# GAUSSIAN Z-INTERFERENCE CHANNEL: AROUND THE CORNER

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Abstract. We investigate the vicinity of the newly established corner point for the capacity region of the Gaussian Z-interference channel. We determine the expression for the slope of the Han and Kobayashi region with Gaussian signaling just around the corner.

## 1. Introduction

Consider a Gaussian $Z$-interference channel described by

$$Y_1 = X_1 + Z_1,$$
$$Y_2 = X_2 + aX_1 + Z_2,$$

with $a < 1$. Here $X_1$ and $X_2$ are transmitted signals, $Z_1$ and $Z_2$ are Gaussian unit-power noises and $Y_1$ and $Y_2$ are received signals. The coefficient $a$ denotes an interference gain. The optimality of Han-Kobayashi achievable region with Gaussian inputs is an (important) unresolved open problem in this area. Part of the difficulty is in the actual computation of the Han-Kobayashi inner bound (even restricting one to Gaussian inputs). In this paper we explicitly compute the slope of the achievable region, around the corner of the newly established corner point of the Gaussian $Z$-interference channel. By investigating the behavior of the capacity region around the corner one may be able to develop new ideas for proving converses or deduce the sub-optimality of Han-Kobayashi achievable region with Gaussian inputs. The idea of using concave envelopes to compute Han-Kobayashi region of interference channels, for a different setting, was done in [2]. The technique here is slightly different from that in [2]. The observation that time sharing using different powers enhances the achievable region in the Gaussian-Z-interference channels was explored in [1]. It turns out that such a time-sharing strategy also affects the slope around the corner of the Gaussian Z interference channel.

### 1.1. **Han and Kobayashi region with Gaussian signals and slope at corner point.**

For a Z-interference channel it is immediate (see [3] for instance) that the Han-Kobayashi region reduces to the union of rate pairs $(R_1, R_2)$ satisfying

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|Q) \\ R_2 &\leq I(X_2; Y_2|U_1, Q) \\ R_1 + R_2 &\leq I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_1, Q) \end{aligned}$$

for some $p(q)p_1(u_1, x_1|q)p_2(x_2|q)$.

It is easy to see that for $\beta > 1$

$$\max_{(R_1,R_2)\in HK} \beta R_2 + R_1 = \max_{p(q)p_1(u_1,x_1|q)p_2(x_2|q)} (\beta-1)I(X_2; Y_2|U_1, Q) + I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_1, Q).$$

That is only the second and third constraints are tight (like a MAC for any $p(q)p_1(u_1, x_1|q)p_2(x_2|q)$ we have a pentagonal region).

We can rewrite the above conditional optimization as follows:

$$\begin{aligned}&(\beta-1)I(X_2:Y_2|U_1,Q)+I(U_1,X_2;Y_2|Q)+I(X_1;Y_1|U_1,Q)\\&\quad=I(X_1,X_2;Y_2|Q)+(\beta-1)I(X_2;Y_2|U_1,Q)-I(X_1;Y_2|U_1,X_2,Q)+I(X_1;Y_1|U_1,Q).\end{aligned}$$

The maximization of the above expression with respect to $p(q)p_1(u_1,x_1|q)p_2(x_2|q)$ can be equivalently seen to be the following computation. For a given pair of independent distributions $p_1(x_1)p_2(x_2)$ compute the quantity

$$I(X_1,X_2;Y_2)+\max_{p_1(u_1|x_1)}(\beta-1)I(X_2;Y_2|U_1)-I(X_1;Y_2|U_1,X_2)+I(X_1;Y_1|U_1)$$

which yields

$$I(X_1,X_2;Y_2)+\mathcal{C}_{X_1}\left[(\beta-1)I(X_2:Y_2)-I(X_1;Y_2|X_2)+I(X_1;Y_1)\right].$$

Using the above function values at independent points, we can create a further concave envelope over the set of joint distributions and the maximum over this yields the Han-Kobayashi weighted sum rate in general. For discrete alphabets with no constraints on the distribution, the latter concave envelope is not necessary, since the maximum value of a function and that of its concave envelope is the same.

Coming back to Gaussian signaling, we set $X_1=U_1+V_1$ where $U_1,V_1$ are independent Gaussians with power $(1-\alpha)Q_1$ and $\alpha Q_1$ respectively. Let $X_2$ be Gaussian with power $Q_2$. We define the following function of $Q_1,Q_2$, say $f_\beta(Q_1,Q_2)$ according to

$$\frac{1}{2}\log(1+a^2Q_1+Q_2)+\max_{\alpha\in[0,1]}\left\{\frac{\beta-1}{2}\log\left(1+\frac{Q_2}{a^2\alpha Q_1+1}\right)-\frac{1}{2}\log(1+a^2\alpha Q_1)+\frac{1}{2}\log(1+\alpha Q_1)\right\}.$$

Then the weighted Han-Kobayashi sum-rate for power $P_1,P_2$ is the concave envelope of $f_\beta(Q_1,Q_2)$ at the point $(P_1,P_2)$.

A simple computation for the optimal alpha leads to the following expression for $f_\beta(Q_1,Q_2)$.

**Lemma 1.** *The function $f_\beta(Q_1,Q_2)$ is given by the following*

$$f_\beta(Q_1,Q_2)=\begin{cases}\frac{1}{2}\log(1+a^2Q_1+Q_2)+\frac{\beta-1}{2}\log\left(1+Q_2\right) & \beta\geq\frac{1-a^2+Q_2}{a^2Q_2}\\ \frac{\beta}{2}\log\left(1+\frac{Q_2}{a^2Q_1+1}\right)+\frac{1}{2}\log(1+Q_1) & \beta\leq\frac{(1-a^2+Q_2)(1+a^2Q_1)}{a^2Q_2(1+Q_1)}\\ \frac{1}{2}\log\frac{(1+a^2Q_1+Q_2)}{a^2Q_2}+\frac{\beta-1}{2}\log(\beta-1) & \\ \quad-\frac{\beta}{2}\log\beta+\frac{\beta}{2}\log\frac{(1-a^2+Q_2)}{1-a^2} & o.w.\end{cases}.$$

As a consequence, we have the following

- When $\beta\geq\frac{1-a^2}{a^2Q_2}+\frac{1}{a^2}$ the function

$$f_\beta(Q_1,Q_2)=\frac{1}{2}\log(1+a^2Q_1+Q_2)+\frac{\beta-1}{2}\log\left(1+Q_2\right),$$

  or the maximum happens at $\alpha=0$.
- When $\beta$ is slightly smaller than $\frac{1-a^2}{a^2Q_2}+\frac{1}{a^2}$ then

$$f_\beta(Q_1,Q_2)>\frac{1}{2}\log(1+a^2Q_1+Q_2)+\frac{\beta-1}{2}\log\left(1+Q_2\right),$$

  or the maximum happens at $\alpha\in(0,1)$ and not at $\alpha=0$. Thus maximum is larger than the value at $\alpha=0$.

*Remark* 1. Thus, from the above analysis, for $\beta$ slightly smaller than $\frac{1-a^2}{a^2P_2}+\frac{1}{a^2}$ clearly the value of $\beta R_2+R_1$ for HK is clearly at least as large as $f_\beta(P_1,P_2)$ which is strictly larger than

$$\frac{1}{2}\log(1+a^2P_1+P_2)+\frac{\beta-1}{2}\log\left(1+P_2\right),$$

thus the hyperplane, $\beta R_2+R_1$ passes above the corner point.

**Theorem 1.** *Consider a Gaussian Z-interference channel. The smallest $\beta$ such that the supporting hyperplane of the form $\beta R_2+R_1$ of Han-Kobayashi signaling scheme with Gaussian inputs passes through the corner point is given by*

$$\beta_{cr}=1+\max\left\{\frac{-\log a^2-\frac{1-a^2}{(1+a^2P_1+P_2)}}{\log(1+P_2)-\frac{P_2}{1+P_2}},\frac{(1-a^2)(1+P_2)}{a^2P_2}\right\}.$$

*Proof.* The theorem is equivalent to showing that the concave envelope of $f_\beta(Q_1,Q_2)$ at $(P_1,P_2)$ equals $f_\beta(P_1,P_2)$ and hence passes through the corner point *if and only if* $\beta\geq\beta_{cr}$. The necessity of the condition

$$\beta\geq 1+\frac{(1-a^2)(1+P_2)}{a^2P_2}=\frac{1-a^2}{a^2P_2}+\frac{1}{a^2}$$

follows from Remark 1.

A simple supporting plane argument (or duality) yields that the concave envelope of $f_\beta(Q_1,Q_2)$ at $(P_1,P_2)$ equals $f_\beta(P_1,P_2)$ will happen *if and only if* the function

$$g_\beta(Q_1,Q_2):=f_\beta(Q_1,Q_2)-\frac{a^2}{2(1+a^2P_1+P_2)}Q_1-\left(\frac{1}{2(1+a^2P_1+P_2)}+\frac{\beta-1}{2(1+P_2)}\right)Q_2$$

has a global maximum at $(Q_1,Q_2)=(P_1,P_2)$.

**Interior analysis.** In the first part of the proof, we will show that when $\beta\geq\frac{1-a^2+P_2}{a^2P_2}$ then $g_\beta(Q_1,Q_2)$ has a unique local maximum at $(Q_1,Q_2)=(P_1,P_2)$.

For any $Q_1$, if $Q_2\geq P_2$ then $f_\beta(Q_1,Q_2)$ (given by first expression in Lemma 1) is jointly concave in $Q_1,Q_2$, hence in this regime there will be a unique stationary point (hence local maximum) at $(Q_1,Q_2)=(P_1,P_2)$.

For $(Q_1,Q_2)$ such that

$$\frac{1-a^2}{a^2P_2}+\frac{1}{a^2}\leq\beta<\frac{(1-a^2+Q_2)(1+a^2Q_1)}{a^2Q_2(1+Q_1)}$$

the function $f_\beta(Q_1,Q_2)$ is given by the second expression in Lemma 1. Hence for a local maximum to exist in this region we must have (first derivative conditions)

$$\frac{a^2}{2(1+a^2P_1+P_2)}=\frac{\beta a^2}{2(1+a^2Q_1+Q_2)}-\frac{\beta a^2}{2(1+a^2Q_1)}+\frac{1}{2(1+Q_1)}$$

$$\frac{1}{2(1+a^2P_1+P_2)}+\frac{\beta-1}{2(1+P_2)}=\frac{\beta}{2(1+a^2Q_1+Q_2)}.$$

By observing that $\frac{\beta-1}{2(1+P_2)}\leq\frac{\beta}{2}-\frac{1}{2a^2}$ we see that the second condition above yields that

$$\frac{1}{2(1+a^2P_1+P_2)}+\frac{\beta}{2}-\frac{1}{2a^2}\geq\frac{\beta}{2(1+a^2Q_1+Q_2)}$$

and plugging above in first equation we obtain that

$$0 \leq \frac{\beta a^2}{2} - \frac{1}{2} - \frac{\beta a^2}{2(1+a^2 Q_1)} + \frac{1}{2(1+Q_1)}$$

or equivalently

$$\frac{\beta a^4}{1+a^2 Q_1} \geq \frac{1}{1+Q_1}. \tag{1}$$

Considering the Hessian (second order conditions for local maximum), note that the sign of the determinant of the Hessian is the opposite of the sign of the second derivative of $\frac{1}{2}\log(1+Q_1) - \frac{\beta}{2}\log(1+a^2 Q_1)$ with respect to $Q_1$. Thus for a point to be the local maximum we need

$$\frac{1}{(1+Q_1)^2} \geq \frac{\beta a^4}{(1+a^2 Q_1)^2}.$$

From (1) above, to satisfy the second derivative condition, we need

$$\frac{1}{(1+Q_1)} \geq \frac{1}{(1+a^2 Q_1)},$$

which cannot hold if $Q_1 > 0$ and $a \in [0,1)$.

In the final (third) regime in Lemma 1 of $(Q_1, Q_2)$, a stationary point exists only if $a^2 Q_1 + Q_2 = a^2 P_1 + P_2$ and

$$\frac{\beta}{1-a^2+Q_2} - \frac{1}{Q_2} = \frac{\beta - 1}{1+P_2} \leq \beta - \frac{1}{a^2},$$

where the last inequality follows from the relation between $P_2$ and $\beta$. If $Q_2 \geq a^2$ then this is equivalent to

$$\beta \geq \frac{1-a^2+Q_2}{a^2 Q_2},$$

but this cannot happen since we are in the final regime. On the other hand if $a^2 > Q_2$ then the sign of the inequality reverses and a stationary point is plausible.

Considering the second order conditions, to be a local maximum we further need that

$$\frac{\beta}{2}\log(1-a^2+Q_2) - \frac{1}{2}\log Q_2$$

is concave in $Q_2$; and this will happen if

$$\beta \frac{1}{(1-a^2+Q_2)^2} \geq \frac{1}{Q_2^2}.$$

However since

$$\beta < \frac{1-a^2+Q_2}{a^2 Q_2}$$

the previous inequality is infeasible when $a^2 > Q_2$. Thus we cannot have a stationary point in the third regme of parameters as well.

**Boundary points.** The above analysis fails to consider the possibility of global maximum happening at boundary points. One can easily eliminate $Q_1$ or $Q_2$ that tends to $\infty$ since the value of the function would tend to $-\infty$. Therefore it suffices to restrict ourselves to $(Q_1, 0)$ and $(0, Q_2)$.

*Along the line* $(Q_1, 0)$:
Maximum along the line $(Q_1, 0)$ is obtained when

$$1 + Q_1 = \frac{1 + a^2 P_1 + P_2}{a^2}.$$

Thus we need

$$\begin{aligned}
&\frac{1}{2}\log(1+Q_1) - \frac{a^2}{2(1+a^2P_1+P_2)}Q_1 \\
&\quad \le \frac{1}{2}\log(1+a^2P_1+P_2) + \frac{\beta-1}{2}\log(1+P_2) - \frac{a^2}{2(1+a^2P_1+P_2)}P_1 \\
&\quad - \left(\frac{1}{2(1+a^2P_1+P_2)} + \frac{\beta-1}{2(1+P_2)}\right)P_2. \qquad \Longleftrightarrow \\
&-\frac{1}{2}\log(a^2) - \frac{1}{2} \le \frac{\beta-1}{2}\log(1+P_2) - \frac{a^2}{2(1+a^2P_1+P_2)}(1+P_1) \\
&\quad - \left(\frac{1}{2(1+a^2P_1+P_2)} + \frac{\beta-1}{2(1+P_2)}\right)P_2 \qquad \Longleftrightarrow \\
&-\log(a^2) \le (\beta-1)\log(1+P_2) + \frac{1-a^2}{(1+a^2P_1+P_2)} - \frac{(\beta-1)P_2}{(1+P_2)}
\end{aligned}$$

Hence this holds *if and only if*

$$\beta_{cr} - 1 \ge \frac{-\log a^2 - \frac{1-a^2}{(1+a^2P_1+P_2)}}{\log(1+P_2) - \frac{P_2}{1+P_2}}. \tag{2}$$

*Along the line* $(0, Q_2)$:
Maximum along this line is obtained at $Q_2 = Q_2^*$ where

$$\frac{\beta}{1+Q_2^*} = \frac{1}{1+a^2P_1+P_2} + \frac{\beta-1}{1+P_2}$$

Thus for the value at $(P_1, P_2)$ to be larger than the value at the boundary point above we need that

$$\begin{aligned}
&\frac{\beta}{2}\log(1+Q_2^*) - \left(\frac{1}{2(1+a^2P_1+P_2)} + \frac{\beta-1}{2(1+P_2)}\right)Q_2^* \\
&\quad \le \frac{\beta-1}{2}\log(1+P_2) + \frac{1}{2}\log(1+a^2P_1+P_2) - \frac{a^2P_1}{2(1+a^2P_1+P_2)} \\
&\quad - \left(\frac{1}{2(1+a^2P_1+P_2)} + \frac{\beta-1}{2(1+P_2)}\right)P_2.
\end{aligned}$$

Plugging in the choice of $Q_2^*$ obtained and elementary manipulations yield that we require

$$\frac{\beta}{2}\log\left(\frac{\beta(1+P_2)(1+a^2P_1+P_2)}{\beta(1+P_2)+(\beta-1)a^2P_1}\right) \leq \frac{\beta-1}{2}\log(1+P_2)+\frac{1}{2}\log(1+a^2P_1+P_2).$$

This is equivalent to requiring

$$\log(1+P_2)+(\beta-1)\log(1+a^2P_1+P_2) \leq \beta\log\left(\frac{\beta(1+P_2)+(\beta-1)a^2P_1}{\beta}\right).$$

But by the convexity of log() the above is true for any $\beta \geq 1$.

Combining the interior analysis, Remark 1 and (2), we see that the critical slope is given by

$$\beta_{cr} = 1+\max\left\{\frac{-\log a^2 - \frac{1-a^2}{(1+a^2P_1+P_2)}}{\log(1+P_2)-\frac{P_2}{1+P_2}}, \frac{(1-a^2)(1+P_2)}{a^2P_2}\right\}.$$

□

## Conclusion

We established the slope of the Han-Kobayashi region (with Gaussian signaling) at the corner point of the Gaussian Z-interference channel. The technique employed may be of interest to other settings since we performed the optimization of the dual functional to compute the concave envelope.

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