Invariance of the Han–Kobayashi region with respect to temporally-correlated Gaussian inputs

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***Abstract*—We establish that the multi-letter extension of the Han–Kobayashi achievable region with temporally correlated vector Gaussian inputs matches the Han–Kobayashi achievable region with scalar Gaussian inputs for the Gaussian interference channel.**

I. Introduction

Determining a computable characterization of the capacity region of the Gaussian interference channel is a central open question in network information theory. In particular, it is not known whether the Han–Kobayashi region [1] with Gaussian inputs yields the capacity region or not. Recently, it was shown [2] that multi-letter extensions of the Han-Kobayashi region for some discrete memoryless interference channels strictly improves on the Han–Kobayashi achievable region. Motivated by this result, it is natural to ask the same question for the Gaussian interference channel: do the multi-letter extensions of the Han–Kobayashi region with temporally correlated Gaussian inputs improve on the Han–Kobayashi achievable region with Gaussian inputs. In this note, we answer this question in the negative.

A. Preliminaries

A Gaussian interference channel is defined by

$$Y_1 = X_1 + bX_2 + Z_1$$
$$Y_2 = X_2 + aX_1 + Z_2$$

where $a, b \in \mathbb{R}$ and $Z_1, Z_2 \sim \mathcal{N}(0,1)$ are independent unit-power Gaussians. We assume the inputs X_1 and X_2 satisfy power constraints P_1 and P_2, respectively. This channel setting has been actively studied in the literature since mid 70s, so a complete literature survey is beyond the scope of this paper. An interested reader can refer to [3, Chapter 6.4] for a detailed introduction to Gaussian interference channels and the relevant literature. In the next paragraph we summarize some known results.

The capacity region has been established for the case $|a|, |b| \geq 1$ [4]. The capacity region has two noted Pareto-optimal points, called "corner" points, of the form: (C_1, R_2^*) and (R_1^*, C_2) where $C_1 = \frac{1}{2}\log(1+P_1)$ and $C_2 = \frac{1}{2}\log(1+P_2)$ denote the interference-free point-to-point capacities to the two receivers. The above corner points have been determined for all ranges of parameters, see [4]–[6]. Additionally, the Pareto-optimal point that maximizes the rate sum $R_1 + R_2$ under the condition: $|a|(1+b^2P_2) + |b|(1+a^2P_1) \leq 1$ has been established independently in [7]–[9]. The result in [10] establishes that the Hausdorff distance (under L^1-norm) between the capacity region and the Han–Kobayashi achievable region with Gaussian inputs is at most 1, for all ranges of parameters.

Theorem 1 (Han–Kobayashi achievable region)**.** *A non-negative rate pair (R_1, R_2) is achievable for a memoryless interference channel if it satisfies*

$$\begin{aligned}
R_1 &\leq I(X_1;Y_1|U_2Q)\\
R_1 &\leq I(X_2;Y_2|U_1Q)\\
R_1 + R_2 &\leq I(U_2X_1;Y_1|Q) + I(X_2;Y_2|U_2U_1Q)\\
R_1 + R_2 &\leq I(U_1X_2;Y_2|Q) + I(X_1;Y_1|U_2U_1Q)\\
R_1 + R_2 &\leq I(U_2X_1;Y_1|U_1Q) + I(U_1X_2;Y_2|U_2Q)\\
2R_1 + R_2 &\leq I(U_2X_1;Y_1|Q) + I(X_1;Y_1|U_1U_2Q)\\
&\quad + I(U_1X_2;Y_2|U_2Q)\\
R_1 + 2R_2 &\leq I(U_1X_2;Y_2|Q) + I(X_2;Y_2|U_1U_2Q)\\
&\quad + I(U_2X_1;Y_1|U_1Q)
\end{aligned}$$

for some $p(q)p(u_1,x_1|q)p(u_2,x_2|q)$.

Remark 1. The region presented above, Theorem 6.4 in [3], is an equivalent characterization of the Han–Kobayashi achievable region obtained in [11]. For the Gaussian interference channel with power constraints the input distributions are required to satisfy $\mathrm{E}(X_1^2) \leq P_1$ and $\mathrm{E}(X_2^2) \leq P_2$.

Definition 1. The *Han–Kobayashi achievable region with Gaussian inputs* refers to the evaluation of the region in Theorem 1, where for each $Q = q$, $X_{1q} = U_1^q + V_1^q$ and $X_{2q} = U_2^q + V_2^q$, where $U_1^q, V_1^q, U_2^q, V_2^q$ are mutually independent Gaussian random variables, and the constraints $\mathrm{E}_Q(X_{1Q}^2) \leq P_1$ and $\mathrm{E}_Q(X_{2Q}^2) \leq P_2$ hold. We denote this region as $\mathcal{R}^{GS}$, where GS represents Gaussian signaling.

Definition 2. *A k-letter extension of the Han–Kobayashi achievable region* refers to the (normalized) region in Theorem 1 evaluated for the interference channel obtained by taking k independent copies of the original interference channel.

Remark 2. By grouping channel uses into blocks of k time-slots one observes that the k-letter extension of the Han–Kobayashi achievable region also yields an achievable region for the original interference channel. Further, it is known (via Fano's inequality) that the (set-theoretic) limit of the k-letter extension of the Han–Kobayashi achievable region goes to the true capacity region. The k-letter extension of the Han–

Kobayashi achievable region with (vector) Gaussian inputs is defined in a similar manner as for the scalar case.

There have been attempts to study the local optimality of Gaussian distributions for the Han–Kobayashi rate region with perturbations using Hermite polynomials [12], as well as using temporally correlated coding schemes. While the former approach yielded interesting insights, so far the approach has not exhibited any rate pair that lay outside the Han–Kobayashi achievable region with Gaussian inputs.

There have been some instances in network information theory, including work by the authors, where multi-letter Gaussian schemes have been shown to match the single-letter scheme, such as [13]–[15]. This work is a natural extension of such results and this result subsumes other results and deals with the Han–Kobayashi region in its entirety.

B. The k-letter extension of the Han–Kobayashi achievable region with (vector) Gaussian inputs

From its definition, we see that the k-letter extension of the Han–Kobayashi achievable region with (vector) Gaussian inputs reduces to the set of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ that satisfy

$$R_1 \le \mathrm{E}_Q\left(\frac{1}{2k}\log\frac{\left|I+K_{U_1}^Q+K_{V_1}^Q+b^2K_{V_2}^Q\right|}{\left|I+b^2K_{V_2}^Q\right|}\right) \tag{1a}$$

$$R_2 \le \mathrm{E}_Q\left(\frac{1}{2k}\log\frac{\left|I+K_{U_2}^Q+K_{V_2}^Q+a^2K_{V_1}^Q\right|}{\left|I+a^2K_{V_1}^Q\right|}\right) \tag{1b}$$

$$\begin{aligned}R_1+R_2 \le \mathrm{E}_Q\Bigg(&\frac{1}{2k}\log\frac{\left|I+K_{U_1}^Q+K_{V_1}^Q+b^2K_{U_2}^Q+b^2K_{V_2}^Q\right|}{\left|I+b^2K_{V_2}^Q\right|}\\ &+\frac{1}{2k}\log\frac{\left|I+K_{V_2}^Q+a^2K_{V_1}^Q\right|}{\left|I+a^2K_{V_1}^Q\right|}\Bigg)\end{aligned} \tag{1c}$$

$$\begin{aligned}R_1+R_2 \le \mathrm{E}_Q\Bigg(&\frac{1}{2k}\log\frac{\left|I+K_{U_2}^Q+K_{V_2}^Q+a^2K_{U_1}^Q+a^2K_{V_1}^Q\right|}{\left|I+a^2K_{V_1}^Q\right|}\\ &+\frac{1}{2k}\log\frac{\left|I+K_{V_1}^Q+b^2K_{V_2}^Q\right|}{\left|I+b^2K_{V_2}^Q\right|}\Bigg)\end{aligned} \tag{1d}$$

$$\begin{aligned}R_1+R_2 \le \mathrm{E}_Q\Bigg(&\frac{1}{2k}\log\frac{\left|I+K_{V_1}^Q+b^2K_{U_2}^Q+b^2K_{V_2}^Q\right|}{\left|I+b^2K_{V_2}^Q\right|}\\ &+\frac{1}{2k}\log\frac{\left|I+K_{V_2}^Q+a^2K_{U_1}^Q+a^2K_{V_1}^Q\right|}{\left|I+a^2K_{V_1}^Q\right|}\Bigg)\end{aligned} \tag{1e}$$

$$\begin{aligned}2R_1+R_2 \le \mathrm{E}_Q\Bigg(&\frac{1}{2k}\log\frac{\left|I+K_{U_1}^Q+K_{V_1}^Q+b^2K_{U_2}^Q+b^2K_{V_2}^Q\right|}{\left|I+b^2K_{V_2}^Q\right|}\\ &+\frac{1}{2k}\log\frac{\left|I+K_{V_1}^Q+b^2K_{V_2}^Q\right|}{\left|I+b^2K_{V_2}^Q\right|}\\ &+\frac{1}{2k}\log\frac{\left|I+K_{V_2}^Q+a^2K_{U_1}^Q+a^2K_{V_1}^Q\right|}{\left|I+a^2K_{V_1}^Q\right|}\Bigg)\end{aligned} \tag{1f}$$

$$\begin{aligned}R_1+2R_2 \le \mathrm{E}_Q\Bigg(&\frac{1}{2k}\log\frac{\left|I+K_{U_2}^Q+K_{V_2}^Q+a^2K_{U_1}^Q+a^2K_{V_1}^Q\right|}{\left|I+a^2K_{V_1}^Q\right|}\\ &+\frac{1}{2k}\log\frac{\left|I+K_{V_2}^Q+a^2K_{V_1}^Q\right|}{\left|I+a^2K_{V_1}^Q\right|}\\ &+\frac{1}{2k}\log\frac{\left|I+K_{V_1}^Q+b^2K_{U_2}^Q+b^2K_{V_2}^Q\right|}{\left|I+b^2K_{V_2}^Q\right|}\Bigg),\end{aligned} \tag{1g}$$

for some Q and symmetric positive semi-definite matrices $K_{U_1}^q, K_{V_1}^q, K_{U_2}^q, K_{V_2}^q \in \mathbb{R}^{k\times k}$ such that $\mathrm{E}_Q\left(\mathrm{tr}\left(K_{U_1}^Q+K_{V_1}^Q\right)\right) \le kP_1$ and $\mathrm{E}_Q\left(\mathrm{tr}\left(K_{U_2}^Q+K_{V_2}^Q\right)\right) \le kP_2$. By a standard application of cardinality-bounding techniques, it suffices to consider $|Q| \le 9$ (not needed in this note). Let $\mathcal{R}_k^{GS}$ denote the above region.

The main result of this note is the following:

Theorem 2. *$\mathcal{R}_k^{GS} = \mathcal{R}_1^{GS}$ for all $k \ge 1$.*

We will prove this theorem in the next section.

II. Proof of Theorem 2

For a $k\times k$ Hermitian matrix A, let $\lambda_1(A) \le \cdots \le \lambda_k(A)$ denote its eigenvalues. The proof uses a couple of standard technical results that we state at the outset.

Theorem 3 (Fiedler [16]). *Let A, B be $k\times k$ Hermitian matrices. Suppose $\lambda_1(A)+\lambda_1(B) \ge 0$. Then*

$$\prod_{i=1}^{k}(\lambda_i(A)+\lambda_i(B)) \le |A+B| \le \prod_{i=1}^{k}(\lambda_i(A)+\lambda_{k+1-i}(B))$$

Lemma 1. *Let A, B be $k\times k$ Hermitian matrices with $B \succeq 0$. Then $\lambda_i(A+B) \ge \lambda_i(A)$ for $i = 1, \cdots, k$.*

Proof. The Courant-Fischer-Weyl min-max principle and $B \succeq 0$ imply that

$$\begin{aligned}\lambda_i(A+B) &= \min_{\substack{V\subseteq\mathbb{C}^k\\ \dim V=i}}\max_{\substack{x\in V\\ \|x\|=1}} x^*(A+B)x\\ &\ge \min_{\substack{V\subseteq\mathbb{C}^k\\ \dim V=i}}\max_{\substack{x\in V\\ \|x\|=1}} x^*Ax\\ &= \lambda_i(A).\end{aligned}$$

□

Given any collection of symmetric positive semi-definite matrices $K_{U_1}^q, K_{V_1}^q, K_{U_2}^q, K_{V_2}^q \in \mathbb{R}^{k\times k}$, define

$$\begin{aligned}\hat{K}_{V_1}^q &:= \mathrm{diag}\left(\{\lambda_i(K_{V_1}^q)\}\right),\\ \hat{K}_{U_1}^q &:= \mathrm{diag}\left(\{\lambda_i(K_{U_1}^q+K_{V_1}^q)-\lambda_i(K_{V_1}^q)\}\right) \succeq 0,\\ \hat{K}_{V_2}^q &:= \mathrm{diag}\left(\{\lambda_{k+1-i}(K_{V_2}^q)\}\right),\\ \hat{K}_{U_2}^q &:= \mathrm{diag}\left(\{\lambda_{k+1-i}(K_{U_2}^q+K_{V_2}^q)-\lambda_{k+1-i}(K_{V_2}^q)\}\right) \succeq 0,\end{aligned}$$

where $\mathrm{diag}(\{a_i\})$ indicates a diagonal matrix with diagonal entries $a_1, \cdots, a_k$. The positive semi-definiteness of $\hat{K}_{U_1}, \hat{K}_{U_2}$ follows from Lemma 1. Note that these are trace preserving operations, i.e. $\mathrm{tr}(\hat{K}_{U_1}^q+\hat{K}_{V_1}^q) = \mathrm{tr}(K_{U_1}^q+K_{V_1}^q)$ and $\mathrm{tr}(\hat{K}_{U_2}^q+\hat{K}_{V_2}^q) = \mathrm{tr}(K_{U_2}^q+K_{V_2}^q)$.

Further note that

$$\left|I + a^2 K^q_{V_1}\right| = \left|I + a^2 \hat{K}^q_{V_1}\right| = \prod_{i=1}^{k} \left(1 + a^2 \lambda_i(K^q_{V_1})\right), \quad (2a)$$

$$\left|I + b^2 K^q_{V_2}\right| = \left|I + b^2 \hat{K}^q_{V_2}\right| = \prod_{i=1}^{k} \left(1 + b^2 \lambda_i(K^q_{V_2})\right). \quad (2b)$$

Corollary 1. *For any $c_1, c_2 \geq 0$, let $(A_1, \hat{A}_1) = (K^q_{V_1}, \hat{K}^q_{V_1})$ or $(K^q_{U_1} + K^q_{V_1}, \hat{K}^q_{U_1} + \hat{K}^q_{V_1})$, and let $(A_2, \hat{A}_2) = (K^q_{V_2}, \hat{K}^q_{V_2})$ or $(K^q_{U_2} + K^q_{V_2}, \hat{K}^q_{U_2} + \hat{K}^q_{V_2})$. Then*

$$|I + c_1 A_1 + c_2 A_2| \leq \left|I + c_1 \hat{A}_1 + c_2 \hat{A}_2\right|.$$

Proof.

$$\begin{aligned}|I + c_1 A_1 + c_2 A_2| &\leq \prod_{i=1}^{k} (1 + c_1 \lambda_i(A_1) + c_2 \lambda_{k+1-i}(A_2)) \\ &= \left|I + c_1 \hat{A}_1 + c_2 \hat{A}_2\right|\end{aligned}$$

where the inequality follows from Theorem 3. □

Corollary 1 and Equation 2 imply that replacing $(K^q_{U_1}, K^q_{V_1}, K^q_{U_2}, K^q_{V_2})$ by $(\hat{K}^q_{U_1}, \hat{K}^q_{V_1}, \hat{K}^q_{U_2}, \hat{K}^q_{V_2})$ cannot decrease any of the right-hand-sides of (1). This shows that $\mathcal{R}^{GS}_k$ can be attained by diagonal covariance matrices.

When the matrices $K^q_{U_2}, K^q_{V_2}, K^q_{U_1}, K^q_{V_1}$ are diagonal with entries $K^q_{U_2}(i), K^q_{V_2}(i), K^q_{U_1}(i), K^q_{V_1}(i), i = 1, \ldots, k$, observe that, for instance, we can express

$$\begin{aligned}&\mathrm{E}_Q \left(\frac{1}{2k} \log \left|I + K^Q_{U_2} + K^Q_{V_2} + a^2 K^Q_{U_1} + a^2 K^Q_{V_1}\right|\right) \\ &= \sum_q \mathrm{P}(Q = q) \left(\frac{1}{2k} \sum_{i=1}^{k} \log\left(1 + K^q_{U_2}(i) + K^q_{V_2}(i)\right.\right. \\ &\qquad \left.\left. + a^2 K^q_{U_1}(i) + a^2 K^q_{V_1}(i)\right)\right) \\ &= \sum_{q,i} \mathrm{P}(\tilde{Q} = (q,i)) \left(\frac{1}{2} \log\left(1 + K^{q,i}_{U_2} + K^{q,i}_{V_2}\right.\right. \\ &\qquad \left.\left. + a^2 K^{q,i}_{U_1} + a^2 K^{q,i}_{V_1}\right)\right) \\ &= \mathrm{E}_{\tilde{Q}} \left(\frac{1}{2} \log\left(1 + K^{\tilde{Q}}_{U_2} + K^{\tilde{Q}}_{V_2} + a^2 K^{\tilde{Q}}_{U_1} + a^2 K^{\tilde{Q}}_{V_1}\right)\right).\end{aligned}$$

In the above we defined a new time-sharing variable $\tilde{Q}$ and set $\mathrm{P}(\tilde{Q} = (q,i)) = \frac{1}{k} \mathrm{P}(Q = q)$, and defined scalar variables $K^{q,i}_{U_1} = K^q_{U_1}(i)$ (and others similarly). Note that the last expression is an expectation over scalar variables and corresponds to the expression in $\mathcal{R}^{GS}_1$. All other terms in $\mathcal{R}^{GS}_k$ also can be expressed similarly (with the consistent choice $\mathrm{P}(\tilde{Q} = (q,i)) = \frac{1}{k} \mathrm{P}(Q = q)$ and $K^{q,i}_{U_1} = K^q_{U_1}(i)$). Now the inclusion $\mathcal{R}^{GS}_k \subseteq \mathcal{R}^{GS}_1$ is immediate, thus establishing Theorem 2.

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