Proof of a conjecture on the Gaussian signaling region for the Gaussian Z-interference channel

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Abstract

We establish a recent conjecture regarding the Gaussian signaling region for the Z-interference channel. This helps us isolate a large class of parameters for which multiplexing, or noisebergs, are not needed for the computation of the optimal Gaussian signaling region.

I. INTRODUCTION

In this paper, we study the (scalar) one-sided interference channel (or Z-interference) given by $Y_1' = X_1' + Z_1'$ and $Y_2' = X_2' + aX_1' + Z_2'$, as depicted in Figure 1. Here, X_1' and X_2' are transmitted signals constrained to have average powers P_1' and P_2', respectively, $a \in (0,1)$ is an interference gain, Z_1' and Z_2' are standard Gaussians, and Y_1' and Y_2' are the two received signals. Thus, this Z-interference channel model is specified using three parameters (a, P_1', P_2').

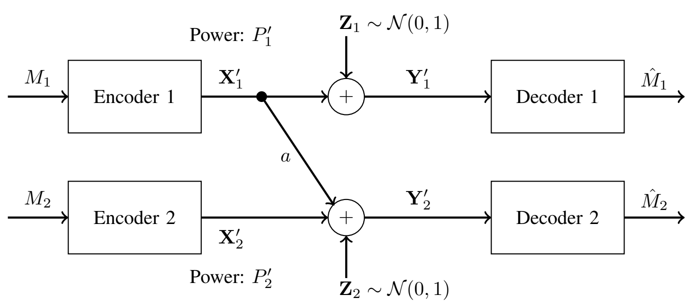

Fig. 1. Gaussian Z-Interference Channel.

An (n, R_1, R_2) code $\mathcal{C}$, for this model, consists of

- two message sets $[1:2^{nR_1}] := \{1, 2, ..., \lfloor 2^{nR_1} \rfloor\}$ and $[1:2^{nR_2}] := \{1, 2, ..., \lfloor 2^{nR_2} \rfloor\}$,
- two encoder functions $[1:2^{nR_i}] \to \mathcal{X}_i^n, i \in \{1,2\}$ mapping each message m_i to a codeword x_i^n, where

$$\frac{1}{\lfloor 2^{nR_i} \rfloor} \sum_{m_i} \|x_i^n(m_i)\|_2^2 \leq nP_i', \ i \in \{1,2\},$$

- two decoder functions $\mathcal{Y}_i^n \to [1:2^{nR_i}], i \in \{1,2\}$ mapping a codeword y_i^n to a message estimate, $\hat{m}_i$.

Assume that the messages (M_1, M_2) are uniformly distributed over $[1:\lfloor 2^{nR_1} \rfloor] \times [1:\lfloor 2^{nR_2} \rfloor]$. The average probability error is defined to be

$$P_e^{(n)} = \Pr((\hat{M}_1, \hat{M}_2) \neq (M_1, M_2)).$$

A rate pair (R_1, R_2) is *achievable* if there is a sequence of (n, R_1, R_2) codes such that $P_e^{(n)} \to 0$ as $n \to \infty$. Then the capacity region $\mathscr{C}$ is defined as the closure of the set of all achievable rate pairs.

Scalar interference channels have been studied since the early 70s [**?**], [1]–[31]. One of the key open questions in this area is whether Han–Kobayashi inner bound with Gaussian signaling achieves the capacity region. As we will discuss below, it is hoped that the results in this paper can bring us closer to resolving the above problem.

In the case of strong interference, when $a \geq 1$, the capacity region was established in [6], [7]. In this case, the unintended receiver can fully decode the interfering message. Also, when $a = 0$, the problem decouples and has a trivial solution. As shown in [8], the Gaussian Z-interference channel with interference parameter a in the range $(0, 1)$ can be regarded as a degraded Gaussian interference channel, a model shown in Figure 2.

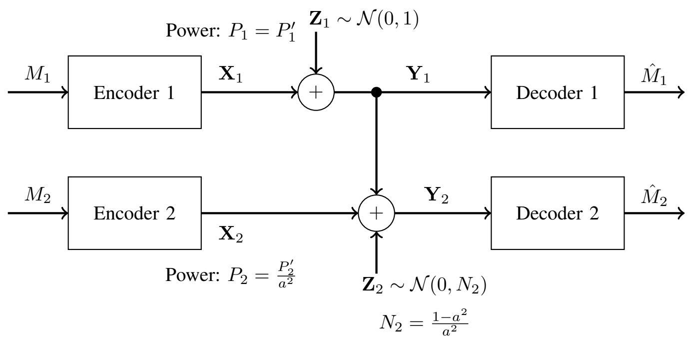

Fig. 2. Degraded Gaussian Interference Channel.

Like the Gaussian Z-interference channel, the degraded Gaussian interference channel is characterized by three parameters: the two transmitter powers P_1 and P_2, and the power of the additional independent noise in the second receiver, power N_2. These parameters are related to the parameters of the original Z-interference channel by $P_1 = P_1'$, $P_2 = P_2'/a^2$ and $N_2 = (1 - a^2)/a^2$. Moreover, since $0 < a < 1$, the additional noise power N_2 is always positive. We choose to use the more common notation, without the primes, to denote the equivalent degraded setting.

A. Han-Kobayashi Region with Gaussian Signaling

It is known that the Han-Kobayashi inner bound reduces to the following three inequalities for a Gaussian Z-interference channel.

Proposition 1 (Han-Kobayashi inner bound for Gaussian Z-interference)**.** *Given a Gaussian* Z*-interference channel* $p(y_1|x_1)p(y_2|x_1, x_2)$ *with parameters* (P_1, P_2, N_2)*, a rate pair* (R_1, R_2) *is achievable if*

$$R_1 < I(X_1; Y_1|Q), \tag{1a}$$

$$R_2 < I(X_2; Y_2|U_1, Q), \tag{1b}$$

$$R_1 + R_2 < I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_1, Q), \tag{1c}$$

for some $p(q)p(u_1, x_1|q)p(x_2|q)$ *satisfying*

$$E\|X_1^n\|^2 \leq nP_1, \tag{1d}$$

$$E\|X_2^n\|^2 \leq nP_2. \tag{1e}$$

Definition 1 (Gaussian Signaling)**.** The Han-Kobayashi achievable region with Gaussian signaling and power control for Gaussian Z-interference channels is the set of all rate pairs $(R_1, R_2) \in \mathbb{R}^2_{\geq 0}$ such that 1 holds with $X_1 = U_1 + V_1$ for any $p(q)p(u_1|q)p(v_1|q)p(x_2|q)$ where $(U_1|Q = q), (V_1|Q = q), (X_2|Q = q)$ are zero-mean Gaussian random variables for each q.

Remark 1. Instead of considering this as a region in $\mathbb{R}^2_{\geq 0}$, it can be effectively described by its supporting hyperplanes since it is convex. Therefore, we will describe the region alternately in terms of the maximum value of $R_1 + \beta R_2$, for $\beta \geq 1$. For $\beta \leq 1$, the value has been established in [7], [8] (also see the next section).

It was conjectured in [16] and proved in [27] that the Han-Kobayashi achievable region with Gaussian signaling, $\mathcal{R}_{\text{HK-GS}}$ reduces to the following.

Theorem 1. *Consider a Gaussian* Z*-interference channel with parameters* (P_1, P_2, N_2)*. Let* $\beta \geq 1$*. Then for all* $P_1, P_2 \geq 0$*,*

$$\sup_{(R_1,R_2)\in\mathcal{R}_{\textit{HK-GS}}} R_1 + \beta R_2 = \max_{\alpha, \tilde{P}} \left(\alpha f_\beta \left(\tilde{P}, \frac{P_2}{\alpha} \right) + (1 - \alpha) f_\beta \left(\frac{P_1 - \alpha\tilde{P}}{1 - \alpha}, 0 \right) \right).$$

subject to $\frac{P_2}{P_1+P_2} \leq \alpha \leq 1$ *and* $0 \leq \tilde{P} \leq P_1 + P_2 - \frac{P_2}{\alpha}$, *where*

$$f_\beta(P_1, P_2) = \frac{1}{2}\begin{cases} \log(P_1+P_2+1+N_2) + (\beta-1)\log(P_2+1+N_2) - \beta\log(1+N_2), & (P_1,P_2) \in \mathcal{R}_1, \\ \beta\log(P_1+P_2+1+N_2) + \log(P_1+1) - \beta\log(P_1+1+N_2), & (P_1,P_2) \in \mathcal{R}_2, \\ \log(P_1+P_2+1+N_2) + \beta\log(P_2+N_2) - \log(P_2) - (\beta-1)\log(N_2) & \\ \quad +(\beta-1)\log(\beta-1) - \beta\log(\beta), & (P_1,P_2) \in \mathcal{R}_3, \end{cases}$$

with

$$\mathcal{R}_1 = \left\{(P_1,P_2) : \beta \geq \frac{P_2+N_2}{P_2}(1+N_2)\right\}, \tag{2a}$$

$$\mathcal{R}_2 = \left\{(P_1,P_2) : \beta \leq \frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{P_1+1}\right)\right\}, \tag{2b}$$

$$\mathcal{R}_3 = \left\{(P_1,P_2) : \frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{P_1+1}\right) < \beta < \frac{P_2+N_2}{P_2}(1+N_2)\right\}. \tag{2c}$$

A recent survey of this region can be found in [32]. This paper establishes a conjecture about this region stated in [32].

B. Outer Bounds to the capacity region of the Gaussian Z-interference channel

One extreme point of the capacity region occurs when X_1 sends information at its maximum possible rate. Here, the rate pair (R_1, R_2) is given by $R_1 = \frac{1}{2}\log(1+P_1)$ and $R_2 = \frac{1}{2}\log(1+\frac{P_2}{1+P_1+N_2})$. There is a slope discontinuity for the capacity region at this extreme point, which follows from the capacity region of an associated degraded broadcast channel [5], [8]. From this, it immediately follows that this point also maximizes $\beta R_1 + R_2$, for $\beta \leq 1$. This corner point will be referred to as the Costa-Sato corner point.

Another extreme point in the achievable region occurs when X_2 sends information at its maximum possible rate. Here, we have $R_1 = \frac{1}{2}\log(1+\frac{P_1}{1+N_2+P_2})$ and $R_2 = \frac{1}{2}\log(1+\frac{P_2}{1+N_2})$. This was established in [21], fixing a gap in [33]. This corner point is referred to as the Costa-Polyanskiy-Wu corner point. There is also a slope discontinuity for the capacity region at this extreme point, which follows from a recent outer bound developed in [26]. This bound is improved in [29].

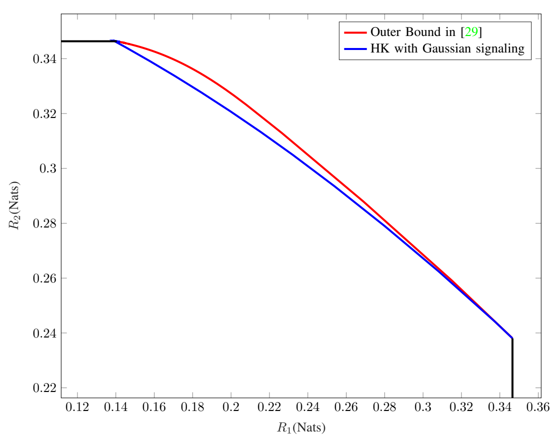

Fig. 3. The bounds to the capacity region when $P_1' = P_2' = 1, a = 0.8$, or equivalently $P_1 = 1, P_2 = \frac{25}{16}, N_2 = \frac{9}{16}$.

C. Gaussian Optimality, and Multiplexing

As surveyed in [32], the optimal transmission strategy (or the solution to the problem in Theorem 1), seems to lie in seven phases potentially. However, the current techniques that establish the optimality of Gaussian distributions for information functionals work only when there is no multiplexing. Therefore, deducing the set of parameters for which one does not need multiplexing in the optimal transmission strategy is imperative. Towards this, the following conjecture was posed in [32].

Conjecture 1. *Consider a degraded Gaussian Z-interference channel with parameters* (P_1, P_2, N_2)*. The noiseberg region consists only of a pure superposition coding strategy (i.e. no time-sharing is required for any* β*-sum-rate) whenever*

$$\frac{(N_2+P_2)(1+N_2+P_1)}{P_2(1+P_1)} \leq \beta^*,$$

where β^* *is the unique positive solution of* $\psi(\beta)=0$*, where*

$$\psi(\beta) := \beta\Bigg(\log\left(1+\frac{P_2}{1+N_2+P_1}\right) - \frac{N_2P_2}{(1+N_2+P_1)(1+N_2+P_1+P_2)}\Bigg) + \log\left(1-\frac{P_2(1+P_1)}{(1+N_2+P_1)(1+N_2+P_1+P_2)}\beta\right). \quad (3)$$

This conjecture proposes a set of parameters (P_1, P_2, N_2) for which no multiplexing is needed for the computation of any $\beta \geq 1$, in $\mathcal{R}_{\text{HK-GS}}$. Figure 4 illustrates this set of parameters.

To see the origin of the function $\psi(\beta)$, we recall the following result.

Theorem 2 ([23]). *Let* $\beta_{sato} = \max\left\{\beta \geq 1 : \sup\{R_1+\beta R_2 : (R_1,R_2)\in\mathcal{R}_{\text{HK}-\text{GS}}\} = \frac{1}{2}\log(1+P_1)+\beta\frac{1}{2}\log(1+\frac{P_2}{1+P_1+N_2})\right\}$ *be the largest value that the hyperplane induced by* $R_1+\beta R_2$ *passes the Costa-Sato corner point. Then*

$$\beta_{sato} = \min\left\{\frac{(P_2+N_2)(1+N_2+P_1)}{P_2(1+P_1)}, \beta^*\right\},$$

where β^* *is defined as in Conjecture 1.*

Using this result, Conjecture 1 can be interpreted as stating that if the capacity region departs from the Costa-Sato corner point (as one increases β) along the pure superposition phase rather than the multiplexing phase, then multiplexing is never needed for the computation of $\mathcal{R}_{\text{HK-GS}}$. The main result of this paper is a proof of Conjecture 1.

We have the following equivalent condition to the assumption in Conjecture 1, which is easier to verify.

Lemma 1. *The following two are equivalent:*

$$\frac{(N_2+P_2)(1+N_2+P_1)}{P_2(1+P_1)} \leq \beta^* \iff \psi\left(\frac{(N_2+P_2)(1+N_2+P_1)}{P_2(1+P_1)}\right) \geq 0,$$

where $\psi(\beta)$ *is defined in Equation (3).*

Proof. This lemma is an immediate corollary of Lemma 2. □

The main theorem of this paper is the following:

Theorem 3. *Conjecture 1 is valid, or equivalently when* (P_1,P_2,N_2) *satisfy* $\psi\left(\frac{(N_2+P_2)(1+N_2+P_1)}{P_2(1+P_1)}\right) \geq 0$*, that is,*

$$\frac{(N_2+P_2)(1+N_2+P_1)}{P_2(1+P_1)}\log\left(1+\frac{P_2}{1+N_2+P_1}\right) - \frac{(N_2+P_2)N_2}{(1+P_1)(1+N_2+P_1+P_2)} + \log\left(1-\frac{N_2+P_2}{1+N_2+P_1+P_2}\right) \geq 0,$$

then

$$\sup_{(R_1,R_2)\in\mathcal{R}_{\textit{HK-GS}}} R_1+\beta R_2 = f_\beta(P_1,P_2),$$

where $f_\beta(P_1,P_2)$ *is defined in Theorem 1.*

We will prove this result in the next section.

Remark 2. While the proof of the theorem is essentially a (non-trivial) exercise in calculus and optimization, it is hoped that the proof should lead to insights that should be of use beyond the theorem. For instance, it may help curate genies or other tools to establish tight converses. Similar results have been obtained previously in the literature. In the very weak interference regime, we know the sum-capacity when $a(1+b^2P_2')+b(1+a^2P_1') \leq 1$, using a genie approach.

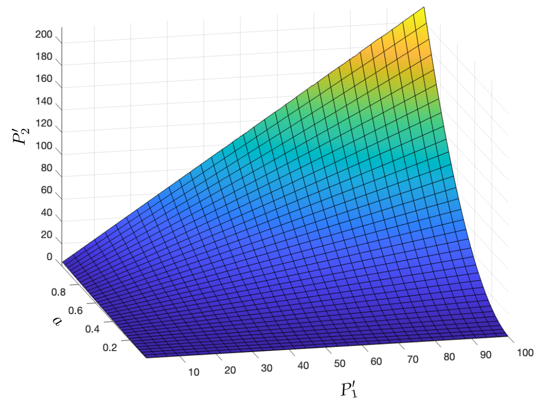

Fig. 4. Any tuple (P_1', a, P_2') above the surface does not require multiplexing to achieve $\mathcal{R}_{\mathrm{HK-GS}}$.

The following result had been obtained concerning the slope of $\mathcal{R}_{\text{HK-GS}}$ at the Costa-Polyanskiy-Wu corner point.

Theorem 4 ([22]). *Consider the Gaussian Z-interference channel. The smallest β such that the supporting hyperplane $R_1 + \beta R_2$ of Han-Kobayashi inner bound with Gaussian inputs passes through the corner point is given by*

$$\beta_{costa} = 1 + \max\left\{ \frac{\log(N_2+1) - \frac{N_2}{1+N_2+P_1+P_2}}{\log\left(1+\frac{P_2}{1+N_2}\right) - \frac{P_2}{1+N_2+P_2}}, \frac{N_2(1+N_2+P_2)}{P_2} \right\}.$$

If the first term above is larger, a multiplexing strategy beats a pure superposition coding scheme for $\beta = \beta_{costa} - \epsilon$, for a sufficiently small and positive ϵ. Therefore, under the assumptions of the conjecture we must have the second term to be larger. This is established in Lemma 3 and shows that, under the assumption of Conjecture 1, we have

$$\beta_{costa} = 1 + \frac{N_2(1+N_2+P_2)}{P_2}.$$

II. Proof of Theorem 3

Lemma 2. *Let $\psi : [0, \beta_{max}) \to \mathbb{R}$ be defined by Equation (3). Here $\beta_{max} = \frac{(1+N_2+P_1)(1+N_2+P_1+P_2)}{P_2(1+P_1)}$. Then, the following hold: $\exists\ \beta^* \in (0, \beta_{max})$, such that $\psi(\beta^*) = 0$, and $\psi(\beta) > 0$ if and only if $\beta \in [0, \beta^*)$.*

Proof. The proof of this Lemma can be found in Section A of the Appendix. □

Therefore, in the rest of the paper, we will assume that (P_1, P_2, N_2) are strictly positive numbers that satisfy

$$\psi\left(\frac{(N_2+P_2)(1+N_2+P_1)}{P_2(1+P_1)}\right) \geq 0. \tag{4}$$

Lemma 3. *If (P_1, P_2, N_2) satisfies (4), they also satisfy $P_1 + 1 \leq P_2$, and*

$$\frac{\log(N_2+1) - \frac{N_2}{1+N_2+P_1+P_2}}{\log\left(1+\frac{P_2}{1+N_2}\right) - \frac{P_2}{1+N_2+P_2}} \leq \frac{N_2(1+N_2+P_2)}{P_2}.$$

Proof. The proof of this can be found in Section B of the Appendix. As stated earlier, the second statement above shows that under the assumptions of the conjecture, the slope at the Costa-Polyanskiy-Wu corner point of the capacity region is also governed by superposition coding and not by multiplexing. □

The remaining part of the proof is to show that for any $\beta : \beta_{sato} < \beta < \beta_{costa}$, as long as the parameters satisfy (4), there is no multiplexing required. In other words, the maximizer of the optimization problem in Theorem 1 occurs at $\alpha = 1$.

A. No multiplexing for $\beta : \beta_{sato} < \beta < \beta_{costa}$

Let the function $f_\beta(Q_1, Q_2)$ be defined as in Theorem 1. Let $a_\beta(Q_1, Q_2)$ be the upper concave envelope of $f_\beta(Q_1, Q_2)$. Let (P_1, P_2) satisfy (4). To show that no multiplexing is required for $\beta : \beta_{sato} < \beta < \beta_{costa}$, is equivalent to showing that $a_\beta(P_1, P_2) = f_\beta(P_1, P_2)$, or in words, the value of upper concave envelope matches the function value at (P_1, P_2) (thus, one does not need any time-sharing).

For $\beta : \beta_{sato} < \beta < \beta_{costa}$ and (P_1, P_2) satisfying (4), we have $(P_1, P_2) \in \mathcal{R}_3$, where $\mathcal{R}_3$ is as defined in Theorem 1. We consider the tangential plane of $f_\beta(Q_1, Q_2)$, at the point (P_1, P_2). If this plane lies above the function, then this implies that $a_\beta(P_1, P_2) = f_\beta(P_1, P_2)$. This is because, in this case, the tangential plane would be a linear (concave) function that lies above the function $f_\beta(Q_1, Q_2)$ (and passes trough $f_\beta(P_1, P_2)$), and $a_\beta(Q_1, Q_2)$ is the pointwise infimum of all concave functions that lie above $f_\beta(Q_1, Q_2)$.

The tangential plane to $f_\beta(Q_1, Q_2)$ at (P_1, P_2) is given by

$$t_\beta(Q_1, Q_2) := f_\beta(P_1, P_2) + \frac{1}{2}\left(\frac{1}{P_1 + P_2 + 1 + N_2}\right)(Q_1 - P_1) + \frac{1}{2}\left(\frac{1}{P_1 + P_2 + 1 + N_2} + \frac{\beta}{P_2 + N_2} - \frac{1}{P_2}\right)(Q_2 - P_2).$$

Therefore $t_\beta(Q_1, Q_2) \geq f_\beta(Q_1, Q_2)$ is equivalent to showing that

$$g_\beta(Q_1,Q_2) := f_\beta(Q_1, Q_2) - \frac{1}{2}\left(\frac{1}{P_1 + P_2 + 1 + N_2}\right)Q_1 - \frac{1}{2}\left(\frac{1}{P_1 + P_2 + 1 + N_2} + \frac{\beta}{P_2 + N_2} - \frac{1}{P_2}\right)Q_2.$$

attains a maximum at (P_1, P_2).

Interior analysis

In this section we will show that (P_1, P_2) is the unique interior local maximizer of $g_\beta(Q_1, Q_2)$.

Lemma 4. *Let (P_1, P_2) satisfy (4) and β satisfy, $\beta_{sato} < \beta < \beta_{costa}$. Then (P_1, P_2) is the unique local maximizer of $g_\beta(Q_1, Q_2)$ in $\mathbb{R}^2_{>0}$.*

Proof. From Lemma 3, we know that $P_2 \geq P_1 + 1$. Note that that the expression for $f_\beta(Q_1, Q_2)$ (and hence $g_\beta(Q_1, Q_2)$) depends on the partition $(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3)$ that (Q_1, Q_2) belongs to. It is easy to verify that $g_\beta(Q_1, Q_2)$ is continuously differentiable in $\mathbb{R}^2_{>0}$.

Since β is in $(\beta_{sato}, \beta_{costa})$, it implies that $(P_1, P_2) \in \mathcal{R}_3$, or equivalently,

$$\frac{P_2 + N_2}{P_2}\left(1 + \frac{N_2}{P_1 + 1}\right) < \beta < \frac{P_2 + N_2}{P_2}(1 + N_2). \tag{5}$$

1) *Case* 1, $(Q_1, Q_2) \in \mathcal{R}_1$.

Therefore, from the definition of $(Q_1, Q_2) \in \mathcal{R}_1$, we have

$$\beta \geq \frac{Q_2 + N_2}{Q_2}(1 + N_2), \tag{6}$$

and

$$\begin{aligned} g_\beta(Q_1, Q_2) = &\frac{1}{2}\log(Q_1 + Q_2 + 1 + N_2) + \frac{(\beta - 1)}{2}\log(Q_2 + 1 + N_2) - \frac{\beta}{2}\log(1 + N_2) \\ &- \frac{1}{2}\left(\frac{1}{P_1 + P_2 + 1 + N_2}\right)Q_1 - \frac{1}{2}\left(\frac{1}{P_1 + P_2 + 1 + N_2} + \frac{\beta}{P_2 + N_2} - \frac{1}{P_2}\right)Q_2. \end{aligned}$$

The first-order conditions for local optimality yields,

$$\begin{aligned} \frac{1}{Q_1 + Q_2 + 1 + N_2} &= \frac{1}{P_1 + P_2 + 1 + N_2}, \\ \frac{1}{Q_1 + Q_2 + 1 + N_2} + \frac{\beta - 1}{Q_2 + 1 + N_2} &= \frac{1}{P_1 + P_2 + 1 + N_2} + \frac{\beta}{P_2 + N_2} - \frac{1}{P_2}. \end{aligned}$$

Solving for β, and observing that $Q_2 > P_2$ from (6) and (5), we obtain

$$\frac{(Q_2 - P_2 + 1 + N_2)(P_2 + N_2)}{P_2(Q_2 - P_2 + 1)} = \beta \geq \frac{Q_2 + N_2}{Q_2}(1 + N_2).$$

The above inequality can be rewritten as

$$\left(1+\frac{N_2}{Q_2-(P_2-1)}\right)\left(1+\frac{N_2}{1+(P_2-1)}\right)\geq\left(1+\frac{N_2}{Q_2}\right)\left(1+\frac{N_2}{1}\right). \tag{7}$$

Define

$$\theta(x):=\left(1+\frac{N_2}{Q_2-x}\right)\left(1+\frac{N_2}{1+x}\right).$$

Simple calculation yields that $\theta(x)$ is strictly convex in $(0,Q_2-1)$, and also observe that $\theta(0)=\theta(Q_2-1)$. Since, $Q_2-1>P_2-1\geq P_1>0$, necessitates (from convexity of θ) that $\theta(P_2-1)\leq\theta(0)=\theta(Q_2-1)$, contradicting (7). Therefore, there cannot be a local maximizer $(Q_1,Q_2)\in\mathcal{R}_1$.

2) *Case* 2, $(Q_1,Q_2)\in\mathcal{R}_2$.

Then

$$\begin{aligned}g_\beta(Q_1,Q_2)=&\frac{\beta}{2}\log(Q_1+Q_2+1+N_2)+\frac{1}{2}\log(Q_1+1)-\frac{\beta}{2}\log(Q_1+1+N_2)\\&-\frac{1}{2}\left(\frac{1}{P_1+P_2+1+N_2}\right)Q_1-\frac{1}{2}\left(\frac{1}{P_1+P_2+1+N_2}+\frac{\beta}{P_2+N_2}-\frac{1}{P_2}\right)Q_2,\end{aligned}$$

and

$$\beta\leq\frac{Q_2+N_2}{Q_2}\left(1+\frac{N_2}{1+Q_1}\right).$$

Suppose (Q_1,Q_2) satisfies the first and second-order (we write the only non–trivial one here) conditions for optimality, i.e.:

$$\begin{aligned}&\frac{\beta}{Q_1+Q_2+1+N_2}+\frac{1}{Q_1+1}-\frac{\beta}{Q_1+1+N_2}=\frac{1}{P_1+P_2+1+N_2},\\&\frac{\beta}{Q_1+Q_2+1+N_2}=\frac{1}{P_1+P_2+1+N_2}+\frac{\beta}{P_2+N_2}-\frac{1}{P_2},\\&-\frac{1}{(Q_1+1)^2}+\frac{\beta}{(Q_1+1+N_2)^2}\leq 0.\end{aligned}$$

The first-order condition implies

$$\beta\left(\frac{P_2-(Q_1+1)}{(Q_1+1+N_2)(P_2+N_2)}\right)=\frac{P_2-(Q_1+1)}{(Q_1+1)P_2}.$$

Suppose $P_2=Q_1+1$, plugging into the second order condition, we have

$$\frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{1+P_1}\right)<\beta\leq\left(\frac{Q_1+1+N_2}{Q_1+1}\right)^2=\left(\frac{P_2+N_2}{N_2}\right)^2\implies P_1+1>P_2,$$

a contradiction.

Suppose $P_2\neq Q_1+1$. Note, as $(Q_1,Q_2)\in\mathcal{R}_2$ and $(P_1,P_2)\in\mathcal{R}_3$, we have

$$\begin{aligned}\frac{(P_2+N_2)(Q_1+1+N_2)}{P_2(Q_1+1)}&=\beta>\frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{1+P_1}\right)\implies Q_1<P_1,\\\frac{(P_2+N_2)(Q_1+1+N_2)}{P_2(Q_1+1)}&=\beta\leq\frac{Q_2+N_2}{Q_2}\left(1+\frac{N_2}{1+Q_1}\right)\implies Q_2\leq P_2,\end{aligned}$$

thus $Q_1+Q_2<P_1+P_2$. Plugging β into the second of first-order conditions above and simplifying yields,

$$\frac{1}{P_1+P_2+1+N_2}=\frac{P_2(Q_1+1)+N_2(P_2-Q_2)}{P_2(Q_1+1)(Q_1+Q_2+1+N_2)}\geq\frac{1}{Q_1+Q_2+1+N_2},$$

using $P_2\geq Q_2$. However, this implies that $Q_1+Q_2\geq P_1+P_2$, contradicting $Q_1+Q_2<P_1+P_2$, obtained above. This, there is not $(Q_1,Q_2)\in\mathcal{R}_2$ that is a local maximizer.

3) *Case* 3, $(Q_1, Q_2) \in \mathcal{R}_3$.
Then

$$g_\beta(Q_1,Q_2) = \frac{1}{2}\log(Q_1+Q_2+1+N_2) + \frac{\beta}{2}\log(Q_2+N_2) - \frac{1}{2}\log(Q_2) - \frac{(\beta-1)}{2}\log(N_2) + \frac{(\beta-1)}{2}\log(\beta-1) - \frac{\beta}{2}\log(\beta) - \frac{1}{2}\left(\frac{1}{P_1+P_2+1+N_2}\right)Q_1 - \frac{1}{2}\left(\frac{1}{P_1+P_2+1+N_2} + \frac{\beta}{P_2+N_2} - \frac{1}{P_2}\right)Q_2,$$

and β satisfies

$$\frac{Q_2+N_2}{Q_2}\left(1+\frac{N_2}{Q_1+1}\right) < \beta < \frac{Q_2+N_2}{Q_2}(1+N_2).$$

The first and second-order (again the non-trivial inequalities only) conditions give

$$\begin{aligned}
&\frac{1}{Q_1+Q_2+1+N_2} = \frac{1}{P_1+P_2+1+N_2},\\
&\frac{1}{Q_1+Q_2+1+N_2} + \frac{\beta}{Q_2+N_2} - \frac{1}{Q_2} = \frac{1}{P_1+P_2+1+N_2} + \frac{\beta}{P_2+N_2} - \frac{1}{P_2},\\
&-\frac{\beta}{(Q_2+N_2)^2} + \frac{1}{Q_2^2} \le 0.
\end{aligned}$$

Note that $P_1+P_2 = Q_1+Q_2$, so if $(Q_1,Q_2) \neq (P_1,P_2)$, then $Q_1 \neq P_1$ and $Q_2 \neq P_2$. Then

$$\begin{aligned}
&\frac{(Q_2+N_2)(P_2+N_2)}{Q_2P_2} = \beta \ge \frac{(Q_2+N_2)^2}{Q_2^2} \implies P_2 \le Q_2,\\
&\frac{(Q_2+N_2)(P_2+N_2)}{Q_2P_2} = \beta \ge \frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{1+P_1}\right) \implies 1+P_1 \ge Q_2.
\end{aligned}$$

In the above, the first inequality follows from the second-order conditions, and the second inequality as $(P_1,P_2) \in \mathcal{R}_3$. Therefore, it follows that

$$P_2 < Q_2 \le 1+P_1,$$

and this is a contradiction of the result we obtained in Lemma 3 on (P_1,P_2). Thus, (P_1,P_2) is the only local maximizer in $\mathcal{R}_3$.
This completes the proof of the lemma. □

To complete the proof of Conjecture 1, we have to show that the global maximizer is not on the boundary, i.e., on the lines $Q_2 = 0$, or $Q_1 = 0$.

Boundary analysis

In this section, we show that $g_\beta(Q_1,Q_2)$ has only one local maximum on the Q_1-axis and Q_2-axis, which is proved to be smaller than the interior maximum.

1) *Case* 1: $Q_2 = 0$.
The first-order conditions for optimality yield

$$\begin{aligned}
\frac{1}{1+Q_1} &= \frac{1}{1+P_1+P_2+N_2},\\
\frac{\beta}{1+Q_1+N_2} &\le \frac{1}{P_1+P_2+1+N_2} + \frac{\beta}{P_2+N_2} - \frac{1}{P_2}.
\end{aligned}$$

Lemma 5. *If (P_1,P_2) satisfy the condition in (4) and $\beta_{sato} < \beta < \beta_{costa}$, then*

$$g_\beta(P_1+P_2+N_2, 0) \le g_\beta(P_1,P_2).$$

The proof can be found in Appendix C.

2) *Case* 2: $Q_1 = 0$.
Case 2.1: $\beta \ge \frac{Q_2+N_2}{Q_2}(1+N_2)$.

In this case $(0, Q_2) \in \mathcal{R}_1$. The first derivative conditions for a local maximizer yields

$$\frac{\beta}{Q_2 + 1 + N_2} = \frac{1}{P_1 + P_2 + 1 + N_2} + \frac{\beta}{P_2 + N_2} - \frac{1}{P_2},$$
$$\frac{1}{Q_2 + 1 + N_2} \le \frac{1}{P_1 + P_2 + 1 + N_2}.$$

This implies

$$\beta \left(\frac{P_1 + 1}{(P_2 + N_2)(P_1 + P_2 + 1 + N_2)} \right) \le \frac{P_1 + 1 + N_2}{P_2(P_1 + P_2 + 1 + N_2)},$$

or equivalently

$$\beta \le \frac{P_2 + N_2}{P_2} \left(1 + \frac{N_2}{1 + P_1} \right).$$

The right-hand-side is β_{sato} for P_1, P_2 satisfying (1) and since we considering $\beta : \beta_{sato} < \beta < \beta_{costa}$, we have a contradiction.

Case 2.2: $\beta < \frac{Q_2+N_2}{Q_2}(1 + N_2)$.

The first derivative conditions for a local maximizer yields

$$\frac{\beta}{Q_2 + 1 + N_2} = \frac{1}{P_1 + P_2 + 1 + N_2} + \frac{\beta}{P_2 + N_2} - \frac{1}{P_2},$$
$$\frac{\beta}{Q_2 + 1 + N_2} + 1 - \frac{\beta}{1 + N_2} \le \frac{1}{P_1 + P_2 + 1 + N_2}.$$

Therefore, we must have

$$\frac{\beta}{P_2 + N_2} - \frac{1}{P_2} + 1 - \frac{\beta}{1 + N_2} \le 0.$$

Since $P_2 \ge 1 + P_1 > 1$, the above condition reduces to $\beta \ge \frac{P_2+N_2}{P_2}(1 + N_2) = \beta_{costa}$. Since we considering $\beta : \beta_{sato} < \beta < \beta_{costa}$, we have a contradiction.

III. Discussion and Conclusion

Determining the capacity region of the Gaussian Z-interference channel is a fundamental open problem in network information theory. It is even more frustrating when there is a candidate, the Han-Kobayashi region with Gaussian signaling, for its capacity region. Such instances are rare, with only Marton's inner bound for a two-receiver broadcast channel as the closest analogy. The principal issue with the Han-Kobayashi region with Gaussian signaling is that, for any weighted sum rate, the value is given by the evaluation of the upper concave envelope of an explicit two-dimensional function, $f_\beta(P_1, P_2)$. If we wish to show that this region is optimal, the current techniques for establishing Gaussian optimality (there are several) work only when the optimizer is a single Gaussian distribution and not when the optimizer involves time-sharing between two Gaussians. One can try to employ Fenchel duality, as proposed in [34], to get around this issue. However, as established in [35], the Gaussian optimality can fail if one works with the (tangential) hyperplanes induced by points that need time-sharing. All of the above makes it imperative that we identify parameters for which one does not require time-sharing, which is one of this paper's primary motivations.

Despite a mathematical proof of Conjecture 1 in this paper, the authors do not yet understand why the behavior at Sato's corner point determines the need (or lack of) for time-sharing. As one may notice by going through the full version [36], the proofs of Lemma 3 and Lemma 5 are rather involved and do not seem to have direct analogies to arguments involving information measures. Understanding this may greatly help in designing converses to the capacity region.

Conclusion

In this paper we studied the Gaussian signaling region for the Han-Kobayashi achievable region for the Gaussian Interference channel. We established a recently proposed conjecture showing that the above region does not involve time-sharing for some specified parameters. It is hoped that by focusing on these parameters, one can either disprove the optimality of the Gaussian signaling region or come up with proof of its optimality, as the absence of time-sharing makes it amenable to standard arguments for showing Gaussian optimality.

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Appendix

A. Proof of Lemma 2

Proof. We first show that function $\psi(\beta) = 0$ has a unique (strictly) positive solution β^*. Note that $\psi(0) = 0$,

$$\psi'(\beta) = \log\left(1 + \frac{P_2}{1 + N_2 + P_1}\right) - \frac{N_2 P_2}{(1 + N_2 + P_1)(1 + N_2 + P_1 + P_2)} - \frac{1}{\frac{(1+P_1+N_2)(1+N_2+P_1+P_2)}{P_2(1+P_1)} - \beta}$$

is decreasing in β, and

$$\begin{aligned}
\psi'(0) &= \log\left(1 + \frac{P_2}{1 + N_2 + P_1}\right) - \frac{P_2}{1 + N_2 + P_1 + P_2} \\
&= -\log\left(1 - \frac{P_2}{1 + N_2 + P_1 + P_2}\right) - \frac{P_2}{1 + N_2 + P_1 + P_2} \\
&> 0.
\end{aligned}$$

Further, observe that as $\beta \uparrow \beta_{max}$, $\psi'(\beta) \to -\infty$. From above considerations, $\psi'(\beta) = 0$, say β_0, has exactly one root in $(0, \beta_{max})$. Therefore $\psi(\beta)$ increases in $(0, \beta_0)$ and decreases in (β_0, β_{max}). Additionally, observe that as $\beta \uparrow \beta_{max}$, $\psi(\beta) \to -\infty$. From this, the lemma follows. □

B. Proof of Lemma 3

Assuming (P_1, P_2, N_2) satisfy (4), we know that $\beta_{sato} = \frac{(N_2+P_2)(1+N_2+P_1)}{P_2(1+P_1)}$, and that

$$g_{\beta_{sato}}(Q_1, Q_2) := f_\beta(Q_1, Q_2) - \frac{1}{2(1 + P_1)} Q_1 + \frac{\beta}{2(1 + P_1 + N_2)} Q_1 - \frac{\beta}{2(1 + P_1 + P_2 + N_2)} Q_2,$$

has a global maximum at (P_1, P_2) (the largest such β is the characterization of β_{sato}). Second derivative conditions for local optimality yield

$$\beta_{sato} \le \frac{(1 + P_1 + N_2)^2}{(1 + P_1)^2} \iff \frac{P_2 + N_2}{P_2} \le \frac{(1 + P_1 + N_2)}{(1 + P_1)},$$

implying $P_2 \ge P_1 + 1$ as desired.

The proof of the second condition is a bit more involved. We define the following function $\phi_1 : (-1, \infty) \to \mathbb{R}$ by:

$$\phi_1(x) = \frac{(N_2 + P_2)(1 + N_2 + x)}{P_2(1 + x)} \log\left(1 + \frac{P_2}{1 + N_2 + x}\right) - \frac{N_2(N_2 + P_2)}{(1 + x)(1 + N_2 + x + P_2)} + \log\left(1 - \frac{N_2 + P_2}{1 + N_2 + x + P_2}\right).$$

It's immediate to verify that

$$\psi\left(\frac{(N_2 + P_2)(1 + N_2 + P_1)}{P_2(1 + P_2)}\right) = \phi_1(P_1).$$

Lemma 6. *If $\phi_1(P_1) \ge 0$, for $P_1 > 0$, then $\phi_1(0) > 0$. Furthermore, there is a unique point $y_0 > 0$ such $\phi_1(y_0) = 0$ and $\phi_1(x) < 0$ $\forall x > y_0$. Further, $\phi_1(x) > 0$ for $0 < x < y_0$.*

Proof. As $x \to \infty$, observe that

$$\begin{aligned}
\phi_1(x) &= \frac{(N_2 + P_2)}{(1 + x)} - \frac{N_2(N_2 + P_2)}{(1 + x)(1 + N_2 + x + P_2)} - \frac{N_2 + P_2}{1 + N_2 + x + P_2} \\
&\quad - \frac{(N_2 + P_2)P_2}{2(1 + N_2 + x)(1 + x)} - \frac{(N_2 + P_2)^2}{2(1 + N_2 + x + P_2)^2} + O\left(\frac{1}{x^3}\right) \\
&= \frac{P_2(N_2 + P_2)}{(1 + x)(1 + N_2 + x + P_2)} - \frac{(N_2 + P_2)P_2}{2(1 + N_2 + x)(1 + x)} - \frac{(N_2 + P_2)^2}{2(1 + N_2 + x + P_2)^2} + O\left(\frac{1}{x^3}\right) \\
&= \frac{1}{x^2}\left(P_2(P_2 + N_2) - \frac{1}{2} P_2(P_2 + N_2) - \frac{1}{2}(P_2 + N_2)^2\right) + O\left(\frac{1}{x^3}\right) \\
&= -\frac{1}{2x^2} N_2(N_2 + P_2) + O\left(\frac{1}{x^3}\right).
\end{aligned}$$

Therefore, eventually, $\phi_1(x)$ is negative and tends to 0 from below as $x \to \infty$. Note that for $x \geq 0$, $\phi_1(x)$ and $\hat{\phi}_1(x) := (1+x)\phi_1(x)$ have the same sign. Further, from the above estimate, we also have that $\hat{\phi}_1(x)$ is negative and tends to 0 from below as $x \to \infty$.

Observe that, rearrangement of terms yields,

$$\hat{\phi}_1(x) = r(1+x) - \frac{N_2+P_2}{P_2} r(1+N_2+x) + \frac{N_2}{P_2} r(1+N_2+P_2+x) - \frac{N_2(N_2+P_2)}{1+x+P_2+N_2},$$

where $r(x) := x\log(x)$. Since $r''(x) = \frac{1}{x}$, we obtain that

$$\begin{aligned}
\hat{\phi}_1''(x) &= r''(1+x) - \frac{N_2+P_2}{P_2} r''(1+N_2+x) + \frac{N_2}{P_2} r''(1+N_2+P_2+x) - \frac{2N_2(N_2+P_2)}{(1+x+P_2+N_2)^3} \\
&= \frac{1}{1+x} - \frac{N_2+P_2}{P_2(1+N_2+x)} + \frac{N_2}{P_2(1+N_2+P_2+x)} - \frac{2N_2(N_2+P_2)}{(1+x+P_2+N_2)^3} \\
&= \frac{N_2(N_2+P_2)}{(1+x)(1+N_2+x)(1+N_2+P_2+x)} - \frac{2N_2(N_2+P_2)}{(1+x+P_2+N_2)^3} \\
&= \frac{N_2(N_2+P_2)}{(1+x)(1+N_2+x)(1+N_2+P_2+x)^3} \left((1+x+P_2+N_2)^2 - 2(1+x)(1+N_2+x)\right)
\end{aligned}$$

Observe that $\hat{\phi}_1''(x) = 0$ has exactly one root in the interval $(-1,\infty)$. Furthermore, $\hat{\phi}_1(x)$ is initially convex and then concave. Therefore, combining with the earlier argument, eventually $\hat{\phi}_1(x)$ is concave, increasing, and tends to 0 from below as $x \to \infty$. Further, a simple calculation yields that, $\lim_{x\to -1^+} \hat{\phi}_1''(-1) = -\infty$. Therefore, in the interval $(-1,\infty)$, the function $\hat{\phi}_1(x)$ is initially convex and decreasing, reaches a local minimum (where the value is negative), starts increasing, turns concave and asymptotes from zero from the negative side. Putting this together, if $\hat{\phi}_1(-1) < 0$, then the function always remains negative in the interval $(-1,\infty)$. On the other hand, if $\hat{\phi}_1(-1) \geq 0$, it first decreases to a local minimum at x_*, where the function takes a negative value. Then, it remains negative for $x > x_*$. Clearly, if $\phi_1(P_1) \geq 0$, for $P_1 > 0$, then we must have $\hat{\phi}_1(-1) \geq 0$, and since it is decreasing initially, $\phi_1(0) > 0$. Further, as argued, it has exactly one positive root $y_0 > 0$ such $\phi_1(y_0) = 0$ and $\phi_1(x) < 0\ \forall x \geq y_0$. This establishes the lemma. □

Now we're ready to state the proof of the desired lemma.

Proof. Define the following function:

$$\phi_2(x) = \frac{N_2(1+N_2+P_2)}{P_2}\log(1+N_2+P_2) - \frac{(P_2+N_2)(1+N_2)}{P_2}\log(1+N_2) - \frac{N_2(N_2+x+P_2)}{1+N_2+x+P_2}.$$

Then the desired inequality is equivalent to $\phi_2(P_1) \geq 0$. However, the following calculation shows that $\phi_2(x) \geq 0$ whenever $\phi_1(x) \geq 0$. Observe that

$$\begin{aligned}
\phi_1'(x) = &-\frac{(N_2+P_2)N_2}{P_2(1+x)^2}\log\left(1+\frac{P_2}{1+N_2+x}\right) - \frac{N_2+P_2}{(1+x)(1+N_2+x+P_2)} \\
&+ \frac{N_2(N_2+P_2)}{(1+x)^2(1+N_2+x+P_2)} + \frac{N_2(N_2+P_2)}{(1+x)(1+N_2+x+P_2)^2} \\
&+ \frac{N_2+P_2}{(x+1)(1+x+P_2+N_2)} \\
= &-\frac{(N_2+P_2)N_2}{P_2(1+x)^2}\log\left(1+\frac{P_2}{1+N_2+x}\right) + \frac{N_2(N_2+P_2)}{(1+x)^2(1+N_2+x+P_2)} + \frac{N_2(N_2+P_2)}{(1+x)(1+N_2+x+P_2)^2} \\
\phi_2'(x) = &-\frac{N_2}{(1+N_2+x+P_2)^2}
\end{aligned}$$

then

$$\begin{aligned}
&(\phi_1-\phi_2)'(x) \\
&= -\frac{(N_2+P_2)N_2}{P_2(1+x)^2}\log\left(1+\frac{P_2}{1+N_2+x}\right) + \frac{N_2(N_2+P_2)}{(1+x)^2(1+N_2+x+P_2)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{N_2(N_2+P_2)}{(1+x)(1+N_2+x+P_2)^2}+\frac{N_2}{(1+N_2+x+P_2)^2}\\
&=-\frac{(N_2+P_2)N_2}{P_2(1+x)^2}\log\left(1+\frac{P_2}{1+N_2+x}\right)+\frac{N_2}{(1+x)^2}\\
&=\frac{N_2}{(1+x)^2}\left(1-\frac{N_2+P_2}{P_2}\log\left(1+\frac{P_2}{1+N_2+x}\right)\right),
\end{aligned}
$$

where the second step follows from

$$
\begin{aligned}
&\frac{1}{(1+N_2+x+P_2)^2}+\frac{(N_2+P_2)}{(1+x)^2(1+N_2+x+P_2)}+\frac{(N_2+P_2)}{(1+x)(1+N_2+x+P_2)^2}\\
&=\frac{1+x}{(1+x)(1+N_2+x+P_2)^2}+\frac{(N_2+P_2)}{(1+x)^2(1+N_2+x+P_2)}+\frac{(N_2+P_2)}{(1+x)(1+N_2+x+P_2)^2}\\
&=\frac{1}{(1+x)(1+N_2+x+P_2)}+\frac{(N_2+P_2)}{(1+x)^2(1+N_2+x+P_2)}\\
&=\frac{1+x}{(1+x)^2(1+N_2+x+P_2)}+\frac{(N_2+P_2)}{(1+x)^2(1+N_2+x+P_2)}\\
&=\frac{1}{(1+x)^2}.
\end{aligned}
$$

Note that the sign of $(\phi_1-\phi_2)'(x)$ depends on the strictly increasing function $\xi:(-1,\infty)\to\mathbb{R}$ defined by

$$
\xi(x):=1-\frac{N_2+P_2}{P_2}\log\left(1+\frac{P_2}{1+N_2+x}\right).
$$

Let

$$
\begin{aligned}
x_1:=\frac{P_2}{e^{\frac{P_2}{P_2+N_2}}-1}-1-N_2&=P_2\left(\frac{1}{1-e^{-\frac{P_2}{P_2+N_2}}}-1\right)-1-N_2\\
&>P_2\left(\frac{1}{\frac{P_2}{P_2+N_2}}-1\right)-1-N_2\\
&=-1,
\end{aligned}
$$

then $\xi(x_1)=0$. We are done if we show that $\phi_1(x_1)<0$. This is because conditioned on $\phi_1(0)>0$, combining with the results of Lemma 6, we have $0\le P_1\le y_0\le x_1$. It is easy to verify from their definition that $\phi_1(0)=\phi_2(0)$. Therefore, as $\int_0^x(\phi_1-\phi_2)'(y)dy\le 0$, for $x\in[0,x_1]$, we have $\phi_2(x)\ge\phi_1(x)$ for such x. Therefore, we have $\phi_2(P_1)\ge\phi_1(P_1)\ge 0$. The following calculations show $\phi_1(x_1)<0$.

Let $y=\frac{P_2}{P_2+N_2}\in(0,1)$. Note that

$$
\begin{aligned}
\phi_1(x_1)&=\frac{(N_2+P_2)(1+N_2+x_1)}{P_2(1+x)}\log\left(1+\frac{P_2}{1+N_2+x_1}\right)-\frac{N_2(N_2+P_2)}{(1+x_1)(1+N_2+x_1+P_2)}\\
&\quad+\log\left(1-\frac{N_2+P_2}{1+N_2+x_1+P_2}\right)\\
&=\frac{1+N_2+x_1}{1+x_1}-\frac{N_2}{1+x_1}+\frac{N_2}{1+x_1+P_2+N_2}\\
&\quad+\log\left(1-\frac{N_2+P_2}{1+N_2+x_1+P_2}\right)\\
&=1+\frac{N_2}{P_2}\frac{1}{\left(1+\frac{1+N_2+x_1}{P_2}\right)}+\log\left(1-\frac{N_2+P_2}{P_2}\frac{1}{\left(1+\frac{1+N_2+x_1}{P_2}\right)}\right)\\
&=1+\frac{(1-y)}{y}\frac{1}{\left(1+\frac{1}{e^y-1}\right)}+\log\left(1-\frac{1}{y}\frac{1}{\left(1+\frac{1}{e^y-1}\right)}\right)
\end{aligned}
$$

$$= 1 + \frac{(1-y)(e^y-1)}{ye^y} + \log\left(1 - \frac{e^y-1}{ye^y}\right).$$

Therefore, we are done if we show that, for $y \in (0,1)$, we have

$$\lambda(y) := 1 + \frac{(1-y)(e^y-1)}{ye^y} + \log\left(1 - \frac{e^y-1}{ye^y}\right) < 0.$$

Note that

$$\lambda'(y) = \frac{(1-e^{-y})}{(e^y(y-1)+1)y^2}\left((e^y-1) - (y^3-y^2+y) - \frac{y^3}{e^y-1}\right).$$

Since

$$\begin{aligned}(e^y-1) - (y^3-y^2+y) - \frac{y^3}{e^y-1} &\geq y + \frac{y^2}{2} + \frac{y^3}{6} - (y^3-y^2+y) - \frac{y^3}{y+\frac{y^2}{2}} \\ &= -\frac{y^2}{12+6y}\,(5y+6)\,(y-1) \\ &\geq 0.\end{aligned}$$

We know $\lambda(y)$ increases in $(0,1)$, and note that $\lambda(1) = 0$; hence the result follows.

□

C. Proof of Lemma 5

Proof. Note that

$$g_\beta(P_1+P_2+N_2, 0) = \frac{1}{2}\left(\log(P_1+P_2+N_2+1) - \frac{P_1+P_2+N_2}{P_1+P_2+N_2+1}\right)$$

and

$$\begin{aligned}g_\beta(P_1,P_2) = \frac{1}{2}\big(&\log(P_1+P_2+1+N_2) + \beta\log(P_2+N_2) - \log(P_2) - (\beta-1)\log(N_2) + (\beta-1)\log(\beta-1) - \beta\log(\beta)\big) \\ &- \frac{1}{2}\left(\frac{P_1+P_2}{P_1+P_2+1+N_2} + \frac{\beta P_2}{P_2+N_2} - 1\right).\end{aligned}$$

Let β be such that $(P_1,P_2) \in \mathcal{R}_3$. Define

$$\phi_3(x) = (\beta-1)\log\left(\frac{P_2(\beta-1)}{N_2}\right) - \beta\log\left(\frac{\beta P_2}{P_2+N_2}\right) + \frac{N_2}{x+P_2+1+N_2} - \frac{\beta P_2}{P_2+N_2} + 1,$$

then $g_\beta(P_1+P_2+N_2,0) \leq g_\beta(P_1,P_2)$ is equivalent to $\phi_3(P_1) \geq 0$. Note that when $\beta = \beta_{sato} = \frac{P_2+N_2}{P_2}(1+N_2)$, we have $\phi_2(P_1) = \phi_3(P_1)$; when $\beta = \beta_{costa} = \frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{1+P_1}\right)$, $\phi_1(P_1) = \phi_3(P_1)$. Recall the result obtained in Lemma 3, since $\phi_1(P_1) \geq 0$ (assumption in the conjecture), then $\phi_2(P_1) \geq \phi_1(P_1) \geq 0$, thus if $\beta \mapsto \phi_3(P_1)$ is monotone in $\left(\frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{1+P_1}\right), \frac{P_2+N_2}{P_2}(1+N_2)\right)$, then we're done. Let $P_1 \geq 0$ be such that $\phi_1(P_1) \geq 0$. Note that

$$\begin{aligned}\frac{\partial}{\partial\beta}\phi_3(P_1) &= \log\left(\frac{\beta-1}{\beta}\right) + \log\left(\frac{P_2+N_2}{N_2}\right) - \frac{P_2}{P_2+N_2}, \\ \frac{\partial^2}{\partial\beta^2}\phi_3(P_1) &= \frac{1}{\beta-1} - \frac{1}{\beta} > 0,\end{aligned}$$

we know that $\beta \mapsto \phi_3(P_1)$ is convex in

$$\left(\frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{1+P_1}\right), \frac{P_2+N_2}{P_2}(1+N_2)\right).$$

Note that if we can show that the mapping is increasing in β in the above interval, then we're done. By convexity, it suffices to show

$$\left.\frac{\partial}{\partial\beta}\phi_3(P_1)\right|_{\beta=\frac{P_2+N_2}{P_2}\left(1+\frac{N_2}{1+P_1}\right)} = \log\left(1+\frac{P_2}{1+P_1+N_2}\right) - \frac{P_2}{P_2+N_2} \geq 0.$$

Recall in the proof of Lemma 3, we showed that $0 \leq P_1 \leq y_0 \leq x_1$, where x_1 is the solution of

$$1 - \frac{N_2 + P_2}{P_2} \log \left(1 + \frac{P_2}{1 + N_2 + x} \right) = 0,$$

hence the result follows. □