On the Achievable Rate Sum for Symmetric Gaussian Interference Channels

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Abstract

We compute the exact rate sum of a symmetric Gaussian interference channel for the Han-Kobayashi region with Gaussian signaling for a subset of parameters in the weak interference regime. In this subset we identify three regimes of operation: an initial set where treating interference as noise is optimal, an intermediate regime where one employs a time-sharing strategy (with power control) between treating interference as noise and time-division multiplexing (TDM), and finally a regime where TDM is optimal. Beyond this regime our techniques do not yield the exact sum-rate, however we do observe that one can achieve rates above those prescribed by TDM, even for values of a that are close to 1.

1 Introduction

The Gaussian interference channel is one of the classical problems in multiple user information theory. Its capacity region has been an open problem for over thirty years. Various authors have considered specific cases of this problem [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. We consider the rate sum of a symmetric Gaussian interference channel in standard format, given by

$$Y_1 = X_1 + aX_2 + Z_1, \tag{1}$$
$$Y_2 = aX_1 + X_2 + Z_2 \tag{2}$$

with X_1 and X_2 constrained to average power P, and Z_1 and Z_2 distributed as $N(0,1)$. This problem has been solved in the strong interference regime ($a \geq 1$) [2, 6, 7] and in a very weak interference regime, $2a(1 + a^2P) \leq 1$ [12, 13, 14]. It has been known that the naïve strategy of treating interference as noise is optimal until $2a(1+a^2P) = 1$ and not optimal beyond $2a^2(1+a^2P) = 1$. In this work we compute the best mixed-Gaussian signaling strategy using power control. It is shown that the naïve approach of treating interference as noise (without power control, i. e., without separately adjusting the power associated with each strategy in a given combination) is not optimal even for values of a smaller than $2a^2(1 + a^2P) = 1$.

In particular this strategy leads to the identification of four interference regimes. In the first regime, associated with *weak* interference (small values of a), the interfering signal should be treated as noise. The second regime is characterized by *intermediate* interference, with the parameter a in a certain intermediary interval. In this transition region the best rate sum requires a convex combination of two strategies: (i) treating interference as noise and (ii) time or frequency division multiplexing (TDM/FDM). Next, for a in a subsequent segment of the $(0, 1)$ interval, interference is said to be *moderate* and requires pure TDM/FDM for maximizing the rate sum. Interestingly,

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this third region always start for a^2 below the midpoint of the $(0,1)$ interval, regardless of the parameters a and P.

Finally, for the remainder portion of the $(0,1)$ interval and high values of P, we may have the case where interference is partially decoded in the best known approach for the rate sum. Thus superposition coding is part of the best strategy in this region. The frontier of this peculiar regime, which manages to supersede TDM/FDM, is still not well characterized, but it is known that it happens with a above $a^2(1+a^2P)=1$. We give a lower bound for the start of this fourth regime characterized by *almost strong* (or *forte ma non troppo*) interference. We note that for high values of P, this regime may dominate over a substantial portion of the $(0,1)$ interval of a.

2 An achievable rate sum via Gaussian signalling

In this section we compute the rate sum given by a naïve signalling strategy. We will show that the same rate is given by the Han-Kobayashi signalling scheme for some initial range of parameters.

Theorem 1. *The maximum value of $I(X_1;Y_1|Q)+I(X_2;Y_2|Q)$ subject to Gaussian inputs and individual power constraints, each being P, is given by the upper concave envelope*[1] *of the following function*

$$\max\left\{\frac{1}{2}\log\left(1+2P\right),\log\left(1+\frac{P}{1+a^2P}\right)\right\}$$

evaluated at P.

Proof. We wish to maximize $I(X_1;Y_1|Q)+I(X_2;Y_2|Q)$ subject to Gaussian inputs and individual power constraints, each being P. From Fenchel-Caratheodory arguments we know that we can restrict $|Q|\le 3$. Hence the maximization takes the form

$$\begin{aligned}
\max\quad & \sum_{i=1}^{3}\alpha_i\left(\frac{1}{2}\log\left(1+\frac{P_{1i}}{1+a^2P_{2i}}\right)+\frac{1}{2}\log\left(1+\frac{P_{2i}}{1+a^2P_{1i}}\right)\right)\\
\text{subject to}: & \\
& \sum_{i=1}^{3}\alpha_iP_{1i}\le P,\ \ \sum_{i=1}^{3}\alpha_iP_{2i}\le P,\ \ \sum_{i=1}^{3}\alpha_i=1,\\
& \alpha_i,P_{1i},P_{2i}\ge 0, i=1,2,3.
\end{aligned}$$

A symmetrization argument: Note that any valid maximizer induces another maximizer by the transformation $P_{1i}\leftrightarrow P_{2i}, i=1,2,3$. Thus by taking a convex combination (weighted by $\frac{1}{2}$) of the two maximizers, we can simplify the above optimization problem to

$$\begin{aligned}
\max\quad & \sum_{i=1}^{3}\alpha_i\left(\frac{1}{2}\log\left(1+\frac{P_{1i}}{1+a^2P_{2i}}\right)+\frac{1}{2}\log\left(1+\frac{P_{2i}}{1+a^2P_{1i}}\right)\right)\\
\text{subject to}: & \\
& \sum_{i=1}^{3}\frac{\alpha_i}{2}(P_{1i}+P_{2i})\le P,\ \ \sum_{i=1}^{3}\alpha_i=1\\
& \alpha_i,P_{1i},P_{2i}\ge 0, i=1,2,3.
\end{aligned}$$

[1] The upper concave function of a function $f(x)$ over a domain $\mathcal{D}$ is defined as

$$\mathfrak{C}[f](x):=\inf\{g(x):g(\cdot)\text{ is concave on }\mathcal{D},g(y)\ge f(y),\forall y\in\mathcal{D}\}.$$

For this problem we can reduce (by Fenchel-Caratheodory arguments) the number of convex combinations to two. Denote $P_i = P_{1i} + P_{2i}$ and $\beta_i P_i = P_{1i}$. We can thus reduce the optimization problem to

$$
\begin{aligned}
&\max \quad \sum_{i=1}^{2} \alpha_i \left(\frac{1}{2} \log\left(1 + \frac{\beta_i P_i}{1 + a^2 \bar{\beta}_i P_i}\right) + \frac{1}{2} \log\left(1 + \frac{\bar{\beta}_i P_i}{1 + a^2 \beta_i P_i}\right)\right) \\
&\text{subject to :} \\
&\qquad \sum_{i=1}^{2} \frac{\alpha_i}{2} P_i \leq P, \quad \sum_{i=1}^{2} \alpha_i = 1 \\
&\qquad \alpha_i, P_i \geq 0, i = 1, 2.
\end{aligned}
$$

Suppose $\beta_i \in (0,1)$, then it must be that the following

$$\frac{1}{2} \log\left(1 + \frac{\beta P_i}{1 + a^2 \bar{\beta} P_i}\right) + \frac{1}{2} \log\left(1 + \frac{\bar{\beta} P_i}{1 + a^2 \beta P_i}\right)$$

viewed as a function of β achieves a local maximum at β_i.

Differentiating w.r.t. to β yields that

$$\frac{P_i(1-a^2)}{1 + a^2 \bar{\beta} P_i + \beta P_i} + \frac{a^2 P_i}{1 + a^2 \bar{\beta} P_i} - \frac{P_i(1-a^2)}{1 + a^2 \beta P_i + \bar{\beta} P_i} - \frac{a^2 P_i}{1 + a^2 \beta P_i} = 0.$$

This implies

$$(1-a^2)^2(\bar{\beta} - \beta)(1 + a^2 \bar{\beta} P_i)(1 + a^2 \beta P_i) = a^4 (\bar{\beta} - \beta)(1 + a^2 \bar{\beta} P_i + \beta P_i)(1 + a^2 \beta P_i + \bar{\beta} P_i).$$

Simple manipulations show that above holds *if and only if* either $(i)\beta = \frac{1}{2}$ or $(ii)(1 - 2a^2 - a^4 P_i)(1 + a^2 P_i) = 0$.

When $1 - 2a^2 - a^4 P_i = 0$ one can verify that the function does not depend on β and hence w.l.o.g we can take $\beta = \frac{1}{2}$. Thus we can replace

$$\frac{1}{2} \log\left(1 + \frac{\beta_i P_i}{1 + a^2 \bar{\beta}_i P_i}\right) + \frac{1}{2} \log\left(1 + \frac{\bar{\beta}_i P_i}{1 + a^2 \beta_i P_i}\right)$$

by the maximum value (over β_i)

$$f^*(\frac{P_i}{2}) := \max\left\{ \frac{1}{2} \log\left(1 + P_i\right), \log\left(1 + \frac{P_i}{2 + a^2 P_i}\right)\right\},$$

which corresponds to the values at $\beta = 0, \beta = \frac{1}{2}$ respectively.

Setting $Q_i = \frac{P_i}{2}$, the optimization problem reduces to

$$
\begin{aligned}
&\max \quad \sum_{i=1}^{2} \alpha_i f^*(Q_i) \\
&\text{subject to :} \\
&\qquad \sum_{i=1}^{2} \alpha_i Q_i \leq P, \sum_{i=1}^{2} \alpha_i = 1 \\
&\qquad \alpha_i, P_i \geq 0, i = 1, 2.
\end{aligned}
$$

which is the upper concave envelope of $f^*(\cdot)$ evaluated at P, as desired. The completes the proof of Theorem 1. $\square$

2.1 Explicit evaluation of the concave envelope

In this section we compute the concave envelope obtained in Theorem 1 explicitly. Let

$$f^*(P) = \max\left\{\frac{1}{2}\log\left(1+2P\right), \log\left(1+\frac{P}{1+a^2P}\right)\right\},$$

The upper concave envelope of $f^*(\cdot)$ is easy to compute. Both the functions

$$\frac{1}{2}\log\left(1+2P\right), \log\left(1+\frac{P}{1+a^2P}\right)$$

are concave and increasing in $P \in [0, \infty]$.

There are two cases of interest here:

Case 1: $2a^2 > 1$.

In this case the function $\frac{1}{2}\log\left(1+2P\right)$ is always larger than $\log\left(1+\frac{P}{1+a^2P}\right)$ for all values of $P \in [0, \infty]$ and hence the concave envelope is given by $\frac{1}{2}\log\left(1+2P\right)$.

Case 2: $2a^2 \leq 1$.

In this case the two functions have exactly one point of intersection which happens at P_0 that satisfies $2a^2(1+a^2P_0) = 1$. Below this point the function $\log\left(1+\frac{P}{1+a^2P}\right)$ is larger than $\frac{1}{2}\log\left(1+2P\right)$ and the contrary happens for $P > P_0$. The concave envelope of the maximum is easy to compute and is explicitly stated below. To note that the stated curve is indeed the concave envelope, note that it is concave and is dominated by any concave function that dominates the underlying function.

The concave envelope of the function $f^*(P)$ is:

1. From 0 to P_I^*: the value is $\log\left(1+\frac{P}{1+a^2P}\right)$,
2. From P_I^* to P_{II}^*: the value is

$$\alpha\log\left(1+\frac{P_I^*}{1+a^2P_I^*}\right) + \frac{1-\alpha}{2}\log(1+2P_{II}^*),$$

   where $\alpha \in [0,1]$ is the solution for $P = \alpha P_I^* + (1-\alpha)P_{II}^*$, and
3. From P_{II}^* to infinity: the value is $\frac{1}{2}\log(1+2P)$.

Here P_I^* and P_{II}^* are defined by the points of intersection of a common tangent to both the curves. In particular, they are given by solving the following two equations:

$$\begin{aligned}\frac{(1+a^2)}{1+P_I^*(1+a^2)} - \frac{a^2}{1+a^2P_I^*} &= \frac{1}{1+2P_{II}^*}\\ \frac{1}{2}\log(1+2P_{II}^*) - \log\left(1+\frac{P_I^*}{1+a^2P_I^*}\right) &= \frac{P_{II}^*-P_I^*}{1+2P_{II}^*}.\end{aligned}$$

with the logarithm in the natural base.

Solving this we obtain

$$(1+2P_{II}^*) = (1+P_I^*(1+a^2))(1+a^2P_I^*),$$

and that P_I^* is the solution of

$$\frac{3}{2}\log(1+a^2P_I^*) - \frac{1}{2}\log(1+P_I^*(1+a^2)) = \frac{P_I^*(2a^2-1)+a^2(1+a^2)(P_I^*)^2}{2(1+P_I^*(1+a^2))(1+a^2P_I^*)}.$$

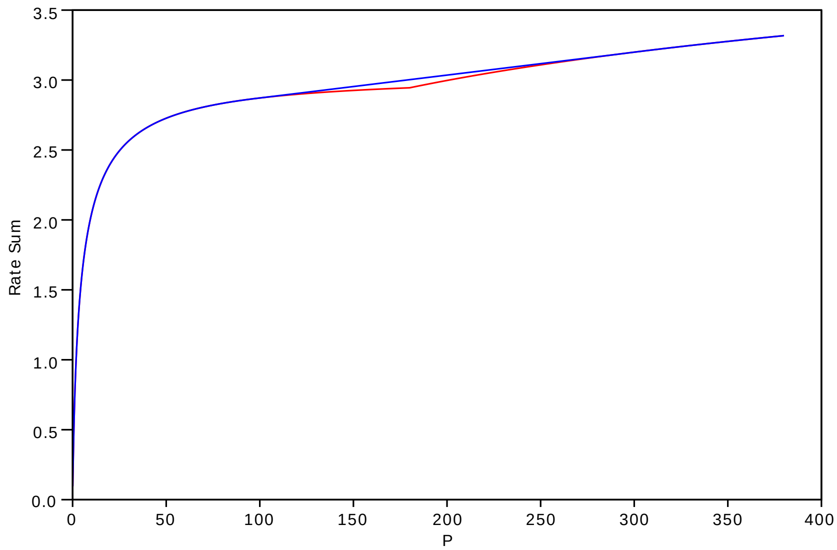

Figure 1: An achievable rate sum using Gaussian signaling

Figure 1 shows the achievable rate sum as a function of P when $a^2 = 0.05$. The red curve denotes the underlying function $f^*(\cdot)$ and the blue curve denotes its concave envelope. Here P_I^* denotes the first point on the X-axis where the red curve and blue curve deviate and P_{II}^* denotes the subsequent point on the X-axis where the curves merge again. An initial part of this curve (extending beyond P_{II}^*) will turn out to be optimal under Gaussian signalling. We will elaborate more on this after the next section, where we compute the optimal Gaussian signalling rate sum for the Han-Kobayashi inner bound in certain parameter regimes.

Summarizing the results we obtain the following achievable region. Let P_I^*, P_{II}^* be as defined earlier.

Theorem 2. *An inner bound to the sum-rate using Gaussian signalling of the Han-Kobayashi scheme in the weak interference regime for the symmetric Gaussian interference channel is given by the following:*

- Case 1: $2a^2 < 1$*:*
  1. *From* 0 *to* P_I^**: the value is* $\log\left(1 + \frac{P}{1+a^2P}\right)$,
  2. *From* P_I^* *to* P_{II}^**: the value is*
  $$\alpha \log\left(1 + \frac{P_I^*}{1+a^2P_I^*}\right) + \frac{1-\alpha}{2}\log(1+2P_{II}^*),$$
  where $\alpha \in [0,1]$ *is the solution for* $P = \alpha P_I^* + (1-\alpha)P_{II}^*$*, and*
  3. *From* P_{II}^* *to infinity: the value is* $\frac{1}{2}\log(1+2P)$.
- Case 2: $2a^2 \geq 1$*: In this case the curve is* $\frac{1}{2}\log(1+2P)$.

3 On sum rate of Han-Kobayashi scheme with Gaussian signalling

In this section we simplify the computation of the sum rate yielded by Han-Kobayashi scheme with Gaussian signalling. From Han-Kobayashi inner bound one can achieve a sum-rate $(R_1 + R_2)$ that satisfies the following four inequalities

$$\begin{aligned}
R_1 + R_2 &\le I(X_1; Y_1|U_2, Q) + I(X_2; Y_2|U_1, Q) \\
R_1 + R_2 &\le I(X_1; Y_1|U_1, U_2, Q) + I(U_1, X_2; Y_2|Q) \\
R_1 + R_2 &\le I(U_2, X_1; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q) \\
R_1 + R_2 &\le I(U_2, X_1; Y_1|U_1, Q) + I(U_1, X_2; Y_2|U_2, Q)
\end{aligned} \tag{3}$$

for any $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$.

Remark: This is obtained using Fourier-Motzkin elimination on the original inner bound and removing the redundant inequalities.

We now specialize this inner bound for Gaussian interference channel with weak interference $(a < 1)$ and restricting ourselves to Gaussian signaling. When $Q = i$ let

$$\begin{aligned}
P_1 &\sim \mathcal{N}(0, \beta_i P_i) \\
P_2 &\sim \mathcal{N}(0, \bar{\beta}_i P_i) \\
U_1 &\sim \mathcal{N}(0, \bar{\alpha}_{1i}\beta_i P_i) \\
U_2 &\sim \mathcal{N}(0, \bar{\alpha}_{2i}\bar{\beta}_i P_i)
\end{aligned}$$

where $\beta_i, \alpha_{1i}, \alpha_{2i} \in [0, 1], P_i \ge 0$ and $\sum_i \beta_i P_i \le P, \sum_i \bar{\beta}_i P_i \le P$.

Remark: Here conditioned on Q, we have $X_1 \sim U_1 + V_1$ where V_1 is a zero-mean Gaussian random variable independent of U_1. Similarly $X_2 \sim U_2 + V_2$.

For this choice of auxiliary random variables, observe that

$$\begin{aligned}
&I(X_1; Y_1|U_2, Q = i) + I(X_2; Y_2|U_1, Q = i) \\
&\quad = I(X_1, X_2; Y_1|U_2, Q = i) + I(X_1, X_2; Y_2|U_1, Q = i) \\
&\qquad - I(X_2; Y_1|X_1, U_2, Q = i) - I(X_1; Y_2|X_2, U_1, Q = i) \\
&\quad = \frac{1}{2}\log(1 + a^2\alpha_{2i}\bar{\beta}_i P_i + \beta_i P_i) + \frac{1}{2}\log(1 + a^2\alpha_{1i}\beta_i P_i + \bar{\beta}_i P_i) \\
&\qquad - I(X_2; Y_1|X_1, U_2, Q = i) - I(X_1; Y_2|X_2, U_1, Q = i)
\end{aligned}$$

It is a routine exercise to verify that (when $\beta_i, \alpha_{1i}, \alpha_{2i} \in [0, 1], P_i \ge 0$)

$$\begin{aligned}
&\frac{1}{2}\log(1 + a^2\alpha_{2i}\bar{\beta}_i P_i + \beta_i P_i) + \frac{1}{2}\log(1 + a^2\alpha_{1i}\beta_i P_i + \bar{\beta}_i P_i) \\
&\ge \frac{1}{2}\log(1 + \alpha_{1i}\beta_i P_i + a^2\bar{\beta}_i P_i) + \frac{1}{2}\log(1 + \alpha_{2i}\bar{\beta}_i P_i + a^2\beta_i P_i) \\
&= I(X_1, X_2; Y_1|U_1, Q = i) + I(X_1, X_2; Y_2|U_2, Q = i)
\end{aligned}$$

holds. Hence the inequality $I(X_1; Y_1|U_2, Q) + I(X_2; Y_2|U_1, Q) \ge I(U_2, X_1; Y_1|U_1, Q) + I(U_1, X_2; Y_2|U_2, Q)$ thus making the first of the four inequalities redundant. Thus under Gaussian signalling we only need to consider the last three inequalities for the sum-rate in (3).

The symmetrization argument can be used in the following way to reduce the number of constraints to two. Split every instance $Q = i$ into two parts $Q' = (i, 1), Q' = (i, 2)$ with equal probability. Now when $Q' = (i, 1)$ generate $U_1', X_1', U_2', X_2' \sim U_1, X_1, U_2, X_2|Q = i$ and when $Q' = (i, 2)$

generate $U_1', X_1' \sim U_2, X_2 | Q = i$ and $U_2', X_2' \sim U_1, X_1 | Q = i$ (i.e when $Q' = (i, 2)$ interchange the auxiliaries between the two receivers). Observe that the new auxiliaries satisfy the following:

$$
\begin{aligned}
&I(X_1'; Y_1' | U_1', U_2', Q') + I(U_1', X_2'; Y_2' | Q') \\
&= I(U_2', X_1'; Y_1' | Q') + I(X_2'; Y_2' | U_1', U_2', Q') \\
&= \frac{1}{2} \left(I(X_1; Y_1 | U_1, U_2, Q) + I(U_1, X_2; Y_2 | Q) + I(U_2, X_1; Y_1 | Q) + I(X_2; Y_2 | U_1, U_2, Q) \right) \\
&I(U_2', X_1'; Y_1' | U_1', Q') + I(U_1', X_2'; Y_2' | U_2', Q') \\
&= I(U_2, X_1; Y_1 | U_1, Q) + I(U_1, X_2; Y_2 | U_2, Q).
\end{aligned}
$$

Hence the sum-rate does not decrease using this symmetrization transformation, and we are now left with the following optimization problem. (By Fenchel-Caratheodory we can take $|Q| = 3$, i.e. $|Q'| = 6$)

$$
\begin{aligned}
&\max \;\; R_1 + R_2 \\
&\text{subject to}: \\
&\qquad R_1 + R_2 \le \sum_{i=1}^{3} \frac{q_i}{2} \frac{1}{2} \log\left(1 + \frac{\alpha_{1i}\beta_i P_i}{1 + a^2 \alpha_{2i} \bar{\beta}_i P_i}\right) + \frac{1}{2} \log\left(1 + \frac{\bar{\beta}_i P_i + a^2 \bar{\alpha}_{1i} \beta_i P_i}{1 + a^2 \alpha_{1i} \beta_i P_i}\right) \\
&\qquad\qquad + \frac{1}{2} \log\left(1 + \frac{\alpha_{2i}\bar{\beta}_i P_i}{1 + a^2 \alpha_{1i} \beta_i P_i}\right) + \frac{1}{2} \log\left(1 + \frac{\beta_i P_i + a^2 \bar{\alpha}_{2i} \bar{\beta}_i P_i}{1 + a^2 \alpha_{2i} \bar{\beta}_i P_i}\right) \qquad (4) \\
&\qquad R_1 + R_2 \le \sum_{i=1}^{3} q_i \left(\frac{1}{2} \log\left(1 + \frac{a^2 \bar{\alpha}_{2i} \bar{\beta}_i P_i + \alpha_{1i} \beta_i P_i}{1 + a^2 \alpha_{2i} \bar{\beta}_i P_i}\right) + \frac{1}{2} \log\left(1 + \frac{a^2 \bar{\alpha}_{1i} \beta_i P_i + \alpha_{2i} \bar{\beta}_i P_i}{1 + a^2 \alpha_{1i} \beta_i P_i}\right)\right) \\
&\qquad \sum_{i=1}^{3} q_i = 1, \sum_{i=1}^{3} \frac{q_i}{2} P_i \le P \\
&\qquad 0 \le q_i, \alpha_{1i}, \alpha_{2i}, \beta_i \le 1, \;\; i = 1, 2, 3 \\
&\qquad P_i \ge 0, \;\; i = 1, 2, 3.
\end{aligned}
$$

The above optimization problem yields the Han-Kobayashi sum-rate with Gaussian signalling.

3.1 Exact computation of Gaussian signalling in certain regimes

To obtain the exact sum-rate in some regimes we will first compute the following upper bound (by ignoring the first constraint) to the solution of the optimization problem described earlier. Since we have one less constraint, we can set $|Q| = 2$.

$$
\begin{aligned}
&\max \;\; R_1 + R_2 \\
&\text{subject to}: \\
&\qquad R_1 + R_2 \le \sum_{i=1}^{2} q_i \left(\frac{1}{2} \log\left(1 + \frac{a^2 \bar{\alpha}_{2i} \bar{\beta}_i P_i + \alpha_{1i} \beta_i P_i}{1 + a^2 \alpha_{2i} \bar{\beta}_i P_i}\right) + \frac{1}{2} \log\left(1 + \frac{a^2 \bar{\alpha}_{1i} \beta_i P_i + \alpha_{2i} \bar{\beta}_i P_i}{1 + a^2 \alpha_{1i} \beta_i P_i}\right)\right) \\
&\qquad \sum_{i=1}^{2} q_i = 1, \sum_{i=1}^{2} \frac{q_i}{2} P_i \le P \\
&\qquad 0 \le q_i, \alpha_{1i}, \alpha_{2i}, \beta_i, \le 1, \\
&\qquad P_i \ge 0.
\end{aligned}
$$

This new optimization problem seeks to compute the upper concave envelope of the following function

$$
\max \;\; \frac{1}{2} \log\left(1 + \frac{a^2 \bar{\alpha}_2 \bar{\beta} P + \alpha_1 \beta P}{1 + a^2 \alpha_2 \bar{\beta} P}\right) + \frac{1}{2} \log\left(1 + \frac{a^2 \bar{\alpha}_1 \beta P + \alpha_2 \bar{\beta} P}{1 + a^2 \alpha_1 \beta P}\right)
$$

subject to $0 \leq \alpha_1, \alpha_2, \beta, \leq 1$. It turns out that this can be computed explicitly.

Let

$$
\begin{aligned}
f(\frac{P}{2}) &= \max_{0\leq\alpha_1,\alpha_2,\beta\leq 1} \frac{1}{2}\log\left(1+\frac{a^2\bar{\alpha}_2\bar{\beta}P+\alpha_1\beta P}{1+a^2\alpha_2\bar{\beta}P}\right)+\frac{1}{2}\log\left(1+\frac{a^2\bar{\alpha}_1\beta P+\alpha_2\bar{\beta}P}{1+a^2\alpha_1\beta P}\right)\\
&= \max_{0\leq\alpha_1,\alpha_2,\beta\leq 1} \frac{1}{2}\log\left(\frac{1+a^2\bar{\beta}P+\alpha_1\beta P}{1+a^2\alpha_2\bar{\beta}P}\right)+\frac{1}{2}\log\left(\frac{1+a^2\beta P+\alpha_2\bar{\beta}P}{1+a^2\alpha_1\beta P}\right)\\
&= \max_{0\leq\alpha_1,\alpha_2,\beta\leq 1} \frac{1}{2}\log\left(\frac{1+a^2\bar{\beta}P+\alpha_1\beta P}{1+a^2\alpha_1\beta P}\right)+\frac{1}{2}\log\left(\frac{1+a^2\beta P+\alpha_2\bar{\beta}P}{1+a^2\alpha_2\bar{\beta}P}\right)
\end{aligned}
$$

Fix $\beta \in [0,1]$ and consider the problem of maximizing $\frac{1+a^2\bar{\beta}P+\alpha_1\beta P}{1+a^2\alpha_1\beta P}$ with respect to α_1. It is straightforward to check that the function is increasing if

$$(1+a^2\alpha_1\beta P)(\beta P)-(1+a^2\bar{\beta}P+\alpha_1\beta P)a^2\beta P>0,$$

else it is decreasing. The above condition is true *iff*

$$1-a^2>a^4\bar{\beta}P.$$

Thus the optimal value is $\alpha_1=1$ if the above condition holds, else it is zero. Similarly the optimal $\alpha_2=1$ if

$$1-a^2>a^4\beta P$$

else it is zero. Thus $f(P)$ can be phrased as the maximum over β of the 4 functions each induced by setting $(\alpha_1,\alpha_2)=(0,0),(0,1),(1,0),(1,1)$. Indeed setting $(\alpha_1,\alpha_2)=(0,1)$ or $(1,0)$ yields a symmetric function of β, hence

$$
\begin{aligned}
f(P) = \max_{0\leq\beta\leq 1}\max\Bigg\{&\frac{1}{2}\log\left(1+a^2\bar{\beta}2P\right)+\frac{1}{2}\log\left(1+a^2\beta 2P\right), \frac{1}{2}\log\left(1+a^2\bar{\beta}2P\right)+\frac{1}{2}\log\left(\frac{1+a^2\beta 2P+\bar{\beta}2P}{1+a^2\bar{\beta}2P}\right),\\
&\frac{1}{2}\log\left(\frac{1+a^2\bar{\beta}2P+\beta 2P}{1+a^2\beta 2P}\right)+\frac{1}{2}\log\left(\frac{1+a^2\beta 2P+\bar{\beta}2P}{1+a^2\bar{\beta}2P}\right)\Bigg\}\\
= \max\Bigg\{&\max_{0\leq\beta\leq 1}\frac{1}{2}\log\left(1+a^2\bar{\beta}2P\right)+\frac{1}{2}\log\left(1+a^2\beta 2P\right), \max_{0\leq\beta\leq 1}\frac{1}{2}\log\left(1+a^2\beta 2P+\bar{\beta}2P\right),\\
&\max_{0\leq\beta\leq 1}\frac{1}{2}\log\left(\frac{1+a^2\bar{\beta}2P+\beta 2P}{1+a^2\beta 2P}\right)+\frac{1}{2}\log\left(\frac{1+a^2\beta 2P+\bar{\beta}2P}{1+a^2\bar{\beta}2P}\right)\Bigg\}\\
= \max\Bigg\{&\log\left(1+a^2P\right), \frac{1}{2}\log\left(1+2P\right), \log\left(1+\frac{P}{1+a^2P}\right)\Bigg\}
\end{aligned}
$$

3.1.1 Explicit evaluation of the concave envelope

The concave envelope can be computed explicitly as follows. There are a few cases to consider:

- $2a^2 \geq 1$. Note that in this regime $\frac{1}{2}\log(1+2P) \geq \log\left(1+\frac{P}{1+a^2P}\right)$ $\forall P \in [0,\infty]$. Hence we are interested in the concave envelope of the maximum of the first two functions. Similar to the analysis earlier, in this case the concave envelope consists of three parts:

1. $0 \leq P \leq P_{III}^*$: The concave envelope matches $\frac{1}{2}\log(1+2P)$.
2. $P_{III}^* \leq P \leq P_{IV}^*$: The concave envelope is given by

$$\frac{\alpha}{2}\log(1+2P_{III}^*) + (1-\alpha)\log(1+a^2P_{IV}^*), \text{ where } \alpha \text{ is the solution of}$$
$$\alpha P_{III}^* + (1-\alpha)P_{IV}^* = P.$$

3. $P_{IV}^* \leq P$: In this range, the concave envelope matches $\log\left(1+a^2P\right)$.

Here P_{III}^* and P_{IV}^* refer to the points where the common tangent to the curves $\frac{1}{2}\log(1+2P)$ and $\log\left(1+a^2P\right)$ meet the curves respectively. A simple calculation shows that P_{III}^* is the solution of the equation

$$\frac{1}{2}\log\left(a^4(1+2P_{III}^*)\right) = \frac{1+P_{III}^* - (1/a^2)}{1+2P_{III}^*},$$

with the logarithm in the natural base, and

$$P_{IV}^* = 1 + 2P_{III}^* - \frac{1}{a^2}.$$

- $2a^2 \leq 1$. It is easy to see that each pair of curves have exactly one point of intersection (and that it is a crossing of curves at this point. More formally, their first derivatives are different). The curves $\log\left(1+\frac{P}{1+a^2P}\right)$ and $\frac{1}{2}\log(1+2P)$ have a crossing point at P_0 that satisfies, $2a^2(1+a^2P_0) = 1$, and the curves $\frac{1}{2}\log(1+2P)$ and $\log(1+a^2P)$ have a crossing point at P_{00} that satisfies $a^2(2+a^2P_{00}) = 2$. Note that $P_0 \leq P_{00}$. Essentially the picture is the following:

  1. $0 \leq P \leq P_0$: Here we have that $\log\left(1+\frac{P}{1+a^2P}\right)$ yields the maximum.
  2. $P_0 \leq P \leq P_{00}$: In this regime, we have that $\frac{1}{2}\log(1+2P)$ yields the maximum.
  3. $P_{00} \leq P$: In this range $\log(1+a^2P)$ yields the maximum.

  Since each of the component function is concave and increasing, there are two possibilities for the concave envelope of the maximum of these three functions.

  - The concave envelope has two straight line segments and three curves: this happens precisely when the common tangent between the curves $\log\left(1+\frac{P}{1+a^2P}\right)$ and $\frac{1}{2}\log(1+2P)$ has a larger slope than that between the curves $\log\left(1+\frac{P}{1+a^2P}\right)$ and $\log(1+a^2P)$.
  - The concave envelope of the maximum of the first and third function dominates the second function and hence contains only one line segment: this happens when the common tangent between the curves $\log\left(1+\frac{P}{1+a^2P}\right)$ and $\log(1+a^2P)$ has a larger slope than that between the curves $\log\left(1+\frac{P}{1+a^2P}\right)$ and $\frac{1}{2}\log(1+2P)$.

  Another equivalent and easier way to distinguish which of the above two cases we are in is to compare the points P_I^* and P_a^*, where P_a^* is the value of P where the common tangent between the curves $\log\left(1+\frac{P}{1+a^2P}\right)$ and $\log(1+a^2P)$ meet the curve $\log\left(1+\frac{P}{1+a^2P}\right)$. The point P_I^* is the same the point described earlier. It is easy to see that if $P_I^* \leq P_a^*$ then we are in the former case, and otherwise we are in the latter case.

  Thus from Figure 2 we see that $P_a^* \geq P_I^*$ and therefore we are always in the former case, i.e. two line segments and three curves. Thus the concave envelope consists of

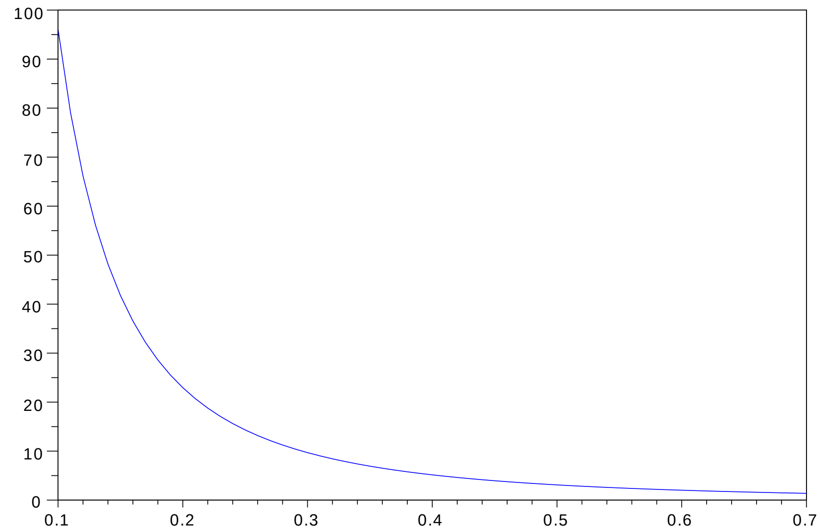

Figure 2: The difference $P_a^* - P_I^*$ as a function of a

1. From 0 to P_I^*: the value is $\log\left(1 + \frac{P}{1+a^2P}\right)$,
2. From P_I^* to P_{II}^*: the value is

$$\alpha \log\left(1 + \frac{P_I^*}{1 + a^2 P_I^*}\right) + \frac{1-\alpha}{2} \log(1 + 2P_{II}^*),$$

   where $\alpha \in [0, 1]$ is the solution for $P = \alpha P_I^* + (1-\alpha)P_{II}^*$, and
3. $P_{II}^* \leq P \leq P_{III}^*$: the value is $\frac{1}{2}\log(1 + 2P)$.
4. $P_{III}^* \leq P \leq P_{IV}^*$: The concave envelope is given by

$$\frac{\alpha}{2}\log(1 + 2P_{III}^*) + (1-\alpha)\log(1 + a^2 P_{IV}^*), \text{ where } \alpha \text{ is the solution of}$$
$$\alpha P_{III}^* + (1-\alpha)P_{IV}^* = P.$$

5. $P_{IV}^* \leq P$: In this range, the concave envelope matches $\log\left(1 + a^2 P\right)$.

Here P_I^*, P_{II}^*, P_{III}^*, and P_{IV}^* are exactly as defined in the earlier sections.

Figure 3 shows the achievable rate sum as a function of P when $a^2 = 0.05$. The red curve denotes the underlying function and the blue curve denotes its concave envelope.

We summarize the results of this section in the following theorem. We assume that P_I^*, P_{II}^*, P_{III}^*, and P_{IV}^* are as defined earlier.

Theorem 3. *An outer bound to the sum-rate using Gaussian signalling of the Han-Kobayashi scheme in the weak interference regime for the symmetric Gaussian interference channel is given by the following:*

- Case 1*:* $2a^2 < 1$*.*

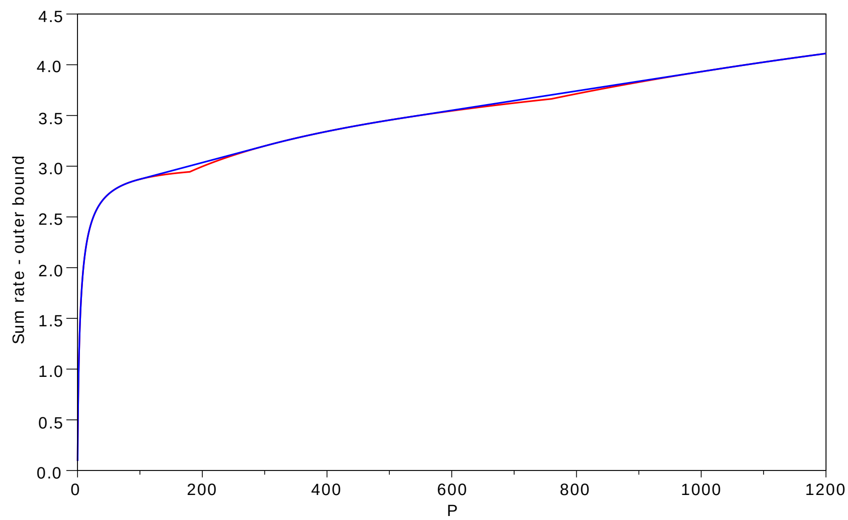

Figure 3: An outer bound to rate sum using Gaussian signaling

1. *From* 0 *to* P_I^**: the value is* $\log\left(1 + \frac{P}{1+a^2P}\right)$,
2. *From* P_I^* *to* P_{II}^**: the value is*

$$\alpha \log\left(1 + \frac{P_I^*}{1 + a^2 P_I^*}\right) + \frac{1-\alpha}{2}\log(1 + 2P_{II}^*),$$

   where $\alpha \in [0,1]$ *is the solution for* $P = \alpha P_I^* + (1-\alpha)P_{II}^*$*, and*
3. $P_{II}^* \leq P \leq P_{III}^*$*: the value is* $\frac{1}{2}\log(1+2P)$.
4. $P_{III}^* \leq P \leq P_{IV}^*$*: The concave envelope is given by*

$$\frac{\alpha}{2}\log(1 + 2P_{III}^*) + (1-\alpha)\log(1 + a^2 P_{IV}^*), \text{ where } \alpha \text{ is the solution of}$$
$$\alpha P_{III}^* + (1-\alpha)P_{IV}^* = P.$$

5. $P_{IV}^* \leq P$*: In this range, the concave envelope matches* $\log\left(1 + a^2 P\right)$.

- Case 2*:* $2a^2 \geq 1$.
   1. $0 \leq P \leq P_{III}^*$*: The concave envelope matches* $\frac{1}{2}\log\left(1+2P\right)$.
   2. $P_{III}^* \leq P \leq P_{IV}^*$*: The concave envelope is given by*

$$\frac{\alpha}{2}\log(1 + 2P_{III}^*) + (1-\alpha)\log(1 + a^2 P_{IV}^*), \text{ where } \alpha \text{ is the solution of}$$
$$\alpha P_{III}^* + (1-\alpha)P_{IV}^* = P.$$

   3. $P_{IV}^* \leq P$*: In this range, the concave envelope matches* $\log\left(1 + a^2 P\right)$.

3.1.2 Optimality in a certain initial regime

A key result of this paper is the following. We assume that P_I^*, P_{II}^*, P_{III}^*, and P_{IV}^* are as defined in the earlier sections.

Theorem 4. *The optimal sum-rate using Gaussian signalling of the Han-Kobayashi scheme in the weak interference regime for the symmetric Gaussian interference channel is given by the following:*

- Case 1*:* $2a^2 < 1$*. The optimal sum-rate for various regimes are*
    1. *From* 0 *to* P_I^**: the value is* $\log\left(1 + \frac{P}{1+a^2P}\right)$,
    2. *From* P_I^* *to* P_{II}^**: the value is*

    $$\alpha \log\left(1 + \frac{P_I^*}{1+a^2P_I^*}\right) + \frac{1-\alpha}{2}\log(1+2P_{II}^*),$$

    where $\alpha \in [0,1]$ *is the solution for* $P = \alpha P_I^* + (1-\alpha)P_{II}^*$*, and*
    3. $P_{II}^* \leq P \leq P_{III}^*$*: the value is* $\frac{1}{2}\log(1+2P)$.
- Case 2*:* $2a^2 \geq 1$*. The optimal sum-rate for the following interval* $[0, P_{III}^*]$ *is given by* $\frac{1}{2}\log(1+2P)$.

Proof. The proof follows by directly comparing the inner and the outer bounds in Theorem 2 and Theorem 3 respectively. □

In Figure 3, note that P_{III}^* denotes the point around $P = 600$ where the red curve and the blue curve start to diverge. Theorem 4 shows that in the interval $[0, P_{III}^*]$ the blue curve is the optimal achievable sum-rate by the Han-Kobayashi scheme using Gaussian signalling.

4 Various regimes of operation

Motivated from our results in the preceding sections we identify various regimes of operation. This is similar to the W-picture that is present in [11]. However our analysis identifies that there are more phase transitions within the Gaussian signalling regime than is indicated by the W-picture.

We fix an $a \in [0,1]$ and divide $P \in [0,\infty]$ into two regimes: *a*) *determined regime* consisting of $P \in [0, P_{III}^*]$, *b*) *nebulous regime* where $P \in (P_{III}^*, \infty)$.

Determined regime

If we consider a value of a such that $2a^2 \leq 1$ we know that within the *determined regime* there are three intervals where the operation modes are different:

1. From 0 to P_I^*: we treat interference as noise (IAN). This is the weak interference condition. Note that the expression "weak interference" is usually associated with the broad region where $a^2 \leq 1$. We prefer to restrict it to the particular region below P_I^* as there are many phase transitions in the entire unit interval of a^2.

2. From P_I^* to P_{II}^*: we do a time-sharing between the IAN strategy with power P_I^* and the time-division multiplexing(TDM) strategy with power P_{II}^*, so that the average power consumed is P. This condition may be labeled intermediate interference.

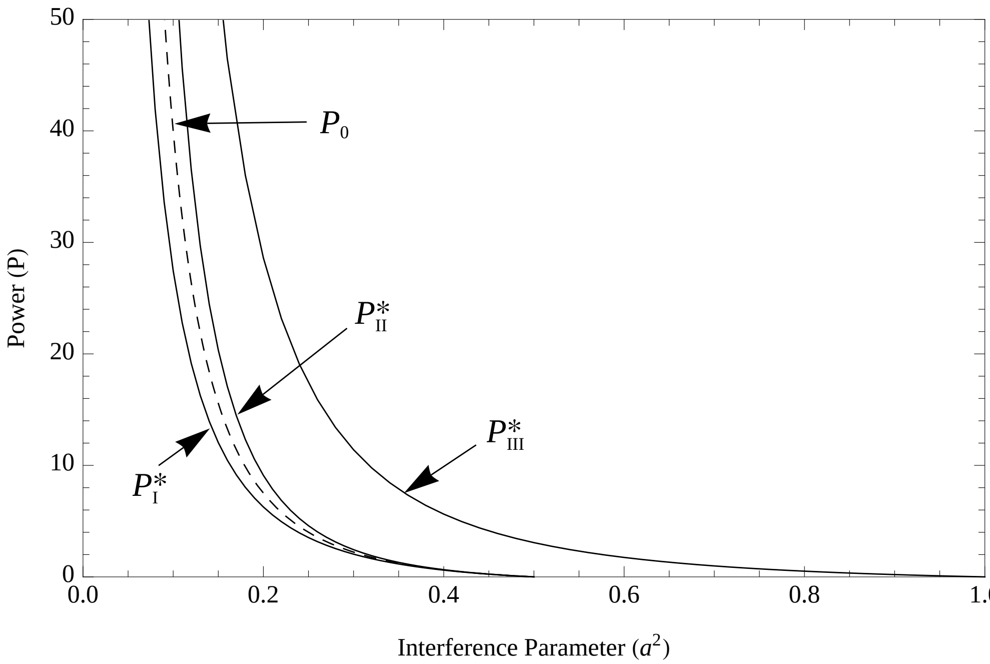

Figure 4: Boundaries of regions of weak interference (interference as noise (IAN)), given by $0 \leq P \leq P_I^*$, intermediate interference (time-sharing between IAN and TDM/FDM), given by $P_I^* \leq P \leq P_{II}^*$, and moderate interference (TDM/FDM), given by $P_{II}^* \leq P \leq P_{III}^*$, in the determined regime.

3. $P_{II}^* \leq P \leq P_{III}^*$: we do a simple TDM strategy. This situation may be referred to as moderate interference.

On the other hand if we consider a value of a such that $2a^2 > 1$, then the optimal strategy in the entire range $P \in [0, P_{III}^*]$ is time-division multiplexing(TDM). Therefore, in this range we encounter moderate interference exclusively.

Figure 4 illustrates the three interference scenarios that occur in the determined regime: weak, intermediate and moderate interference conditions.

Nebulous regime

For any fixed $a \in (0,1)$ in the regime $P > P_{III}^*$ we are unable to determine the optimal sum-rate using Gaussian signaling. Clearly the sum rate of $\frac{1}{2}\log(1+2P)$ achievable using TDM is an inner bound for this regime.

The first observation that the TDM rate sum could be beaten in high signal to noise ratios (SNRs) was made by Sason [10]. This observation could also be made from Etkin, Tse and Wang's [11] result of approximating the capacity to within 1 bit. Both use a partial superposition coding scheme with both decoders decoding identical proportions of the interfering signals. The sum-rate given using this scheme can be computed by setting $q_1 = 1, \beta_1 = \frac{1}{2}, \alpha_{11} = \alpha_{21} = \alpha$ in the optimization problem described by (4). Substituting these choices into (4) we obtain

$$\begin{aligned}
&\max \quad R_1 + R_2 \\
\text{subject to}:& \\
&R_1 + R_2 \leq \frac{1}{2}\log\left(1 + \frac{\alpha P}{1 + a^2\alpha P}\right) + \frac{1}{2}\log\left(1 + \frac{P + a^2\bar{\alpha}P}{1 + a^2\alpha P}\right) \\
&R_1 + R_2 \leq \log\left(1 + \frac{a^2\bar{\alpha}P + \alpha P}{1 + a^2\alpha P}\right)\Bigg) \\
&0 \leq \alpha \leq 1.
\end{aligned} \tag{5}$$

Sason's choice of parameters: Set α such that $\alpha P = a^2(1 + a^2P) - 1$, when $a^2(1 + a^2P) \geq 1$. Indeed this is the optimal choice of α in (5) for a certain range of a^2 and yields a rate sum given by

$$R_1 + R_2 = \log\left(\frac{a^2(1 + P + a^2P)}{1 - a^2 + a^4(1 + a^2P)}\right).$$

Etkin-Tse-Wang's choice of parameters: Set $\alpha = \frac{1}{a^2P}$ whenever $a^2P \geq 1$.

This astute heuristical choice of α gives a rate sum of

$$R_1 + R_2 = \frac{1}{2}\log\left(\frac{(1 + P + a^2P)(1 + 2a^2)}{4a^2}\right)$$

for a certain range of a^2, and the authors show that this scheme is able to beat TDM/FDM for high SNR.

Optimal choice of parameters: When $a^2(1 + a^2P) \geq 1$, set α such that

$$\alpha P = \min\left\{a^2(1 + a^2P) - 1, \frac{1 - a^2}{a^2(1 + a^2)}\right\}.$$

For this choice the rate sum is given by

$$\log\left(\frac{a^2(1 + P + a^2P)}{1 - a^2 + a^4(1 + a^2P)}\right),$$

when $0 \leq \alpha P = a^2(1 + a^2P) - 1 \leq \frac{1-a^2}{a^2(1+a^2)}$, a range that we call Sason's Band. Subsequently, when αP is set equal to $\frac{1-a^2}{a^2(1+a^2)}$, i.e., for $P \geq \frac{1-a^6}{a^6(1+a^2)}$, the rate sum is given by

$$\frac{1}{2}\log\left(\frac{(1 + P + a^2P)(1 + a^2)^2}{4a^2}\right).$$

When $a^2(1 + a^2P) < 1$, set α equal to 1, disallowing any common information. For this choice of α the rate sum is given by

$$\log\left(1 + \frac{P}{1 + a^2P}\right),$$

i.e., the interference as noise (IAN) approach. Naturally, in this region IAN competes with TDM/FDM, and one of the three scenarios, weak, intermediate or moderate interference, shall prevail.

Partial superposition with high SNR and a close to 1

The schemes considered above fail to beat TDM for high SNR. More precisely, when $a \in (a^*, 1)$ one can find P large enough so that

$$\frac{1}{2}\log\left(\frac{(1+P+a^2P)(1+a^2)^2}{4a^2}\right) < \frac{1}{2}\log\left(1+2P\right),$$

where $a^* = \sqrt{\sqrt{5}-2} \approx 0.4858683$.

To beat the TDM rate when a is close to 1, an asymmetrical selection of α_{1a} and α_{2a} is advantageous. To this end, select the parameters as follows: let $q = 1$, so that there is no need for $\alpha_{1b}, \alpha_{2b}, \beta_b, P_b$. Choose $P_a = 2P, \beta_a = \frac{1}{2}, \alpha_{1a} = 0, \alpha_{2a} = \alpha$ for some $\alpha \in (0, 1)$.

With these choice of parameters one can achieve

$$\begin{aligned} &\max \quad (R_1 + R_2) \\ \text{subject to :}& \\ &R_1 + R_2 \le \frac{1}{2}\log\left(1+P+a^2P\right) + \frac{1}{4}\log\left(\frac{1+\alpha P}{1+a^2\alpha P}\right) \\ &R_1 + R_2 \le \frac{1}{2}\log\left(\frac{(1+P+a^2P)(1+a^2P+\alpha P)}{1+a^2\alpha P}\right), \\ &0 \le \alpha \le 1. \end{aligned}$$

Now one can always choose $\alpha \in (0, 1)$ to make these two constraints equal as long as $a^2(1 + a^2P) > 1$. Choose this α^*. Not let $P \to \infty$ we see that the first constraint grows like (independent of α as long as $\alpha > 0$)

$$\frac{1}{2}\log P(\frac{1}{a} + a).$$

By our choice of α we also have that the second constraint matches the first constraint. Hence the sum-rate is at least as large as

$$\frac{1}{2}\log P(\frac{1}{a} + a).$$

This is larger than the growth of $\frac{1}{2}\log(1 + 2P)$.

Thus the approach of partially decoding one of the interfering signals and fully decoding the other can produce higher rate sums than achieved by TDM/FDM.

In this nebulous regime of interference, the conditions are not quite the same as what prevails when $a^2 \ge 1$, which is usually named strong interference. But there is a sense that some of the interference energy needs to be decoded and subsequently eliminated. For this reason we name this interference scenario as “almost strong" interference, or interference that is “forte ma non troppo”.

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