Some results on the scalar Gaussian interference channel

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***Abstract*—We study the optimality of Gaussian signaling (with power control) for the two-user scalar Gaussian interference channel. The capacity region is shown to exhibit a discontinuity of slope around the sum-rate point for a subset of the very weak interference channel. We also show that using colored Gaussians (multi-letter) does not improve on the single-letter region of Gaussian signaling with power control. Finally, we also present an approach to test the optimality of Gaussian signaling motivated by some calculations of the slope of Han-Kobayashi region near the corner point of the Z-interference channel.**

I. Introduction

Determining the capacity region of the scalar Gaussian interference channel remains one of the central open problems in multi-user information theory. Last year, Han-Kobayashi achievable region was shown to be a strict subset of the capacity region [12]. However, when restricted to additive Gaussian noise model - commonly referred to as the Gaussian interference channel - the optimality of Han-Kobayashi scheme is still unresolved. Furthermore, it is not even established whether Gaussian signaling with power control (represented by the time sharing variable Q) can be improved upon.

We restrict our attention to the symmetric setting of the Gaussian interference channel (or the Z-interference channel), though the arguments presented here can be easily generalized to the asymmetric case as well. Consider the symmetric Gaussian interference channel

$$\begin{aligned} Y_1 &= X_1 + aX_2 + Z_1 \\ Y_2 &= aX_1 + X_2 + Z_2 \end{aligned} \tag{1}$$

where both transmitters operate under a power constraint P.

The scenario $a \geq 1$ reduces to the strong interference setting and the capacity region is completely characterized. When $a < 1$ the setting is called weak interference, and only certain points on the capacity region have been established.

- *Corner point*: A recent work [13] establishes the corner points of the capacity region, i.e., the maximum R_2 when $R_1 = \frac{1}{2}\log(1+P)$ and its symmetric point.
- *Sum-capacity*: Treating interference as noise (with Gaussian signaling) is shown to yield the maximum sum rate [2], [11], [15] when

$$2a(1+a^2P) \leq 1.$$

Since the capacity region is convex, the region can be effectively characterized as the intersection of its supporting hyperplanes. We use this approach to investigate the optimality of Gaussian signaling for the Gaussian interference channel.

Our first result (Theorem 1) extends the results concerning the sum-rate and shows that even certain weighted sum-rates pass through the treating interference as noise point. This is an easy generalization of the arguments used to show the sum-capacity.

We have tried many approaches, including the idea of using Hermite polynomials [1], to show that non-Gaussian signals improve on Gaussian signaling with power control. However, we were unsuccessful in our attempts, and it was seen that the technique of using Hermite polynomials fails to achieve any improvement. A different approach was to consider multi-letter treating interference as noise strategy and to consider Gaussian signals as inputs. Our second result (Theorem 2) is to prove that this strategy does not improve on the single letter Gaussian signaling with power control. The techniques used here may be of independent interest.

Recently [7] has evaluated the slope of Han–Kobayashi region for the Gaussian-Z-interference channel around the corner point. Han–Kobayashi has a finite slope around the corner point while the outer bound developed in [13] has an infinite slope. In Section IV we outline an information inequality (Hypothesis 1) whose veracity is essentially equivalent to the optimality of Gaussian signaling around the corner point.

II. Symmetric Gaussian Interference

Consider a symmetric Gaussian interference channel setting (1) where the parameters satisfy

$$2a(1+a^2P) \leq 1. \tag{2}$$

Using genie-aided technique, in Theorem 1, we prove an outer bound which shows that even for a set of weighted sum rates, the supporting hyperplanes to the capacity region pass through the maximum sum-rate point. The proof ideas are not novel but the result may be of interest since one normally does not expect a discontinuity of slope at the sum-rate maximal point of the weak interference.

Theorem 1. *For a symmteric Gaussian interference channel, consider any $a \leq 1$ satisfying* (2), *$\forall \lambda \geq 1$ such that*

$$\lambda \leq 1 + \frac{1 - 2a(1+a^2P)}{a^2((1+a^2P)^2 + P(1 - a(1+a^2P)))}.$$

Then we have the following

$$\max_{(R_1,R_2)\in\mathcal{C}} \lambda R_1 + R_2 \leq \frac{\lambda+1}{2}\log\left(1+\frac{P}{1+a^2P}\right),$$

where $\mathcal{C}$ refers to the capacity region.

Remark: Clearly equality can be achieved by "treating interference as noise". So the expression above is indeed tight. Further, by symmetric considerations, the above (tight) bound also holds for the maximum of $R_1+\lambda R_2$, for the same range of λ.

Proof. We parameterize the Gaussian noise at each receiver as follows:

$$Z_1 = Z_{11}\sin\theta_1 + Z_{12}\cos\theta_1,$$
$$Z_2 = Z_{21}\sin\theta_2 + Z_{22}\cos\theta_2,$$

where $0 \leq \theta_i \leq \frac{\pi}{2}$ and Z_{ij}s are independent standard Gaussians. Define the following "virtual" receivers

$$T_1 = X_1 + \eta_1 Z_{11}, \quad \hat{Y}_1 = aX_1 + Z_{22}\cos\theta_2,$$
$$T_2 = X_2 + \eta_2 Z_{21}, \quad \hat{Y}_2 = aX_2 + Z_{12}\cos\theta_1, \qquad (3)$$

where the parameters are assumed to satisfy

$$a^2\eta_1^2 \leq \cos^2\theta_2, \qquad (4)$$
$$a^2\eta_2^2 \leq \cos^2\theta_1, \qquad (5)$$

In other words $\hat{Y}_1$ is a stochastically degraded version of T_1 and $\hat{Y}_2$ is a stochastically degraded version of T_2.
Starting from Fano's inequality:

$$\begin{aligned}
&n(\lambda R_1+R_2)-n\epsilon\\
&\leq \lambda I(X_1^n;Y_1^n)+I(X_2^n;Y_2^n)\\
&\leq \lambda I(X_1^n;Y_1^n,T_1^n)+I(X_2^n;Y_2^n,T_2^n)\\
&= \lambda h(Y_1^n|T_1^n)-\lambda h(T_1^n|X_1^n)+h(Y_2^n|T_2^n)-h(T_2^n|X_2^n)\\
&\quad +\lambda h(T_1^n)-h(Y_2^n|T_2^n,X_2^n)+h(T_2^n)-\lambda h(Y_1^n|T_1^n,X_1^n)\\
&= \lambda h(Y_1^n|T_1^n)-\lambda h(T_1^n|X_1^n)+h(Y_2^n|T_2^n)-h(T_2^n|X_2^n)\\
&\quad +\lambda h(T_1^n)-h(\hat{Y}_1^n)+h(T_2^n)-\lambda h(\hat{Y}_2^n)\\
&\overset{(a)}{\leq} n\big(\lambda h(Y_1^*|T_1^*)-\lambda h(T_1^*|X_1^*)+h(Y_2^*|T_2^*)-h(T_2^*|X_2^*)\big)\\
&\quad +\lambda h(T_1^n)-h(\hat{Y}_1^n)+h(T_2^n)-\lambda h(\hat{Y}_2^n)\\
&\overset{(b)}{\leq} n\big(\lambda h(Y_1^*|T_1^*)-\lambda h(T_1^*|X_1^*)+h(Y_2^*|T_2^*)-h(T_2^*|X_2^*))\\
&\quad +\lambda h(T_1^*)-h(\hat{Y}_1^*)+h(T_2^*|U_2^*)-\lambda h(\hat{Y}_2^*|U_2^*)\big)\\
&= n\big(\lambda h(Y_1^*|T_1^*)-\lambda h(T_1^*|X_1^*)+h(Y_2^*|T_2^*)-h(T_2^*|X_2^*)\\
&\quad +\lambda h(T_1^*)-\lambda h(Y_1^*|U_2^*,T_1^*,X_1^*)+h(T_2^*|U_2^*)-h(Y_2^*|T_2^*,X_2^*)\big)\\
&= n\big(\lambda I(X_1^*;T_1^*)+\lambda I(X_1^*,U_2^*;Y_1^*|T_1^*)\\
&\qquad +I(X_2^*;T_2^*|U_2^*)+I(X_2^*;Y_2^*|T_2^*).\big)
\end{aligned} \qquad (6)$$

In the above, X_i^* (similarly U_2^*, T_i^*, Y_i^*), $i=1,2$, denote a Gaussian random variable with the same power as X_i (similarly U_2, T_i, Y_i respectively); (a) follows by maximum conditional differential entropy property [8] and (b) follows from Lemma1 and Lemma 2.

Thus we have that for any choice of virtual receivers, the weighted sum-rate must satisfy

$$(\lambda R_1+R_2) \leq \lambda I(X_1^*;T_1^*,Y_1^*)+I(X_2^*;Y_2^*,T_2^*)$$
$$+\lambda I(U_2^*;Y_1^*|X_1^*,T_1^*)-I(U_2^*;T_2^*)$$

for some $U_2^* \sim \mathcal{N}(0,\alpha P)$ and $X_2^* = U_2^*+V_2^*$ where $V_2^* \sim \mathcal{N}(0,(1-\alpha)P)$ independent of U_2^*. (This outer bound may be of independent interest).

To reduce the outer bound to treating interference as noise inner bound, two conditions need to be met. One is that the optimal U_2^* must be a trivial random variable; and the other is that $X_1^* \to Y_1^* \to T_1^*$ and $X_2^* \to Y_2^* \to T_2^*$ are Markov chains.

For U_2^* to be trivial, a simple calculation reveals that we need

$$\frac{\cos^2\theta_1-\lambda a^2\eta_2^2}{a^2P(\lambda-1)} \geq 1. \qquad (7)$$

Further, for the following Markov chains to hold, one needs

$$\begin{cases} X_1^*\to Y_1^*\to T_1^* & \Rightarrow \eta_1=\frac{1+a^2P}{\sin\theta_1}\\ X_2^*\to Y_2^*\to T_2^* & \Rightarrow \eta_2=\frac{1+a^2P}{\sin\theta_2}\end{cases} \qquad (8)$$

Integrating (8) into (4) and (5), one requires

$$2a(1+a^2P) \leq \sin(\theta_1+\theta_2),$$

which suggests the choice $\theta_1+\theta_2=\frac{\pi}{2}$.
Thus the outer-bound reduces to treating interference as noise, provided, there exists a θ_1 such that

$$a(1+a^2P) \leq \sin^2\theta_1,$$
$$a(1+a^2P) \leq \cos^2\theta_1,$$
$$a^2P(\lambda-1)\cos^2\theta_1 \leq \cos^4\theta_1-\lambda a^2(1+a^2P)^2,$$

so as to satisfy (4), (5) and (7), respectively. Simple algebra shows that such θ_1 exists as long as

$$a^2P(\lambda-1)+\sqrt{a^4P^2(\lambda-1)^2+4\lambda a^2(1+a^2P)^2}$$
$$\leq 2(1-a(1+a^2P)).$$

Solving for λ yields the condition in the theorem. □

Lemma 1. *For T_1 and $\hat{Y}_1$ defined in* (3)*, satisfying (4)*

$$\lambda h(T_1^n)-h(\hat{Y}_1^n) \leq n(\lambda h(T_1^*)-h(\hat{Y}_1^*))$$

Proof. W.l.o.g let $X_1 \to T_1 \to \hat{Y}_1$ be physically degraded (note that the terms only depend on the marginals). Thus $\hat{Y}_1 = T_1+\gamma\hat{Z}$, for some appropriate γ. Hence

$$\begin{aligned}
&\lambda h(T_1^n)-h(\hat{Y}_1^n)\\
&= \lambda h(T_1^n,\hat{Y}_1^n)-\lambda h(\hat{Y}_1^n|T_1^n)-h(\hat{Y}_1^n)\\
&= (\lambda-1)h(\hat{Y}_1^n)+\lambda h(T_1^n|\hat{Y}_1^n)-\lambda nh(\hat{Y}_1^*|T_1^*)\\
&\leq n((\lambda-1)h(\hat{Y}_1^*)+\lambda h(T_1^*|\hat{Y}_1^*)-\lambda h(\hat{Y}_1^*|T_1^*))\\
&= n(\lambda h(T_1^*)-h(\hat{Y}_1^*)),
\end{aligned}$$

where the inequality follows from maximum conditional differential entropy property [8]. We also used the degraded structure, i.e., $h(\hat{Y}_1^n|T_1^n)=h(\gamma\hat{Z}^n)=nh(\hat{Z})=nh(\hat{Y}_1^*|T_1^*)$. □

Lemma 2. *For T_2 and $\hat{Y}_2$ defined in* (3)*, satisfying (5)*

$$h(T_2^n)-\lambda h(\hat{Y}_2^n) \leq n(h(T_2^*|U_2^*)-\lambda h(\hat{Y}_2^*|U_2^*))$$

where U_2^* is some appropriately chosen Gaussian random variable with power $\alpha_2 P_2$, for some $\alpha_2 \in [0,1]$ and $X_2 = U_2^* + V_2^*$ where V_2^* is a Gaussian independent of U_2^* having power $(1-\alpha_2)P_2$.

Proof. There are several proofs of this lemma in the literature. This can be inferred from the entropy power inequality a la Bergman's technique [3]; the extremal inequality of Liu and Vishwanath [10], or from the factorization property of the difference of weighted mutual informations and its Gaussian maximizers, established in [9]. □

III. On Colored Gaussian Inputs

In this section we show that using colored Gaussians over a treating interference as noise scheme will not improve on the single letter scheme with power control. It is known from [5] that treating interference as noise with power control strictly improves on the rate without power control for certain regimes of parameters; and further that it matches Han–Kobayashi scheme restricted to Gaussian inputs, but allowing for power control [6] for a large range of parameters (many regimes for which a converse in not available).

Since colored Gaussians (over several letters) can perform at least as well as single-letter Han–Kobayashi scheme restricted to Gaussian inputs and allowing for power control; we investigated the advantage of allowing correlation across time slots. Theorem 2 below shows that we do not get any improvement by using colored Gaussians. The proof technique is novel and useful in similar settings in other scenarios.

Theorem 2. *Consider a symmetric Gaussian interference channel. To compute*

$$\max_{X_i^n \sim \mathcal{N}(0,K_i), i=1,2} \frac{1}{n}(I(X_1^n;Y_1^n) + I(X_2^n;Y_2^n)),$$

subject to $tr(K_i) = nP, \quad i=1,2$, it suffices to restrict K_i in (9) *to diagonal matrices.*

Proof. Re-writing, we wish to show that the following optimization problem

$$\max_{\substack{K_1,K_2 \succeq 0 \\ tr(K_1),tr(K_2) \leq nP}} \frac{|K_1 + a^2K_2 + I|}{|a^2K_2 + I|}\frac{|K_2 + a^2K_1 + I|}{|a^2K_1 + I|}, \tag{9}$$

has diagonal matrices as maximizers.

Let $\lambda^{\downarrow}(A)$ be the n-tuple of eigenvalues (with multiplicity) of A arranged in descending order, i.e. $\lambda_j^{\downarrow}(A)$ is the j-th largest eigenvalue. Fix $\lambda_j^{\downarrow}(K_i)$. Then $|a^2K_i + I| = \prod_j (a^2\lambda_j^{\downarrow}(K_i) + 1)$ is fixed. It remains to maximize the numerator. Applying a common form of Lidskii-Weidlandt inequality (see equation (26) in [4]), we have

$$\lambda^{\downarrow}(K_1) + \lambda^{\uparrow}(a^2K_2 + I) \prec \lambda^{\downarrow}(K_1 + a^2K_2 + I),$$

where $\lambda^{\uparrow}(A)$ is the n-tuple of eigenvalues of A in ascending order and for n-tuple x, y, $x \prec y$ is the standard notation for x being majorized by y.

Consider $f : \mathbb{R} \to \mathbb{R}$, a real-valued convex function. For any n-tuple $x \in \mathbb{R}^n$, let $f(x) \in \mathbb{R}^n$ be the n-tuple obtained by applying f on each component of x. The condition $x \prec y$, is equivalent to $x = Ay$ for some doubly stochastic matrix A. Hence convexity of f yields

$$f(x) = f(Ay) \leq Af(y) \prec f(y),$$

where the inequality is a component by component one. This means $f(x)$ is weakly majorized by $f(y)$, expressed as $f(x) \prec_w f(y)$; i.e., for all $1 \leq k \leq n$,

$$\sum_{i=1}^k f(x)_i^{\downarrow} \leq \sum_{i=1}^k f(y)_i^{\downarrow}.$$

Now let $f(x) = -\log x$, we have

$$-\log(\lambda^{\downarrow}(K_1) + \lambda^{\uparrow}(a^2K_2 + I)) \prec -\log(\lambda^{\downarrow}(K_1 + a^2K_2 + I)).$$

In particular,

$$\begin{aligned}\prod_{i=1}^n (\lambda_i^{\downarrow}(K_1) + a^2\lambda_i^{\uparrow}(K_2) + 1) &\geq \prod_{i=1}^n \lambda_i^{\downarrow}(K_1 + a^2K_2 + I) \\ &= |K_1 + a^2K_2 + I|.\end{aligned}$$

Hence we have

$$\begin{aligned}&|K_1 + a^2K_2 + I||K_2 + a^2K_1 + I| \\ &\leq \prod_{i=1}^n (\lambda_i^{\downarrow}(K_1) + a^2\lambda_i^{\uparrow}(K_2) + 1)(\lambda_i^{\downarrow}(K_2) + a^2\lambda_i^{\uparrow}(K_1) + 1),\end{aligned}$$

with equality when $K_1 = \text{diag}(\lambda_i^{\downarrow}(K_1)), K_2 = \text{diag}(\lambda_i^{\downarrow}(K_2))$, i.e., they are diagonal matrices with anti-aligned (one decreasing, the other increasing) diagonal entries. □

Remark: Since the optimal colored Gaussians are diagonal, their sum-rate is upper bounded by the convex hull (over powers) of the single-letter sum-rates achieved by Gaussian signaling and treating interference as noise. The above analysis also goes through for weighted sum-rates.

IV. Gaussian Z interference: around the corner point

Intrigued by the previous result which showed that colored Gaussians do not improve on single-letter Han–Kobayashi Gaussian signaling with power control, we tried to investigate the optimality of Han–Kobayashi scheme around the corner point of the Gaussian Z-interference channel. The Gaussian Z-interference channel is described by

$$\begin{aligned}Y_1 &= X_1 + Z_1 \\ Y_2 &= aX_1 + X_2 + Z_2,\end{aligned} \tag{10}$$

where $0 < a < 1$, $Z_i \sim \mathcal{N}(0,1)$ and the power constraints

$$\mathbb{E}[X_1^2] \leq P_1, \quad \mathbb{E}[X_2^2] \leq P_2$$

The following corner point of the capacity region was claimed in [5] and a missing lemma was formally established in [13]:

$$(C_1', C_2') = (\frac{1}{2}\log(1 + \frac{a^2P_1}{1+P_2}), \frac{1}{2}\log(1+P_2)). \tag{11}$$

In a similar vein to the attack in sectionII, we try to find the minimum $\lambda \geq 1$ such that $R_1 + \lambda R_2$ passes through the above corner point. For the outer bound obtained in [13], for no finite λ does the hyper-plane pass through the point (i.e. corner point is an extreme point but not an exposed point, and there is no discontinuity in the derivative).

Subsequently, in [7], it is shown that max $R_1 + \lambda R_2$ for all rates in Han–Kobayashi rate-region with Gaussian signaling and power control, passes through the corner point (11) precisely for all λ larger than λ_{cr} given by

$$\max \left\{ \frac{-\log a^2 - \frac{1-a^2}{(1+a^2P_1+P_2)}}{\log(1+P_2) - \frac{P_2}{1+P_2}}, \frac{(1-a^2)(1+P_2)}{a^2P_2} \right\} + 1. \quad (12)$$

We are concerned with the behavior of the capacity region around this corner point. In particular we wish to resolve the question whether Han–Kobayashi with Gaussian signaling is optimal in the neighborhood of the corner point or not?

Consider the following statement:

Hypothesis 1. *For some choice of P_1, P_2 there exists independent random vectors $\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{R}^n$ for some n, satisfying $\mathrm{E}(\|\mathbf{X}_1\|^2) \leq nP_1, \mathrm{E}(\|\mathbf{X}_2\|^2) \leq nP_2$ such that for some $\lambda \geq \lambda_{cr}$ (given by (12))*

$$n\frac{\lambda-1}{2}\log(1+P_2) < (\lambda-1)h(\mathbf{X}_2 + a\mathbf{X}_1 + \mathbf{Z}) - \lambda h(a\mathbf{X}_1 + \mathbf{Z}) + h(\mathbf{X}_1 + \mathbf{Z}). \quad (13)$$

Our next two lemmas shows the relationship between Hypothesis 1 and the optimlaity of Han–Kobayashi region with Gaussian signaling (and power control).

Lemma 3. *If Hypothesis 1 holds then (single-letter) Han–Kobayashi with Gaussian signaling (and power control) is not optimal.*

Proof. Suppose there exists some $a, P_1, P_2, n, \lambda \geq \lambda_{cr}$, and independent random vectors $\mathbf{X}_1, \mathbf{X}_2$ satisfying power constraints such that

$$n\frac{\lambda-1}{2}\log(1+P_2) + n\delta = (\lambda-1)h(\mathbf{X}_2 + a\mathbf{X}_1 + \mathbf{Z}) - \lambda h(a\mathbf{X}_1 + \mathbf{Z}) + h(\mathbf{X}_1 + \mathbf{Z}),$$

for some $\delta > 0$.

Let $\hat{P}_1 = P_1 + Q_1$ be the true power constraint on the transmitters. Take the transmitted sequence to be $\hat{\mathbf{X}}_1 = \mathbf{X}_1 + \mathbf{U}_1$ where $\mathbf{U} \sim \mathcal{N}(0, Q_1 I)$ independent of $\mathbf{X}_1$. Notice that the λ_{cr} for the parameters$(a, \hat{P}_1, P_2)$ is smaller than that of (a, P_1, P_2); therefore the inequality $\lambda \geq \lambda_{cr}$ continues to hold for the new parameter set.

By using multi-letter Han–Kobayashi scheme one can achieve the weighted sum-rate

$$\begin{aligned} &n(R_1 + \lambda R_2) \\ &= I(\hat{\mathbf{X}}_1, \mathbf{X}_2; \mathbf{Y}_2) + (\lambda-1)I(\mathbf{X}_2; \mathbf{Y}_2|\mathbf{U}_1) \\ &\quad + I(\hat{\mathbf{X}}_1; \mathbf{Y}_1|\mathbf{U}_1) - I(\hat{\mathbf{X}}_1; \mathbf{Y}_2|\mathbf{U}_1, \mathbf{X}_2) \\ &= h(\mathbf{X}_2 + a\mathbf{U}_1 + a\mathbf{X}_1 + \mathbf{Z}) - h(\mathbf{Z}) \\ &\quad + (\lambda-1)h(\mathbf{X}_2 + a\mathbf{X}_1 + \mathbf{Z}) - \lambda h(a\mathbf{X}_1 + Z) \\ &\quad + h(\mathbf{X}_1 + \mathbf{Z}) \\ &= h(\mathbf{X}_2 + a\mathbf{U}_1 + a\mathbf{X}_1 + \mathbf{Z}) - h(\mathbf{Z}) \\ &\quad + n\frac{\lambda-1}{2}\log(1+P_2) + n\delta. \end{aligned}$$

Since $\lambda \geq \lambda_{cr}$ the single-letter Han–Kobayashi with Gaussian signaling (and power control) yields a weighted sum-rate given by

$$\frac{\lambda-1}{2}\log(1+P_2) + \frac{1}{2}\log(1 + a^2(Q_1 + P_1) + P_2).$$

Therefore to show the sub-optimality of the above expression it suffices to show that

$$\frac{n}{2}\log 2\pi e(1 + a^2(Q_1+P_1) + P_2) - h(\mathbf{X}_2 + a\mathbf{U}_1 + a\mathbf{X}_1 + \mathbf{Z}) \to 0$$

as $Q_1 \to \infty$. Clearly since

$$\begin{aligned} h(\mathbf{X}_2 + a\mathbf{U}_1 + a\mathbf{X}_1 + \mathbf{Z}) &\geq h(a\mathbf{U}_1 + \mathbf{Z}) \\ &= \frac{n}{2}\log 2\pi e(1 + a^2 Q_1), \end{aligned}$$

we are done. □

The interesting aspect is that the converse is also true, as can be seen from the following lemma.

Lemma 4. *If Hypothesis 1 is not true then (single-letter) Han–Kobayashi with Gaussian signaling (and power control) yields the optimal slope around the corner point given in (11).*

Proof. Clearly by Fano's inequality we obtain that any achievable rates R_1, R_2 must satisfy

$$\begin{aligned} &R_1 + \lambda R_2 \\ &\leq \lim_n \frac{1}{n} \sup_{\mathbf{X}_1, \mathbf{X}_2} I(\mathbf{X}_1; \mathbf{Y}_1) + \lambda I(\mathbf{X}_2; \mathbf{Y}_2) \\ &\leq \lim_n \frac{1}{n}\big(\sup h(\mathbf{Y}_2) + \sup\big((\lambda-1)h(\mathbf{Y}_2) \\ &\qquad - \lambda h(\mathbf{Y}_2|\mathbf{X}_2) + h(\mathbf{Y}_1) - h(\mathbf{Y}_1|\mathbf{X}_1)\big)\big) \\ &\overset{(a)}{\leq} \frac{1}{2}\log(1 + P_2 + a^2P_1) + \frac{\lambda-1}{2}\log(1+P_2), \end{aligned}$$

and the last expression matches the sum-rate of the (single-letter) Han–Kobayashi with Gaussian signaling (and power control). Inequality (a) follows since the hypothesis is false. □

A. Remarks on the Hypothesis

(*i*) The inequality in Hypothesis 1 does not hold if either $\mathbf{X}_2 \sim \mathcal{N}(0, P_2 I)$ or if $\mathbf{X}_1 \sim \mathcal{N}(0, P_1 I)$. It is immediate that when $\mathbf{X}_1 \sim \mathcal{N}(0, P_1 I)$, the maximizing choice of $\mathbf{X}_2$ is $\mathbf{X}_2 \sim \mathcal{N}(0, P_2 I)$; and then one can verify that the inequality does not hold. For the case when $\mathbf{X}_2 \sim \mathcal{N}(0, P_2 I)$, one can deduce that the inequality cannot hold from the concavity in t of $h(\sqrt{t}\mathbf{X}_1 + \mathbf{Z})$ when $\mathbf{Z} \sim \mathcal{N}(0, I)$. (see [14] for instance).

(*ii*) On the other hand, suppose the expression on the right hand side of (13) has colored Gaussian maximizers for each n; then the same argument as that used in the previous section will yield that the hypothesis is false; in particular implying the optimality of Gaussian signaling.

Conclusion

This paper presents a collection of results on the capacity region of the scalar Gaussian interference channel, a central open problem in network information theory. We demonstrate a discontinuity of slope of the capacity region at the sum-rate point for a certain range of parameters. We showed that using colored Gaussians does not improved on the single-letter strategy using a novel technique that can easily be extended to other situations. Finally we present an information inequality that is equivalent to the optimality of Han–Kobayashi region with Gaussian signaling around the corner point of the Gaussian Z-interference channel.

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