ESTR 1007: Single-Variable Calculus

Midterm Examination: Chapters 1–4

Fall 2026 Time allowed: 90 minutes Total: 100 marks

Instructions. Answer all seven questions. Give complete proofs and justify limit calculations; an unsupported numerical answer may receive little credit. You may cite results proved in Chapters 1–4 of the course notes, except where a proof from the definition is explicitly requested. Do not use derivatives, integrals, or theorems from later chapters. No calculators or electronic devices.

1. Completeness and the real numbers (14 marks). Let $S \subset \mathbb{R}$ be nonempty and bounded above, and set $s = \sup S$.

(a) Prove that for every $\varepsilon > 0$ there exists $u \in S$ such that $s - \varepsilon < u \le s$. (4 marks)

(b) Suppose that $T \subset \mathbb{R}$ is nonempty and bounded above, and that $S \subseteq T$. Prove $\sup S \le \sup T$. (5 marks)

(c) Using the Archimedean property, show that for every $a < b$ there is a rational number q with $a < q < b$. You may use the fact that every real number lies between consecutive integers. (5 marks)

2. Functions and domains (11 marks). Define $f(x) = \sqrt{x+2}$ on $[-2, \infty)$, and $g(x) = 1/(x-1)$ on $\mathbb{R} \setminus \{1\}$.

(a) Find the largest domain of $f \circ g$, and give a simplified formula for it. (6 marks)

(b) Find the largest domain of $g \circ f$, and give a formula for it. (5 marks)

3. Limits from the definitions (18 marks).

(a) Prove directly from the ε–δ definition that $\lim_{x \to 2} x^2 = 4$. Give an explicit δ in terms of ε. (5 marks)

(b) Prove directly from the ε–δ definition that

$$\lim_{x \to 1} \frac{\sqrt{2x+7} - 3}{x - 1} = \frac{1}{3}.$$

Give an explicit δ in terms of ε. (7 marks)

(c) Prove directly from the definition of a limit as $x \to +\infty$ that

$$\lim_{x \to +\infty} \left(\sqrt{x^2 + 6x} - x \right) = 3.$$

Give an explicit M in terms of ε. (6 marks)

4. Further limit calculations (15 marks). Evaluate each limit. Show the algebraic steps that justify your answer; do not use derivatives or L'Hôpital's rule.

(a)
$$\lim_{x\to 1} \frac{x^3-1}{\sqrt{2x+7}-3}.$$
(5 marks)

(b)
$$\lim_{x\to+\infty} \left(\sqrt{x^2+5x+1}-\sqrt{x^2+2x+3}\right).$$
(5 marks)

(c)
$$\lim_{x\to 0} \frac{\sqrt{1+x^2\sin^2(1/x)}-1}{x}.$$
(5 marks)

5. Nonexistence of limits (15 marks). Let
$$F(x)=\begin{cases} x^2, & x\in\mathbb{Q},\\ 2x-x^2, & x\notin\mathbb{Q}.\end{cases}$$

(a) Determine every $a\in\mathbb{R}$ for which $\lim_{x\to a} F(x)$ exists, and give the value of each existing limit. Justify your answer using density of rational and irrational numbers and limit properties. (10 marks)

(b) Determine every point at which F is continuous. (5 marks)

6. Continuity and intermediate values (17 marks).

(a) Find all $c\in\mathbb{R}$ for which the function
$$h_c(x)=\begin{cases} \dfrac{x^2-4}{x-2}, & x<2,\\ c, & x=2,\\ x^2-1, & x>2\end{cases}$$
is continuous at 2. Explain your answer using one-sided limits. (7 marks)

(b) Let $f:[0,1]\to[0,1]$ be continuous. Prove that f has a fixed point, i.e. there is $c\in[0,1]$ with $f(c)=c$. State precisely where you use the Intermediate Value Theorem. (4 marks)

(c) Let $p(x)=x^3+x-1$. Prove that p has exactly one real zero, without using derivatives. (6 marks)

7. Equal values at separated points (10 marks). Let $f:[a,b]\to\mathbb{R}$ be continuous, where $a<b$, and suppose that $f(a)=f(b)$. Prove that there is a point $c\in[a,(a+b)/2]$ such that
$$f(c)=f\left(c+\frac{b-a}{2}\right).$$
You may use the Intermediate Value Theorem, but you must explain why it applies.

End of midterm examination