A New Proof of Parisi's Conjecture for the Finite Random Assignment Problem

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Abstract — **Consider the problem of minimizing cost when assigning n jobs to n machines. An assignment is a one-to-one mapping of jobs onto the machines. Assume that the cost of executing job i on machine j is $c_{ij},\ i,j = 1,\ldots,n$. When the c_{ij} are i.i.d. exponentials of mean 1, Parisi conjectured that the average cost of the minimum assignment equals $\sum_{i=1}^{n} \frac{1}{i^2}$. Recently, the authors, and independently, Linusson and Wästlund, have proved this conjecture. In the above work the authors also made a refined conjecture that, if established, would yield another proof of the Parisi's conjecture. This paper establishes the refined conjecture, thus providing a new proof of Parisi's conjecture.**

I. Introduction

Consider a system with n jobs and n machines where the cost of executing job i on machine j is c_{ij}. The assignment problem concerns the determination of a 1-to-1 assignment of jobs onto machines that minimizes the cost of executing all the jobs. The cost of the minimizing assignment is given by $A_n = \min_\pi \sum_{i=1}^{n} c_{i,\pi(i)}$. In the *random* assignment problem the c_{ij} are i.i.d. random variables drawn from some distribution, and the quantity of interest is the expected minimum cost, $\mathbb{E}(A_n)$. For $c_{ij} \sim$ i.i.d. exp(1) variables, Parisi [9] conjectured that:

$$\mathbb{E}(A_n) = \sum_{i=1}^{n} \frac{1}{i^2}. \qquad (1)$$

Let $C = [c_{ij}]$ be an $n \times n$ cost matrix with i.i.d. exp(1) entries. Delete the top row of C to obtain the rectangular matrix L of dimensions $(n-1) \times n$. For each $i = 1,\ldots,n$, let S_i be the cost of the minimum-cost permutation in the sub-matrix obtained by deleting the i^{th} column of L. These quantities are illustrated below.

$$\underset{\text{L}}{\begin{array}{|c|c|c|}\hline 3 & 6 & 11 \\ \hline 9 & 2 & 20 \\ \hline\end{array}} \quad \underset{S_1 = 13}{\begin{array}{|c|c|}\hline 6 & \mathbf{11} \\ \hline \mathbf{2} & 20 \\ \hline\end{array}} \quad \underset{S_2 = 20}{\begin{array}{|c|c|}\hline 3 & \mathbf{11} \\ \hline \mathbf{9} & 20 \\ \hline\end{array}} \quad \underset{S_3 = 5}{\begin{array}{|c|c|}\hline \mathbf{3} & 6 \\ \hline 9 & \mathbf{2} \\ \hline\end{array}}.$$

Let σ be the random permutation of $\{1,\ldots,n\}$ such that $S_{\sigma(1)} \leq \ldots \leq S_{\sigma(n)}$. Define $T_i = S_{\sigma(i)}$. We shall refer to the sequence $\{T_i, i = 1,\ldots,n\}$ as the T-matchings of L. In the above example, $T_1 = 5$, $T_2 = 13$ and $T_3 = 20$.

In [8] we prove the following

Theorem 1 *For $j = 1,\ldots,n-1$, $\ T_{j+1} - T_j \sim \exp(j(n-j))$ and these increments are independent of each other.*

Theorem 2 $\mathbb{E}(A_n) = \sum_{i=1}^{n} \frac{1}{i^2}$.

In [8], we use Theorem 1 to establish Theorem 2.

II. Main result

Let L be an $(n-1) \times n$ matrix of i.i.d. exp(1) entries and let $\{T_i\}_1^n$ denote its T-matchings, as defined in the previous section. Let Υ denote the set of all placements of the row-wise minimum entries of L; for example, all the row-wise minima in the same column, all in distinct columns, etc. Now consider any fixed placement of the row minima $\xi \in \Upsilon$. We prove the following conjecture made in [8]

Theorem 3 *Conditioned on a particular placement ξ,*

$$T_{j+1} - T_j \sim \exp(j(n-j)) \text{ for } j = 1,\ldots,n-1.$$

Furthermore, these increments are independent of each other.

The proof of Theorem 3 uses the memoryless property of the exponential distribution and some combinatorial observations to reduce the computations to that of Theorem 2.

Clearly, if we average over all $\xi \in \Upsilon$ then we recover Theorem 1. Hence Theorem 3 is a refinement of Theorem 1. It turns out that Theorem 3 is simple to prove in the case when ξ is the placement corresponding to all row-wise minima being in distinct columns.

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