with a counterexample that Han–Kobayashi (HK) inner bound is sub-optimal for interference channel.

The capacity region is known under a small set of interference instantiations such as strong interference and injective deterministic interference. The sum capacity is established for a larger class of channels such as Gaussian interference channel with mixed or very weak interference. In all the cases mentioned above the capacity region (or the sum-capacity) matches the one given by $\mathscr{H}$. Furthermore, it was not known whether $\mathscr{H}$ is the capacity region $\mathscr{C}$ or not. In this thesis, we show that there are channel instances where $\mathscr{H} \subsetneq \mathscr{C}$; thus showing the sub-optimality of the HK region.

The main innovation of our work lies in the choice of the channel realizations because the computation of the HK region is not particularly straightforward. We study a class of interference channels, defined as CZI channels in the next section, where the evaluation of $\mathscr{H}$ becomes significantly simplified[1]. We take particular channels inside this class and compute a (normalized) two-letter achievable region of the corresponding two-letter product channel. We show that there are many examples where the (normalized) two-letter achievable region considered is strictly larger than $\mathscr{H}$, which indicates $\mathscr{H} \subsetneq \mathscr{C}$.

[1] The analysis in Chapter 3 is along very similar lines but we were unable to identify examples where the (normalized) two-letter achievable region of a two-letter product channel becomes larger than the original $\mathscr{H}$.

## 4.1 CZI Channel

We say that an interference channel has clean Z interference (CZI) if one of the sub channels is a clean channel. We choose the channel from $X_2$ to $Y_2$ to be clean as depicted in Figure 4.1 and study its HK region.

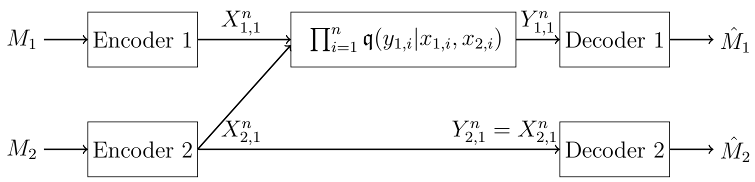


Figure 4.1: Discrete memoryless CZI channel

The following proposition reveals an equivalent characterization of the HK region for CZI channels which simplifies its evaluation.

**Proposition 7.** *The HK region of a CZI channel is identical to the union of rate pairs* $(R_1, R_2)$ *that satisfy*

$$R_1 < I(X_1; Y_1|U_2, Q), \tag{4.1}$$

$$R_2 < H(X_2|Q), \tag{4.2}$$

$$R_1 + R_2 < I(X_1, U_2; Y_1|Q) + H(X_2|U_2, Q) \tag{4.3}$$

*for some pmf* $p(q)p(u_2|q)p(x_2|u_2)p(x_1|q)$*, where* $|U_2| \leq |X_2|$ *and* $|Q| \leq 2$*.*

*Proof.* First of all, it is a simple exercise to note that the HK region of a CZI channel reduces to the three constraints above by setting $U_1 = \phi$. Hence, the above region is a subset of the HK region.

Conversely, (4.1) is identical to (1.1) of the HK region. (4.2) and (4.3) are respectively looser constraints than (1.2) and (1.3) of the HK region, which makes the above region larger than the original HK region. Thus proving equivalence.

Note that the changes in cardinality of $U_2$ and $Q$ follow from standard applications of cardinality reduction techniques all while the underlying region remains the same. Therefore, we do not have to take these changes into account when talking about the two regions' equivalence. □

The first result that we present below is a result that shows the *optimality* of the HK region along certain directions.

**Proposition 8.** *For a CZI channel,*

$$\max_{\mathscr{H}}(\lambda R_1 + R_2) = \max_{\mathscr{C}}(\lambda R_1 + R_2), \forall \lambda \leq 1.$$

*Proof.* A standard converse/outer-bound argument proves that treating interference as noise is optimal.

$$
\begin{aligned}
& n(\lambda R_1 + R_2) - n\epsilon_n \\
& \overset{(a)}{\leq} H(X_{2,1}^n | X_{1,1}^n) + \lambda I(X_{1,1}^n; Y_{1,1}^n) \\
& = \sum_{i=1}^{n} H(X_{2,i}|X_{1,i}) - I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}|X_{1,i}) \\
& \qquad + \lambda I(X_{1,1}^n; Y_{1,i}|Y_{1,i+1}^n) \\
& \leq \sum_{i=1}^{n} H(X_{2,i}|X_{1,i}) - I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}|X_{1,i}) \\
& \qquad + \lambda I(X_{1,1}^n, Y_{1,i+1}^n; Y_{1,i}) \\
& = \sum_{i=1}^{n} H(X_{2,i}|X_{1,i}) - I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}|X_{1,i}) \\
& \qquad + \lambda \left( I(X_{1,1}^n, Y_{1,i+1}^n, X_{2,1}^{i-1}; Y_{1,i}) - I(X_{2,1}^{i-1}; Y_{1,i}|X_{1,1}^n, Y_{1,i+1}^n) \right) \\
& \overset{(b)}{=} \sum_{i=1}^{n} H(X_{2,i}|X_{1,i}) - I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}|X_{1,i}) \\
& \qquad + \lambda \left( I(X_{1,1}^n, Y_{1,i+1}^n, X_{2,1}^{i-1}; Y_{1,i}) - I(Y_{1i+1}^n; X_{2,i}|X_{1,1}^n, X_{2,1}^{i-1}) \right) \\
& = \sum_{i=1}^{n} H(X_{2,i}|X_{1,i}) - (1-\lambda) I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}|X_{1,i}) \\
& \qquad - \lambda \Big( I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}, Y_{1i+1}^n|X_{1,i}) \\
& \qquad\qquad - I(X_{1,1}^n, Y_{1,i+1}^n, X_{2,1}^{i-1}; Y_{1,i}) \Big) \\
& = \sum_{i=1}^{n} H(X_{2,i}|X_{1,i}) - (1-\lambda) I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}|X_{1,i}) \\
& \qquad + \lambda I(X_{1,i}; Y_{1,i}) - \lambda \Big( I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}, Y_{1,i+1}^n|X_{1,i}) \\
& \qquad\qquad\qquad - I(X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}, Y_{1,i+1}^n; Y_{1,i}|X_{1,i}) \Big) \\
& \overset{(c)}{=} \sum_{i=1}^{n} H(X_{2,i}|X_{1,i}) + \lambda I(X_{1,i}; Y_{1,i}) \\
& \qquad - (1-\lambda) I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}|X_{1,i}) \\
& \qquad\quad - \lambda I(X_{2,i}; X_{1,1}^{n\setminus i}, X_{2,1}^{i-1}, Y_{1,i+1}^n|Y_{1,i}, X_{1,i}) \\
& \leq n \left( \max(H(X_2) + \lambda I(X_1; Y_1) \right),
\end{aligned}
$$

where (a) follows from Fano's inequality, (b) Csiszar sum identity and (c) properties of the Markov chain formed by $Y_{1,i} \to (X_{1,i}, X_{2,i}) \to \big(X_{2,1}^{i-1}, X_{1,1}^{n\setminus i}, Y_{1,i+1}^n\big)$.

Since $\epsilon > 0$ is arbitrary, we see that any achievable rate pair must satisfy

$$\lambda R_1 + R_2 \le \max_{p_1(x_1)p_2(x_2)} H(X_2) + \lambda I(X_1; Y_1),$$

which is achievable by treating interference as noise, or more precisely, setting $U_2 = \phi$ in the HK region. Hence, the proposition is established. □

On the contrary, we will see that, for some channels,

$$\max_{\mathscr{H}}(\lambda R_1 + R_2) < \max_{\mathscr{C}}(\lambda R_1 + R_2)$$

when $\lambda$ becomes larger than 1. The following lemma helps us evaluate the quantity $\max_{\mathscr{H}}(\lambda R_1 + R_2)$.

**Lemma 5.** *For a CZI channel, for all $\lambda > 1$*

$$\begin{aligned}
&\max_{\mathscr{H}}(\lambda R_1 + R_2) \\
&\quad = \max_{p_1(x_1)p_2(x_2)} \Big\{ I(X_1, X_2; Y_1) + \mathop{\mathfrak{C}}_{p_2(x_2)} \big( H(X_2) - I(X_2; Y_1|X_1) + (\lambda - 1) I(X_1; Y_1) \big) \Big\},
\end{aligned} \tag{4.4}$$

*where $\mathop{\mathfrak{C}}_{x}[f(x)]$ of $f(x)$ is the upper concave envelope of $f(x)$ over $x$.*

*Proof.* For any $(R_1, R_2) \in \mathscr{H}$, there must exist a distribution $p(q)p_2(u_2, x_2|q)p_1(x_1|q)$ such that

$$\begin{aligned}
\lambda R_1 + R_2 &\le (\lambda - 1) I(X_1; Y_1|U_2, Q) \\
&\qquad + I(X_1, U_2; Y_1|Q) + H(X_2|U_2, Q) \\
&= I(X_1, X_2; Y_1|Q) + H(X_2|U_2, Q) \\
&\qquad - I(X_2; Y_1|U_2, X_1, Q) + (\lambda - 1) I(X_1; Y_1|U_2, Q) \\
&\overset{(d)}{\le} I(X_1, X_2; Y_1|Q) + \mathop{\mathfrak{C}}_{p_2(x_2|q)} \big( H(X_2|Q) \\
&\qquad - I(X_2; Y_1|X_1, Q) + (\lambda - 1) I(X_1; Y_1|Q) \big),
\end{aligned}$$

where $(d)$ follows directly from the definition of the upper concave envelope.

Since $Q$ computes an average, and since the average is less than the maximum, we obtain that

$$\begin{aligned}
&\max_{\mathscr{H}}(\lambda R_1 + R_2) \\
&\quad \le \max_{p_1(x_1)p_2(x_2)} \Big\{ I(X_1, X_2; Y_1) + \mathop{\mathfrak{C}}_{p_2(x_2)} \big( H(X_2) - I(X_2; Y_1|X_1) + (\lambda - 1) I(X_1; Y_1) \big) \Big\}.
\end{aligned}$$

On the other hand, for any $p_2(u_2, x_2)p_1(x_1)$, the following rate pair

$$(R_1, R_2) = (I(X_1; Y_1|U_2),\ H(X_2|U_2) + I(U_2; Y_1))$$

belongs to $\mathscr{H}$ as it satisfies the constraints.

Thus,

$$\begin{aligned}
&\max_{\mathscr{H}}(\lambda R_1 + R_2)\\
&\quad\geq \max_{p_2(u_2,x_2)p_1(x_1)} \lambda I(X_1; Y_1|U_2) + H(X_2|U_2) + I(U_2; Y_1)\\
&\quad= \max_{p_2(u_2,x_2)p_1(x_1)} I(X_1, U_2; Y_1) + H(X_2|U_2) + (\lambda - 1)I(X_1; Y_1|U_2)\\
&\quad= \max_{p_2(u_2,x_2)p_1(x_1)} I(X_1, X_2; Y_1) + H(X_2|U_2)\\
&\qquad\qquad - I(X_2; Y_1|U_2, X_1) + (\lambda - 1)I(X_1; Y_1|U_2)\\
&\quad\overset{(e)}{=} \max_{p_2(x_2)p_1(x_1)} I(X_1, X_2; Y_1)\\
&\qquad + \mathfrak{C}_{p_2(x_2)}\big(H(X_2) - I(X_2; Y_1|X_1) + (\lambda - 1)I(X_1; Y_1)\big),
\end{aligned}$$

where (e) follows directly from the definition of the upper concave envelope. This establishes the converse and completes the proof of the lemma. □

By viewing the channel use across two consecutive time-slots as the channel use of a single time-slot of the corresponding product channel, we obtain what is usually termed the *two-letter* realization of the original channel. For the two letter product channel of a CZI channel, the transition probability satisfies

$$\tilde{\mathfrak{q}}(y_{1,1}, y_{1,2}|x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = \mathfrak{q}(y_{1,1}|x_{1,1}x_{2,1})\mathfrak{q}(y_{1,2}|x_{1,2}, x_{2,2}),$$

where $\mathfrak{q}$ is the transition probability of the CZI channel.

**Proposition 9.** *The set of rate pairs satisfying*

$$\begin{aligned}
R_1 &= \frac{1}{2}I(X_{1,1}, X_{1,2}; Y_{1,1}, Y_{1,2}|Q),\\
R_2 &= \frac{1}{2}H(X_{2,1}, X_{2,2}|Q),
\end{aligned}$$

*for some pmf* $p(q)p(x_{1,1}, x_{1,2}|q)p(x_{2,1}, x_{2,2}|q)$ *with* $|Q| \leq 2$ *is achievable by the original channel.*

*Proof.* This rate pair is precisely the *treating-interference-as-noise* rate pair of the two-letter channel, and the normalization by $\frac{1}{2}$ indicates is due to the fact that we code over two time-slots of the original channel. □

We denote this (normalized) region as $\mathscr{H}^2$. Note that this is not the (normalized) two-letter HK Region but rather a region contained inside the two-letter HK region. HK region is sub-optimal if $\max_{\mathscr{H}}(\lambda R_1 + R_2) < \max_{\mathscr{H}^2}(\lambda R_1 + R_2)$.

### 4.1.1 Sub-optimality of the HK region

In this part we provide several CZI channels for which, for some fixed ($\lambda > 1$), $\max_{\mathscr{H}^2}(\lambda R_1 + R_2)$ becomes larger than $\max_{\mathscr{H}}(\lambda R_1 + R_2)$, which proves the sub-optimality of the HK region.

Examples are of channels with binary input/output. A $2 \times 2$ matrix is used to represent the channel:

$$\begin{aligned}&\mathfrak{q}(y_1|x_1, x_2)\\&= \begin{bmatrix} P(Y_1 = 0|X_1, X_2 = 0, 0) & P(Y_1 = 0|X_1, X_2 = 0, 1) \\ P(Y_1 = 0|X_1, X_2 = 1, 0) & P(Y_1 = 0|X_1, X_2 = 1, 1) \end{bmatrix}.\end{aligned}$$

The fact that $X_2$ is binary allows us to compute the upper concave envelope in Lemma 5 with extremely high precision.

The channels in Table 4.1 are obtained using numerical methods. We prove, as a demonstration, in the Appendix that the difference in rates of the first channel listed above is not due to numerical errors and that the maximum single-letter rate is indeed strictly smaller than the maximum (normalized) two-letter rate achieved by the corresponding two-letter product channel.

### 4.1.2 Intuition and a natural modification

In this section, we present an intuition as well as a coding strategy motivated by this intuition that indicates how one may improve on the Han–Kobayashi encoding scheme.

The counterexamples we generated in the last section had the following feature: even though $\lambda$ was strictly larger than one, the optimal $U_2$ that yielded $\max_{\mathscr{H}}(\lambda R_1 + R_2)$ was still the trivial random variable; implying that there were distributions $p_1(x_1)$ and $p_2(x_2)$ such that

$$R_1 = I(X_1; Y_1), \quad R_2 = I(X_2; Y_2) = H(X_2)$$

yielded the maximum weighted sum-rate.

Suppose we now go to the two-letter product channel and take the product distribution of the marginals that yielded the one letter maximum as the transmitter distribution, clearly we would get the same rate. It is an easy exercise

Table 4.1: Table of counter-examples

| $\lambda$ | channel | $\max_{\mathscr{H}}(\lambda R_1 + R_2)$ | $\max_{\mathscr{H}^2}(\lambda R_1 + R_2)$ |
|---|---|---|---|
| 2 | $\begin{bmatrix} 1 & 0.5 \\ 1 & 0 \end{bmatrix}$ | 1.107516 | 1.108141 |
| 2.5 | $\begin{bmatrix} 0.204581 & 0.364813 \\ 0.030209 & 0.992978 \end{bmatrix}$ | 1.159383 | 1.169312 |
| 3 | $\begin{bmatrix} 0.591419 & 0.865901 \\ 0.004021 & 0.898113 \end{bmatrix}$ | 1.241521 | 1.255814 |
| 3 | $\begin{bmatrix} 0.356166 & 0.073253 \\ 0.985504 & 0.031707 \end{bmatrix}$ | 1.292172 | 1.311027 |
| 3 | $\begin{bmatrix} 0.287272 & 0.459966 \\ 0.113711 & 0.995405 \end{bmatrix}$ | 1.117253 | 1.123151 |
| 4 | $\begin{bmatrix} 0.429804 & 0.147712 \\ 0.948192 & 0.002848 \end{bmatrix}$ | 1.181392 | 1.196189 |
| 4 | $\begin{bmatrix} 0.068730 & 0.443630 \\ 0.011377 & 0.954887 \end{bmatrix}$ | 1.223409 | 1.243958 |
| 5 | $\begin{bmatrix} 0.969199 & 0.564440 \\ 0.954079 & 0.061409 \end{bmatrix}$ | 1.351229 | 1.372191 |
| 5 | $\begin{bmatrix} 0.943226 & 0.447252 \\ 0.950791 & 0.024302 \end{bmatrix}$ | 1.231254 | 1.250564 |
| 6 | $\begin{bmatrix} 0.943292 & 0.045996 \\ 0.589551 & 0.202487 \end{bmatrix}$ | 1.069405 | 1.076932 |
| 6 | $\begin{bmatrix} 0.714431 & 0.019375 \\ 0.955918 & 0.448539 \end{bmatrix}$ | 1.528508 | 1.541781 |
| 7 | $\begin{bmatrix} 0.058449 & 0.558649 \\ 0.194915 & 0.959172 \end{bmatrix}$ | 1.424974 | 1.452769 |
| 7 | $\begin{bmatrix} 0.033312 & 0.876067 \\ 0.286125 & 0.992825 \end{bmatrix}$ | 1.179438 | 1.187867 |
| 10 | $\begin{bmatrix} 0.307723 & 0.874843 \\ 0.032090 & 0.710535 \end{bmatrix}$ | 1.370830 | 1.388674 |
| 15 | $\begin{bmatrix} 0.946802 & 0.311909 \\ 0.730770 & 0.155075 \end{bmatrix}$ | 1.391596 | 1.406325 |
| 100 | $\begin{bmatrix} 0.382410 & 0.081474 \\ 0.584797 & 0.241840 \end{bmatrix}$ | 3.754016 | 3.789316 |
| 100 | $\begin{bmatrix} 0.673979 & 0.194596 \\ 0.781192 & 0.285216 \end{bmatrix}$ | 1.711938 | 1.730715 |

to verify that $I(X_1;Y_1)$ is convex in $X_2$ (utilizing the fact that $X_1$ and $X_2$ are independent). Thus a perturbation of the product distribution into two distributions that preserve the average would reduce $R_2 = \frac{1}{2}H(X_{2,1}, X_{2,2})$ but increase $R_1 = \frac{1}{2}I(X_{1,1}, X_{1,2}; Y_{1,1}, Y_{1,2})$. Since we are interested in $\lambda R_1 + R_2$ with $\lambda > 1$, it is conceivable that such a perturbation would increase the weighted sum-rate.

Note that $X_2$ acts like a state variable on the communication of the channel between $X_1$ and $Y_1$. If the channel from $X_1 \to Y_1$, with $X_2$ as the state, is not memoryless, we know that the optimal code distributions on $X_{1,1}^n$ are not independent distributions.

For instance, if one creates $X_{2,1}^n$ according to a first-order Markov process, the channel from $X_{1,1}^n$ to $Y_{1,1}^n$ becomes a channel whose state varies like a first order Markov process. For such a coding strategy, one could achieve $R_2 = \bar{H}(X_2)$, $R_1 = \bar{C}(X_1;Y_1)$, where $\bar{H}(X_2)$ denotes the entropy rate of the Markov process $X_{2,1}^n$ and $\bar{C}(X_1;Y_1)$ denotes the capacity of the channel whose state varies according to $X_{2,1}^n$.

Note that in general $\bar{C}(X_1;Y_1)$ does not have a closed form and is quite hard to compute; but this scheme, as opposed to block coding, appears to be a natural fit for interference channels. It would also explain why i.i.d. coding (in the sense of Han–Kobayashi) might not be optimal for a CZI channel.

## 4.2 Conclusion

We have shown in the paper that Han–Kobayashi achievable region is strictly sub-optimal, which makes finding new ways of modeling achievable regions for interference channels almost a necessity in the future.

# Appendix

### Analysis of a particular example

Consider the CZI channel depicted in Figure 4.2 where $\mathfrak{q}(y_1|x_1,x_2)$ is illustrated as two point-to-point channels $X_1 \to Y_1$ depending on the choice of $X_2$. We show the details of computing $\mathscr{H}$ when $\lambda = 2$.

By Lemma 4.2

$$\max_{\mathscr{H}}(2R_1 + R_2) = \max_{p_1(x_1)p_2(x_2)} \Big\{ I(X_1, X_2; Y_1) + \mathop{\mathfrak{C}}_{p_2(x_2)} \big( H(X_2) - I(X_2;Y_1|X_1) + I(X_1;Y_1) \big) \Big\}. \tag{4.5}$$

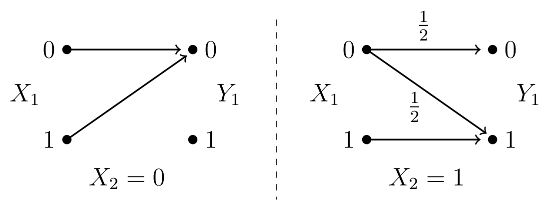


Figure 4.2: Binary CZI channel

Let $P(X_1 = 0) = p$ and $P(X_2 = 0) = q$. Define

$$\begin{aligned} f(p,q) :=& H(X_2) - I(X_2; Y_1|X_1) + I(X_1; Y_1) \\ =& H_b(q) - 2pH_b(\frac{q+1}{2}) - 2\bar{p}H_b(q) + H_b(q + \frac{p}{2}\bar{q}) + p\bar{q}. \end{aligned} \tag{4.6}$$

Here $H_b(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ denotes the binary entropy function.

Thus, we obtain that

$$\max_{\mathscr{H}}(2R_1 + R_2) = \max_{p,q} \left\{ H_b(q + \frac{p}{2}\bar{q}) - p\bar{q} + \mathfrak{C}_q\big(H_b(q + \frac{p}{2}\bar{q}) - p\bar{q}\big) \right\}. \tag{4.7}$$

Clearly the main computational imprecision may[2] arise from the estimation of the concave envelope; however as the next result shows; for this channel we obtain an explicit characterization of the concave envelope.

**Lemma 6.** *Consider the bivariate function $f(p,q)$ as defined in (4.6) where $(p,q) \in [0,1] \times [0,1]$. Then*

*(i) if $p > \frac{1}{2}$,*

$$\mathfrak{C}_q[f(p,q)] = f(p,q).$$

*(ii) if $p \leq \frac{1}{2}$,*

$$\mathfrak{C}_q[f(p,q)] = \begin{cases} f(p,q) & q \geq 1-2p \\ \frac{f(p,1-2p)-f(p,0)}{1-2p}q + f(p,0) & o.w. \end{cases}.$$

*Proof.* The second derivative with respect to $q$ is

$$\frac{\partial^2 f(p,q)}{\partial q^2} = \frac{p}{q\bar{q}\ln 2}\frac{(1-3q-2p\bar{q})}{(1+q)(2q+p\bar{q})} \tag{4.8}$$

If $p \in (\frac{1}{2}, 1)$, then (4.8) is negative for $q \in (0,1)$, i.e., if $p > \frac{1}{2}$, $f(p,q)$ is concave in $q$ and $\mathfrak{C}_q[f(p,q)] = f(p,q)$.

[2] In general since the concave envelope is computed over a single variable and the function is rather well behaved (at most two inflection points) when $X_2$ is binary, numerical computations using Matlab have yielded very high precision results even for the other counter examples listed.

If $p \in (0, \frac{1}{2})$, then (4.8) has one solution, $q^* \in (0,1)$.

$$q^* = \frac{1-2p}{3-2p}.$$

In fact, $f(p,q)$ is convex for $q \in (0, q*)$ and concave for $q \in (q*, 1)$. Thus $\underset{q}{\mathfrak{C}}[f(p,q)]$ consists of two parts. First part is a tangent line from the point $f(p,0)$ to the function $f(p,\hat{q})$ and the second part is equal to $f(p,q)$.

To find the point where the tangent line meets the function, $(\hat{q})$, we need to solve the following equation

$$\frac{f(p,\hat{q}) - f(p,0)}{\hat{q}} = \frac{\partial f(p,q)}{\partial q}\Big|_{\hat{q}}.$$

Because the function is initially convex and then concave, the above equation will have at most one solution $\hat{q} \neq 0$. One can verify that $\hat{q} = 1-2p$ is the required solution, and this completes the proof. $\square$

Define $F(p,q)$ for $(p,q) \in [0,1] \times [0,1]$ as

$$F(p,q) = \begin{cases} H_b(q + \frac{p}{2}\bar{q}) - p\bar{q} + f(p,q) & q \geq 1-2p \\ H_b(q + \frac{p}{2}\bar{q}) - p\bar{q} + \frac{f(p,1-2p)-f(p,0)}{1-2p} q + f(p,0) & \text{o.w.} \end{cases} \tag{4.9}$$

where $f(p,q)$ is defined in (4.6).

From Lemma 6 and (4.7), we know that

$$\max_{\mathscr{H}}(2R_1 + R_2) = \max_{p,q} F(p,q). \tag{4.10}$$

A plot of $F(p,q)$ is shown in Figure 4.3 along with a zoom-in plot on the maximizing point.

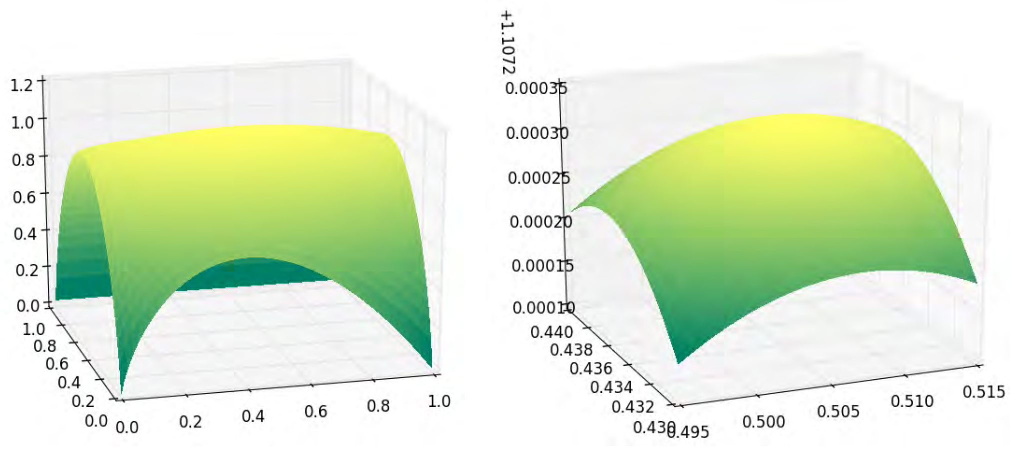


Figure 4.3: $F(p,q)$

We see that $F(p,q)$ is very well behaved and hence it is easy to achieve high precisions in calculations. A tedious exercise shows that the concave envelope of $F(p,q)$ w.r.t. $(p,q)$ matches the function value $F(p_0,q_0)$ at[3] $(p_0,q_0) = (0.507829413, 0.436538150)$. Hence an upper bound on $\max_{\mathscr{H}}(2R_1+R_2)$ is given by maximum value of the supporting hyperplane to $F(p,q)$ at $p_0, q_0$, which is in turn upper bounded by $F(p_0,q_0)+|a|+|b|$ where $a = \frac{\partial F}{\partial p}\Big|_{p_0}$, and $b = \frac{\partial F}{\partial q}\Big|_{q_0}$. Evaluating the values we obtain an upper bound given by

$$\max_{\mathscr{H}}(2R_1 + R_2) \le 1.107577. \tag{4.11}$$

On the other hand consider the following point in $\mathscr{H}^2$ given by

$$R_1 = \frac{1}{2} I(X_{1,1}, X_{1,2}; Y_{1,1}, Y_{1,2}), \quad R_2 = \frac{1}{2} H(X_{2,1}, H_{2,2}|Q),$$

where $P(X_{1,1}, X_{1,2} = (0,0)) = p_0$, $P(X_{1,1}, X_{1,2} = (1,1)) = 1-p_0$, $P((X_{2,1}, X_{2,2}) = (0,0)) = 0.36q_0$, $P((X_{2,1}, X_{2,2}) = (0,1)) = P((X_{2,1}, X_{2,2}) = (1,0)) = 0.64q_0$ and $P((X_{2,1}, X_{2,2}) = (1,1)) = 1 - 1.64q_0$. For this choice of distribution we get $2R_1 + R_2 = 1.1080356$, which is strictly larger than the bound given in (4.11). This establishes the sub-optimality of the Han–Kobayashi region for the particular example considered in the Appendix.

As mentioned in Section 4.1.2 the distribution of $(X_{2,1}, X_{2,2})$ that outperforms the one-letter region is not the product distribution; but more surprisingly one is doing *repetition coding* on $X_{1,1}, X_{1,2}$.

[3] We choose a point that is numerically very close to the true maximum.

# Chapter 5

# Summary

This thesis considers two of the most basic settings in network information theory where the capacity regions are unknown, namely the broadcast channel and the interference channel.

For broadcast channel, we showed via counterexamples that superposition-coding region is sub-optimal for three-receiver more-capable channel. Furthermore, we showed that Marton's inner bound actually achieves capacity of the counterexamples.

For interference channel, we proposed the concept of very weak interference which significantly simplified the expression of achievable sum-rate of Han–Kobayashi inner bound and made simulations possible at last. Han–Kobayahi inner bound is Marton's inner bound's counter part in interference channel in the sense that it subsumes all known inner bound and achieves the capacity for all interference channels where capacity is known. In fact, we showed that it also achieves sum-rate capacity for a new set of channels with a newly developed genie-based outer bound.

In the future, we would like to find the intrinsic reasons as to why some information theoretic expressions tensorize while others do not. We know that only those achievable regions which tensorizes can represent capacity region but we do not have efficient ways of identifying them. This thesis demonstrates that computational techniques, coupled with identifying extremal distributions, can be useful both in proving sub-optimality of certain achievable regions as well as reduction of outer bounds to achievable regions for special channel structures. This is still a largely underutilized and relatively unknown direction of research in network information theory.

# Appendix A

# Information measures and properties

Let $X \sim p(x)$, then we define entropy of $X$ as

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x) = -E_X(\log p(X))$$

which measures the uncertainty in the outcome of $X$.

Further, let $Y|\{X = x\} \sim p(y|x)$ for every $x$, we define the conditional entropy $H(Y|X)$ as the average of $H(Y|X = x)$ over $X$, i.e.,

$$\begin{aligned} H(Y|X) &= \sum_{x \in \mathcal{X}} p(x) \left( -\sum_{y \in \mathcal{Y}} p(y|x) \log p(y|x) \right) \\ &= -\sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p(x,y) \log p(y|x) \\ &= -E_{X,Y}(\log p(Y|X)). \end{aligned}$$

Let $(X, Y) \sim p(x, y)$ be the pair of discrete random variables defined as above. We define the joint entropy of $X$ and $Y$ as

$$H(X, Y) = -E_{X,Y}(\log p(X, Y)).$$

We define the mutual information between $X$ and $Y$ as

$$I(X; Y) = \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}$$

which measures the amount of information on $X$ obtainable from observing $Y$, or vice versa.

Similarly, let $(X, Y, Z) \sim p(x, y, z)$, we define conditional mutual information between $X$, $Y$ given $Z$ as

$$I(X;Y|Z) = \sum_{(x,y,z)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{Y}\mathcal{Z}} p(x,y,z)\log\frac{p(x,y|z)}{p(x|z)p(y|z)}$$

**Theorem 14** (Jensen's inequality)**.** *Let $X \in \mathcal{X}$ (or $\mathbb{R}$) be a random variable with finite mean $E(X)$ and $g$ be a real-valued convex function over $\mathcal{X}$ (or $\mathbb{R}$) with finite expectation $E(g(X))$. Then*

$$E(g(X)) \geq g(E(X)).$$

The following properties are used frequently and could be derived from the definitions and Jensen's inequality:

i $0 \leq H(X) \leq \log|\mathcal{X}|$.

ii $H(Y|X) \leq H(Y)$.

iii $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y) \leq H(X) + H(Y)$.

iv $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = H(X) + H(Y) - H(X,Y)$.

v $I(X;Y|Z) = H(X|Z) - H(X|Y,Z) = H(Y|Z) - H(Y|X,Z) = H(X|Z) + H(Y|Z) - H(X,Y|Z)$.

vi If $X \to Y \to Z$ forms a Markov chain, i.e. $(X,Y,Z) \sim p(x)p(y|x)p(z|y)$, then $H(Z|Y,X) = H(Z|Y)$, $I(X;Z|Y) = 0$ and $I(X;Z) \leq I(X;Y)$, where the last one is called data processing inequality.

vii Let $X^n \sim p(x^n)$, then

$$\begin{aligned} H(X^n) &= H(X_1) + H(X_2|X_1) + \cdots + H(X_1|X_1,\ldots,X_{n-1}) \\ &= \sum_{i=1}^{n} H(X_i|X_1,\ldots,X_{i-1}) \\ &= \sum_{i=1}^{n} H(X_i|X^{i-1}). \end{aligned}$$

viii $I(X^n;Y) = \sum_{i=1}^{n} I(X_i;Y|X^{i-1})$.

The following two results are used in converse proofs frequently:

**Theorem 15** (Fano's inequality)**.** *Let $(X,Y) \sim p(x,y)$ and $P_e = P\{X \neq Y\}$, then*

$$H(X|Y) \leq H(P_e) + P_e\log|\mathcal{X}| \leq 1 + P_e\log|\mathcal{X}|.$$

**Theorem 16** (Csiszár's sum lemma)**.** *For any* $p(U, y_{1,1}^n, y_{2,1}^n)$ *we have*

$$\sum_{i=1}^{n} I(Y_{1,1}^{i-1}; Y_{2,i}|U, Y_{2,i+1}^{n}) = \sum_{i=1}^{n} I(Y_{2,i+1}^{\;n}; Y_{1,i}|U, Y_{1,1}^{i-1}).$$

Csiszár's sum lemma, originally presented in [5], is one of the most commonly used identities to derive outer bounds for discrete memoryless broadcast channels and is considered the bottleneck of traditional techniques.

# Appendix B

# Cardinality bounding techniques

We introduce the frequently used convex cover method for bounding the cardinalities of auxiliary random variables.

The following theorem is the basis of cardinality bounding.

**Theorem 17** (Fenchel-Eggleston-Carathéodory theorem)**.** *Any point in the convex closure of a connected set $\mathscr{R} \in \mathbb{R}^d$ can be represented as a convex combination of at most d points in $\mathscr{R}$.*

The following lemma is a direct consequence of Fenchel-Eggleston-Carathéodory theorem.

**Lemma 7** (Support lemma)**.** *Let $\mathcal{X}$ be a finite set and $\mathscr{U}$ be an arbitrary set. Let $\mathscr{P}$ be a connected compact subset of pmfs on $\mathcal{X}$. Suppose that $g_j(\pi)$, $j = 1, \ldots, d$, are real-valued continuous functions of $\pi \in \mathscr{P}$. Then for every $U \sim F(u)$ defined on $\mathscr{U}$, there exist a random variable $U' \sim p(u')$ with $|\mathscr{U}'| \leq d$ and a collection of conditional pmfs $p(x|u') \in \mathscr{P}$, indexed by $u' \in \mathscr{U}'$, such that for $j = 1, \ldots, d$,*

$$\int_{\mathscr{U}} g_j(p(x|u))dF(u) = \sum_{u' \in \mathscr{U}'} g_j(p(x|u'))p(u').$$

We use the following example to demonstrate the general steps to take in bounding the cardinalities of auxiliary random variables.

Let $U \sim F(u)$ be defined on $\mathscr{U}$. Let $(X, Y_1, Y_2)|\{U = u\} \sim p(x|u)p(y_1, y_2|x)$ and $\mathscr{R}$ be the union of all non-negative rate pairs $(R_1, R_2)$ such that

$$R_1 \leq I(X; Y_1|U),$$
$$R_2 \leq I(U; Y_2),$$

for some $F(u)$. We show that it is sufficient to consider $U$ define on $\mathcal{U}'$ where $|\mathcal{U}'| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\} + 1$ to completely characterize $\mathscr{R}$.

To show this, we prove that given any $(U, X)$, there exists $(U', X)$ with $|\mathcal{U}'| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\}+1$ such that $I(X; Y_1|U') = I(X; Y_1|U)$ and $I(U'; Y_2) = I(U; Y_2)$.

We first show that it suffices to take $|\mathcal{U}| \leq |\mathcal{X}|+1$. Without loss of generality, we take $\mathcal{X} = \{1, 2, \ldots, |\mathcal{X}|\}$. Given $(U, X)$, the set $\mathscr{P}$ of all pmfs on $\mathcal{X}$ is connected and compact. Consider the following $|\mathcal{X}| + 1$ continuous functions on $\mathscr{P}$:

$$g_j(p(x)) = \begin{cases} p(j), & j = 1, 2, \ldots, |\mathcal{X}| - 1 \\ H(Y_1), & j = |\mathcal{X}| - 1 \\ H(Y_2), & j = |\mathcal{X}| + 1 \end{cases}$$

All $|\mathcal{X}|+1$ functions are continuous in $p(x)$. Now by the support lemma mentioned above, there exists a random variable $U'$ defined on $|\mathcal{X}| + 1$ such that

$$H(Y_1|U) = \int_{\mathscr{U}} H(Y_1|U = u)dF(u) = \sum_{u' \in \mathscr{U}'} g_j(p(x|u'))p(u') = H(Y_1|U'),$$
$$H(Y_2|U) = \int_{\mathscr{U}} H(Y_2|U = u)dF(u) = \sum_{u' \in \mathscr{U}'} g_j(p(x|u'))p(u') = H(Y_2|U'),$$
$$\int_{\mathscr{U}} p(x|U = u)dF(u) = \sum_{u' \in \mathscr{U}'} p_{X|U'}(x|u')p(u'), \quad \forall x \in \{1, 2, \ldots, |\mathcal{X}| - 1\}.$$

Because $p(x)$ uniquely determines $H(Y_1|X)$ and $H(Y_2)$, we have

$$I(X; Y_1|U) = H(Y_1|U) - H(Y_1|X) = H(Y_1|U') - H(Y_1|X) = I(X; Y_1|U'),$$
$$I(U; Y_2) = H(Y_2) - H(Y_2|U) = H(Y_2) - H(Y_2|U') = I(U'; Y_2)$$

Therefore, there exists $(U', X)$ with $|\mathcal{U}'| \leq |\mathcal{X}| + 1$ such that $I(X; Y_1|U') = I(X; Y_1|U)$ and $I(U'; Y_2) = I(U; Y_2)$.

Following similar arguments, we can eventually get $(U', X)$ with $|\mathcal{U}'| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\} + 1$ such that $I(X; Y_1|U') = I(X; Y_1|U)$ and $I(U'; Y_2) = I(U; Y_2)$.

Therefore, it is sufficient to consider $U$ defined on $\mathcal{U}'$ where $|\mathcal{U}'| \leq \min\{|\mathcal{X}|, |\mathcal{Y}_1|, |\mathcal{Y}_2|\} + 1$ to completely characterize $\mathscr{R}$.

# Bibliography


[1] V Annapureddy and V Veeravalli, *Gaussian interference networks: sum capacity in the low-interference regime and new outer bounds on the capacity region*, IEEE TRANSACTIONS ON INFORMATION THEORY **55** (2009), no. 7, 3032–3050.

[2] P F Bergmans, *Coding theorem for broadcast channels with degraded components*, IEEE Trans. Info. Theory **IT-15** (March, 1973), 197–207.

[3] M. H M Costa and A. El Gamal, *The capacity region of the discrete memoryless interference channel with strong interference (corresp.)*, Information Theory, IEEE Transactions on **33** (1987), no. 5, 710–711.

[4] T Cover, *Broadcast channels*, IEEE Trans. Info. Theory **IT-18** (January, 1972), 2–14.

[5] I Csizár and J Körner, *Broadcast channels with confidential messages*, IEEE Trans. Info. Theory **IT-24** (May, 1978), 339–348.

[6] A El Gamal, *The capacity of a class of broadcast channels*, IEEE Trans. Info. Theory **IT-25** (March, 1979), 166–169.

[7] Abbas El Gamal and Young-Han Kim, *Network information theory*, Cambridge University Press, 2012.

[8] R G Gallager, *Capacity and coding for degraded broadcast channels*, Probl. Peredac. Inform. **10(3)** (1974), 3–14.

[9] Y Geng, C Nair, S Shamai, and Z V Wang, *On broadcast channels with binary inputs and symmetric outputs*, International Symposium on Information Theory (2010).

[10] Amin Aminzadeh Gohari and Venkat Anantharam, *Evaluation of Marton's inner bound for the general broadcast channel*, IEEE Trans. Info. Theory **IT-58** (2012), no. 2, 608–619.

[11] Te Han and K. Kobayashi, *A new achievable rate region for the interference channel*, Information Theory, IEEE Transactions on **27** (1981), no. 1, 49–60.

[12] J. B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of convex analysis*, Springer, 2001.

[13] J Körner and K Marton, *Comparison of two noisy channels*, Topics in Inform. Theory(ed. by I. Csiszar and P.Elias), Keszthely, Hungary (August, 1975), 411–423.

[14] Sida Liu, Chandra Nair, and Lingxiao Xia, *Interference channels with very weak interference*, 2014 IEEE International Symposium on Information Theory (ISIT'2014) (Honolulu, USA), June 2014, pp. 1031–1035.

[15] K Marton, *A coding theorem for the discrete memoryless broadcast channel*, IEEE Trans. Info. Theory **IT-25** (May, 1979), 306–311.

[16] C Nair, *Capacity regions of two new classes of two-receiver broadcast channels*, Information Theory, IEEE Transactions on **56** (2010), no. 9, 4207–4214.

[17] C. Nair and Z.V. Wang, *The capacity region of the three receiver less noisy broadcast channel*, Information Theory, IEEE Transactions on **57** (2011), no. 7, 4058–4062.

[18] C Nair, L Xia, and M Yazdanpanah, *Sub-optimality of Han-Kobayashi achievable region for interference channels*, Information Theory (ISIT), 2015 IEEE International Symposium on, June 2015, pp. 2416–2420.

[19] Chandra Nair, Abbas El Gamal, and Yeow-Khiang Chia, *An achievability scheme for the compound channel with state noncausally available at the encoder*, CoRR **abs/1004.3427** (2010).

[20] Chandra Nair and Lingxiao Xia, *On Three-Receiver more capable channels*, 2012 IEEE International Symposium on Information Theory (ISIT'2012) (Cambridge, Massachusetts, USA), July 2012, pp. 383–387.

[21] H Sato, *An outer bound to the capacity region of broadcast channels*, IEEE Trans. Info. Theory **IT-24** (May, 1978), 374–377.

[22] X Shang, G Kramer, and B Chen, *A new outer bound and the noisy-interference sum–rate capacity for gaussian interference channels*, IEEE TRANSACTIONS ON INFORMATION THEORY **55** (2009), no. 2, 689–699.

[23] C E Shannon, *A mathermatical theory of communication*, Bell System Technical Journal **27** (July and October, 1948), 379–423 and 623–656.