Single Variable Calculus

Notes for ESTR 1007

Instructor: Prof. Chandra Nair

The Chinese University of Hong Kong

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Remark: Material marked with ‡ may be skipped on a first reading. It supplies additional justification, technical detail, or a more complete treatment of the preceding discussion, and may be read after the main ideas have been mastered. Where a ‡ item is a *deferred proof* of a statement used in core arguments, this is stated explicitly; such proofs may be skipped, but the statement itself remains available.

Conventions: On a closed or half-open interval, continuity at an included endpoint is understood one-sidedly. Unless explicitly stated otherwise, differentiability refers to interior points; endpoint derivatives are identified explicitly as one-sided derivatives. A function is called *increasing* if it is nondecreasing, and *strictly increasing* if it preserves strict inequalities; the same convention applies to *decreasing* and to sequences.

Part I

Foundations of Calculus

Chapter 1

The Real Number System

This chapter specifies the properties of the real number system that will be used throughout the course. We work axiomatically: the real numbers are assumed to form a complete ordered field. From these axioms we derive the elementary algebraic, order-theoretic, and completeness properties needed later for limits, continuity, differentiation, and integration.

Remark 1.1. Throughout these notes we assume elementary facts about the integers: the principle of mathematical induction; parity and divisibility; that every rational number can be written as a fraction in lowest terms; that a nonempty finite set has a maximum and a minimum (finite extrema); and the pigeonhole principle.

1.1 The field axioms

The real numbers form a set $\mathbb{R}$ containing two distinct elements 0 and 1. There are two operations, addition and multiplication, which assign to each pair $a, b \in \mathbb{R}$ real numbers $a + b$ and ab, respectively.

1.1.1 Axioms for addition

For all $a, b, c \in \mathbb{R}$, the following hold.

A1. *Associativity:*

$$a + (b + c) = (a + b) + c.$$

A2. *Additive identity:*

$$a + 0 = 0 + a = a.$$

A3. *Additive inverse:* there exists $-a \in \mathbb{R}$ such that

$$a + (-a) = (-a) + a = 0.$$

A4. *Commutativity:*

$$a + b = b + a.$$

We define subtraction by

$$a - b := a + (-b).$$

Lemma 1.2 (Uniqueness of the additive identity)**.** *If $x \in \mathbb{R}$ satisfies $a + x = a$ for some $a \in \mathbb{R}$, then $x = 0$.*

Proof. Add $-a$ to both sides of $a + x = a$. By associativity,

$$(-a) + (a + x) = ((-a) + a) + x = 0 + x = x.$$

On the other hand,

$$(-a) + a = 0.$$

Hence $x = 0$. □

Corollary 1.3 (Uniqueness of additive inverses)**.** *If $a + b = 0$, then $b = -a$.*

Proof. Since $a + (-a) = 0 = a + b$, adding $-a$ to both sides gives

$$(-a) + (a + (-a)) = (-a) + (a + b).$$

Using associativity and the additive identity,

$$-a = b.$$

□

Lemma 1.4 (Cancellation law for addition)**.** *If $a + b = a + c$, then $b = c$.*

Proof. Add $-a$ to both sides. Then

$$(-a) + (a + b) = (-a) + (a + c).$$

Associativity and the additive identity give $b = c$. □

1.1.2 Axioms for multiplication

For all $a, b, c \in \mathbb{R}$, the following hold.

M1. *Associativity:*

$$a(bc) = (ab)c.$$

M2. *Multiplicative identity:*

$$a \cdot 1 = 1 \cdot a = a.$$

M3. *Multiplicative inverse:* if $a \neq 0$, then there exists $a^{-1} \in \mathbb{R}$ such that

$$aa^{-1} = a^{-1}a = 1.$$

M4. *Commutativity:*

$$ab = ba.$$

For $b \neq 0$, division is defined by

$$\frac{a}{b} := ab^{-1}.$$

Lemma 1.5 (Uniqueness of the multiplicative identity)**.** *If $a \neq 0$ and $ax = a$, then $x = 1$.*

Proof. Since $a \neq 0$, the inverse a^{-1} exists. Multiplying $ax = a$ by a^{-1} gives

$$a^{-1}(ax) = a^{-1}a.$$

Associativity and the multiplicative identity give $x = 1$. □

Corollary 1.6 (Uniqueness of multiplicative inverses)**.** *If $a \neq 0$ and $ax = 1$, then $x = a^{-1}$.*

Proof. Since $aa^{-1} = 1 = ax$, multiply both sides by a^{-1} and use associativity. □

Lemma 1.7 (Cancellation law for multiplication)**.** *If $ab = ac$, then either $a = 0$ or $b = c$.*

Proof. If $a = 0$, the conclusion holds. Suppose $a \neq 0$. Multiplying $ab = ac$ by a^{-1} gives

$$a^{-1}(ab) = a^{-1}(ac).$$

Associativity and the multiplicative identity imply $b = c$. □

1.1.3 Distributivity

Addition and multiplication are connected by the following axiom.

D *Distributivity:*

$$a(b + c) = ab + ac \qquad \text{for all } a, b, c \in \mathbb{R}.$$

Lemma 1.8. *For every $a \in \mathbb{R}$,*

$$a \cdot 0 = 0.$$

Proof. Since $0 + 0 = 0$, multiplying by a gives

$$a(0 + 0) = a \cdot 0.$$

By distributivity,

$$a \cdot 0 + a \cdot 0 = a \cdot 0.$$

Add $-(a \cdot 0)$ to both sides. Associativity gives

$$\big(a \cdot 0 + (-(a \cdot 0))\big) + a \cdot 0 = a \cdot 0 + (-(a \cdot 0)),$$

and hence $0 + a \cdot 0 = 0$. Therefore $a \cdot 0 = 0$. □

Lemma 1.9 (Zero-product property)**.** *If $ab = 0$, then $a = 0$ or $b = 0$.*

Proof. If $a = 0$, there is nothing to prove. If $a \neq 0$, multiply the equation $ab = 0$ by a^{-1}. Then

$$a^{-1}(ab) = a^{-1} \cdot 0.$$

By associativity, the multiplicative identity, and Lemma 1.8,

$$b = 0.$$

□

Lemma 1.10. *For all $a, b \in \mathbb{R}$,*

$$(-a)b = -(ab), \qquad a(-b) = -(ab), \qquad (-a)(-b) = ab.$$

Proof. Since $a + (-a) = 0$, distributivity gives

$$ab + (-a)b = (a + (-a))b = 0b = 0.$$

Thus $(-a)b$ is an additive inverse of ab. By Corollary 1.3,

$$(-a)b = -(ab).$$

The identity $a(-b) = -(ab)$ is proved in the same way. Finally,

$$(-a)(-b) = -\big(a(-b)\big) = -\big(-(ab)\big) = ab.$$

□

1.2 The order axioms

Let $\mathcal{P} \subseteq \mathbb{R}$ denote the set of positive real numbers. We assume the following axioms.

O1. *Trichotomy:* for every $a \in \mathbb{R}$, exactly one of

$$a = 0, \qquad a \in \mathcal{P}, \qquad -a \in \mathcal{P}$$

holds.

O2. *Closure under addition:* if $a, b \in \mathcal{P}$, then $a + b \in \mathcal{P}$.

O3. *Closure under multiplication:* if $a, b \in \mathcal{P}$, then $ab \in \mathcal{P}$.

We say that $a > 0$ when $a \in \mathcal{P}$, and that $a < 0$ when $-a \in \mathcal{P}$. For $a, b \in \mathbb{R}$, define

$$a > b \iff a - b > 0, \qquad a < b \iff b > a.$$

We further define

$$a \geq b \iff (a > b) \text{ or } (a = b), \qquad a \leq b \iff (a < b) \text{ or } (a = b).$$

Lemma 1.11. *If $a > 0$, then $-a < 0$. If $a < 0$, then $-a > 0$.*

Proof. If $a > 0$, then $-a$ cannot be positive, since otherwise closure under addition would imply

$$0 = a + (-a) > 0,$$

contradicting trichotomy. Also $-a \neq 0$, since $-a = 0$ would imply $a = 0$. Hence $-a < 0$. The second assertion follows by applying the first to $-a$. □

Lemma 1.12. *If $a \neq 0$, then $a^2 > 0$.*

Proof. By trichotomy, either $a > 0$ or $-a > 0$. If $a > 0$, then $a^2 > 0$ by closure under multiplication. If $-a > 0$, then

$$a^2 = (-a)(-a) > 0$$

by Lemma 1.10 and closure under multiplication. □

Corollary 1.13. *We have $1 > 0$. Consequently, $1 + 1 > 0$, and in particular $1 + 1 \neq 0$.*

Proof. Since $1 \neq 0$, Lemma 1.12 gives $1^2 > 0$. Since $1^2 = 1$, the first assertion follows. The second follows from closure of positive numbers under addition. □

Lemma 1.14 (Order rules)**.** *Let $a, b, c \in \mathbb{R}$.*

(i) *If $a < b$ and $b < c$, then $a < c$.*

(ii) *If $a < b$, then $a + c < b + c$.*

(iii) *If $a < b$ and $c > 0$, then $ac < bc$.*

(iv) *If $a < b$ and $c < 0$, then $ac > bc$.*

Proof. For *(i)*, $b - a > 0$ and $c - b > 0$. Hence

$$c - a = (c - b) + (b - a) > 0,$$

so $a < c$. For *(ii)*,

$$(b + c) - (a + c) = b - a > 0,$$

so $a + c < b + c$. For *(iii)*,

$$bc - ac = (b - a)c > 0,$$

so $ac < bc$. For *(iv)*, $-c > 0$. By *(iii)*,

$$a(-c) < b(-c).$$

Using Lemma 1.10, this becomes $-ac < -bc$, which is equivalent to $ac > bc$. □

1.3 Absolute value

Definition 1.15. For $a \in \mathbb{R}$, the *absolute value* of a is

$$|a| = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

Lemma 1.16. *For every $a, b \in \mathbb{R}$,*

(i) $|a| \geq 0$, *with equality if and only if* $a = 0$;

(ii) $|-a| = |a|$;

(iii) $|ab| = |a|\,|b|$;

(iv) $|a|^2 = a^2$.

Proof. Parts *(i)* and *(ii)* follow immediately from the definition and trichotomy. For *(iii)*, consider the four possible sign combinations for a and b, and use Lemma 1.10. Part *(iv)* follows from *(ii)* and the definition of absolute value. □

Theorem 1.17 (Triangle inequality)**.** *For all $a, b \in \mathbb{R}$,*

$$|a + b| \leq |a| + |b|\,.$$

Proof. By definition,

$$-|a| \leq a \leq |a|\,, \qquad -|b| \leq b \leq |b|\,.$$

Adding the two left inequalities and the two right inequalities gives

$$-(|a| + |b|) \leq a + b \leq |a| + |b|\,.$$

Thus $|a + b| \leq |a| + |b|$. □

Corollary 1.18 (Reverse triangle inequality)**.** *For all $a, b \in \mathbb{R}$,*

$$\big||a| - |b|\big| \leq |a - b|\,.$$

Proof. From $a = (a - b) + b$ and Theorem 1.17,

$$|a| \leq |a - b| + |b|\,.$$

Hence $|a| - |b| \leq |a - b|$. Interchanging a and b gives $|b| - |a| \leq |a - b|$. The two inequalities together give the result. □

1.4 Completeness

Definition 1.19. Let $S \subseteq \mathbb{R}$. A number $M \in \mathbb{R}$ is an *upper bound* for S if

$$s \leq M \qquad \text{for every } s \in S.$$

The set S is *bounded above* if it has an upper bound.

An upper bound α of S is the *least upper bound*, or *supremum*, of S if every upper bound M of S satisfies $\alpha \leq M$. We write $\alpha = \sup S$.

Definition 1.20. A number $m \in \mathbb{R}$ is a *lower bound* for $S \subseteq \mathbb{R}$ if

$$m \leq s \qquad \text{for every } s \in S.$$

The set S is *bounded below* if it has a lower bound.

A lower bound β of S is the *greatest lower bound*, or *infimum*, of S if every lower bound m of S satisfies $m \leq \beta$. We write $\beta = \inf S$.

Axiom 1.1 (Completeness)**.** *Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.*

Proposition 1.21. *Every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum. More precisely, if $S \subseteq \mathbb{R}$ is nonempty and bounded below, then*

$$\inf S = -\sup(-S), \qquad -S := \{-s : s \in S\}.$$

Proof. Let S be nonempty and bounded below. Then $-S$ is nonempty. If m is a lower bound for S, then $-m$ is an upper bound for $-S$, since

$$s \geq m \quad \Longrightarrow \quad -s \leq -m.$$

Thus $-S$ is bounded above. By Axiom 1.1, let $\alpha = \sup(-S)$.

We claim that $-\alpha = \inf S$. Since α is an upper bound for $-S$,

$$-s \leq \alpha \qquad \text{for all } s \in S.$$

Equivalently,

$$-\alpha \leq s \qquad \text{for all } s \in S,$$

so $-\alpha$ is a lower bound for S. If m is any lower bound for S, then $-m$ is an upper bound for $-S$. Hence

$$\alpha \leq -m,$$

or equivalently $m \leq -\alpha$. Therefore $-\alpha$ is the greatest lower bound of S. □

Theorem 1.22 (Nested Interval Property)**.** *If*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

are closed bounded intervals $I_n = [a_n, b_n]$, then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof. Since $I_{n+1} \subseteq I_n$, we have

$$a_n \le a_{n+1} \qquad \text{and} \qquad b_{n+1} \le b_n \qquad \text{for every } n \in \mathbb{N}.$$

Thus (a_n) is increasing and (b_n) is decreasing. Moreover, $a_n \le b_n \le b_1$, so (a_n) is bounded above by b_1; and $a_1 \le a_n \le b_n$, so (b_n) is bounded below by a_1. By Axiom 1.1 and Proposition 1.21, the supremum

$$a = \sup\{a_n : n \in \mathbb{N}\}$$

and the infimum

$$b = \inf\{b_n : n \in \mathbb{N}\}$$

exist.

We claim that $a \le b$. First note that

$$a_k \le b_n \qquad \text{for all } k, n \in \mathbb{N}:$$

if $k \le n$, then $a_k \le a_n \le b_n$, while if $k \ge n$, then $a_k \le b_k \le b_n$. Thus, for each fixed n, the number b_n is an upper bound for $\{a_k : k \in \mathbb{N}\}$. Since a is the least upper bound, $a \le b_n$ for every n. Hence a is a lower bound for $\{b_n : n \in \mathbb{N}\}$, and since b is the greatest lower bound, $a \le b$.

Let $x \in [a, b]$. For every n,

$$a_n \le a \le x \le b \le b_n,$$

so $x \in [a_n, b_n] = I_n$. Therefore $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is nonempty. □

Remark 1.23. We have derived the Nested Interval Property from the least-upper-bound property Axiom 1.1. Conversely, in an Archimedean ordered field the Nested Interval Property implies the least-upper-bound property by bisection, as follows. Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Choose $s_0 \in S$ and an upper bound t_0, and bisect $[s_0, t_0]$: if the right half contains a point of S, keep the right half; otherwise keep the left half. Repeating this choice produces nested closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots$$

whose lengths $(t_0 - s_0)/2^{n-1}$ shrink to 0. The shrinking of the lengths to 0 requires the Archimedean property, which is proved later in this chapter (Theorem 1.28); this is why the converse is stated only for Archimedean ordered fields, and not as an unqualified equivalence in an arbitrary ordered field. The Nested Interval Property supplies a point $\alpha \in \bigcap_{n=1}^{\infty} I_n$, and a straightforward check shows that $\alpha = \sup S$.

1.5 Square roots and irrationality

Theorem 1.24. *There are no integers p, q, with $q \neq 0$, such that*

$$\left(\frac{p}{q}\right)^2 = 2.$$

Consequently, $\sqrt{2} \notin \mathbb{Q}$.

Proof. Suppose, for a contradiction, that $p^2 = 2q^2$ for some integers p, q with $q \neq 0$. After cancelling common factors if necessary, assume that p and q have no common factor greater than 1.

The equation $p^2 = 2q^2$ implies that p^2 is even. Hence p is even: if p were odd, then $p = 2k + 1$ for some integer k, and

$$p^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

would be odd. Thus $p = 2r$ for some integer r. Substituting into $p^2 = 2q^2$ gives

$$4r^2 = 2q^2, \qquad \text{so} \qquad q^2 = 2r^2.$$

Thus q^2, and hence q, is even. Therefore both p and q are even, contradicting the assumption that they have no common factor greater than 1. The contradiction proves the theorem. □

Theorem 1.25. *There exists a positive real number x such that $x^2 = 2$.*

Proof. Define

$$S = \left\{ y \in \mathbb{R} : y > 0 \text{ and } y^2 < 2 \right\}.$$

The set S is nonempty because $1 \in S$. It is bounded above: if $y \geq 2$, then $y^2 \geq 4 > 2$, so $y \notin S$. By completeness, S has a supremum; denote it by x.

We first note that $x > 0$, since $x \geq 1$. We prove that $x^2 = 2$ by ruling out the alternatives $x^2 < 2$ and $x^2 > 2$.

Suppose first that $x^2 < 2$. Define

$$y = \frac{3x + 4}{2x + 3}.$$

Since $x > 0$, the denominator is positive. Moreover,

$$y - x = \frac{3x + 4 - x(2x + 3)}{2x + 3} = \frac{2(2 - x^2)}{2x + 3} > 0.$$

Thus $y > x$. A direct calculation gives

$$2 - y^2 = \frac{2(2x + 3)^2 - (3x + 4)^2}{(2x + 3)^2} = \frac{2 - x^2}{(2x + 3)^2} > 0.$$

Hence $y^2 < 2$, so $y \in S$. This contradicts the fact that x is an upper bound for S.

Suppose next that $x^2 > 2$, and define the same number

$$y = \frac{3x + 4}{2x + 3}.$$

Then

$$x - y = \frac{x(2x + 3) - (3x + 4)}{2x + 3} = \frac{2(x^2 - 2)}{2x + 3} > 0,$$

so $y < x$. The same calculation now yields

$$y^2 - 2 = \frac{(3x + 4)^2 - 2(2x + 3)^2}{(2x + 3)^2} = \frac{x^2 - 2}{(2x + 3)^2} > 0.$$

Thus $y^2 > 2$. We claim that y is an upper bound for S. If $s \in S$ and $s \geq y$, then $s > 0$, so $s^2 \geq y^2 > 2$, contradicting $s \in S$. Hence $s \leq y$ for every $s \in S$. This contradicts the fact that x is the least upper bound of S. Therefore neither $x^2 < 2$ nor $x^2 > 2$ holds. By trichotomy, $x^2 = 2$. □

1.5.1 Existence of n-th roots

Remark 1.26. The same completeness argument extends from square roots to n-th roots: for every $x > 0$ and every $n \in \mathbb{N}$ there is a unique $y > 0$ with

$$y^n = x,$$

again obtained as the supremum of the set

$$\{t > 0 : t^n < x\}.$$

A shorter proof, based on the Intermediate Value Theorem, is given in Theorem 4.24. The number y is denoted

$$x^{1/n} = \sqrt[n]{x}.$$

Rational and real powers x^r for a general exponent r are defined and studied in Chapter 9.

1.6 The Archimedean property

Theorem 1.27. *The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.*

Proof. Suppose, for a contradiction, that $\mathbb{N}$ is bounded above. By Axiom 1.1, let $\alpha = \sup \mathbb{N}$. Since $n + 1 \in \mathbb{N}$ whenever $n \in \mathbb{N}$, we have

$$n + 1 \leq \alpha \qquad \text{for every } n \in \mathbb{N}.$$

Subtracting 1 gives

$$n \leq \alpha - 1 \qquad \text{for every } n \in \mathbb{N}.$$

Thus $\alpha - 1$ is an upper bound for $\mathbb{N}$, contradicting the fact that α is its least upper bound. □

Theorem 1.28 (Archimedean property)**.** *For every $r > 0$ and every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

$$n\varepsilon > r.$$

Proof. Suppose no such n exists. Then $n\varepsilon \leq r$ for every $n \in \mathbb{N}$. Since $\varepsilon > 0$, multiplying by $1/\varepsilon$ gives

$$n \leq \frac{r}{\varepsilon} \qquad \text{for every } n \in \mathbb{N}.$$

Thus $\mathbb{N}$ is bounded above, contradicting Theorem 1.27. □

Corollary 1.29. *For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

$$\frac{1}{n} < \varepsilon.$$

Proof. Apply Theorem 1.28 with $r = 1$. □

Proposition 1.30 (Bernoulli's inequality)**.** *For every $x \geq -1$ and every $n \in \mathbb{N}$,*

$$(1 + x)^n \geq 1 + nx,$$

with equality if and only if $x = 0$ or $n = 1$.

Proof. We first prove the inequality by induction on n. For $n = 1$, both sides equal $1 + x$, so equality holds.

Suppose $(1+x)^n \geq 1+nx$ for some $n \in \mathbb{N}$. Since $x \geq -1$, we have $1+x \geq 0$. Multiplying the induction hypothesis by the nonnegative number $1 + x$ gives

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x,$$

where the final step uses $nx^2 \geq 0$. This proves the inequality for every $n \in \mathbb{N}$.

It remains to analyze equality. If $x = 0$, then

$$(1+x)^n = 1 = 1 + nx,$$

and if $n = 1$, then

$$(1+x)^n = 1 + x = 1 + nx.$$

Conversely, suppose $x \neq 0$ and $n \geq 2$. We show that strict inequality holds. If $x = -1$, then

$$(1+x)^n = (1-1)^n = 0 > 1 - n = 1 + nx$$

because $n \geq 2$. Now suppose $x \neq -1$. Since $x \geq -1$, we have $x > -1$, so $1 + x > 0$; also $x^2 > 0$. We prove $(1+x)^n > 1 + nx$ by induction on $n \geq 2$. For $n = 2$,

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x.$$

If $(1+x)^n > 1 + nx$ for some $n \geq 2$, then

$$(1+x)^{n+1} = (1+x)(1+x)^n > (1+x)(1+nx) = 1 + (n+1)x + nx^2 > 1 + (n+1)x,$$

because $1 + x > 0$ and $nx^2 > 0$. Hence strict inequality holds for every $n \geq 2$. Therefore equality occurs exactly when $x = 0$ or $n = 1$. □

1.7 Density

Definition 1.31. A set $D \subseteq \mathbb{R}$ is *dense in* $\mathbb{R}$ if every open interval (x, y), with $x < y$, contains an element of D.

Theorem 1.32 (Density of $\mathbb{Q}$)**.** *The rational numbers are dense in* $\mathbb{R}$*. Equivalently, if* $x < y$*, then there is a rational number* q *satisfying*

$$x < q < y.$$

Proof. Let $x < y$. By Corollary 1.29, choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < y - x.$$

Since $\mathbb{N}$ is unbounded above, there exists an integer m with $m > nx$. The set $\{m \in \mathbb{Z} : m > nx\}$ is nonempty and bounded below by nx, so it has a least element. (More generally, a nonempty set of integers that is bounded below has a least element, and a nonempty set of integers that is bounded above has a greatest element; these facts follow from the elementary integer background and the Archimedean property Theorem 1.28.) Let m be the least such integer. Then

$$m - 1 \leq nx < m.$$

Dividing by $n > 0$ gives

$$\frac{m-1}{n} \le x < \frac{m}{n}.$$

If $m/n \ge y$, then

$$\frac{1}{n} = \frac{m}{n} - \frac{m-1}{n} \ge y - x,$$

contradicting $1/n < y - x$. Therefore

$$x < \frac{m}{n} < y.$$

The number m/n is rational, completing the proof. □

Theorem 1.33 (Density of the irrational numbers)**.** *The irrational numbers are dense in* $\mathbb{R}$*. Equivalently, if* $x < y$*, then there is an irrational number* z *satisfying*

$$x < z < y.$$

Proof. Since $x/\sqrt{2} < y/\sqrt{2}$, Theorem 1.32 gives a rational number q satisfying

$$\frac{x}{\sqrt{2}} < q < \frac{y}{\sqrt{2}}.$$

Then

$$x < q\sqrt{2} < y.$$

The number q can be chosen nonzero, and then $q\sqrt{2}$ is irrational. Thus $z = q\sqrt{2}$ has the required properties. □

1.7.1 ‡ Integer combinations of an irrational number

Definition 1.34. For $x \in \mathbb{R}$, the *floor* of x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. Thus

$$\lfloor x \rfloor \le x < \lfloor x \rfloor + 1.$$

The *fractional part* of x is

$$\{x\} := x - \lfloor x \rfloor.$$

Hence

$$0 \le \{x\} < 1 \qquad \text{and} \qquad x = \lfloor x \rfloor + \{x\}.$$

Remark 1.35 (‡ Existence of the floor). For completeness, we justify that $\lfloor x \rfloor$ is well defined. By the unboundedness of $\mathbb{N}$, choose $M, N \in \mathbb{N}$ with

$$-M < x < N.$$

The finite nonempty set

$$B_x := \{n \in \mathbb{Z} : -M \le n \le x\}$$

has a greatest element k. Every integer $n \le x$ either lies in B_x, in which case $n \le k$, or satisfies $n < -M \le k$. Thus k is the greatest integer at most x. Moreover, $k + 1 > x$, since otherwise $k + 1 \in B_x$, contradicting the maximality of k. Therefore $k = \lfloor x \rfloor$.

Remark 1.36 (‡ Basic properties of fractional parts). For every $x \in \mathbb{R}$ and every $n \in \mathbb{Z}$,

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n \qquad \text{and} \qquad \{x + n\} = \{x\}.$$

Also,

$$\{x\} = 0 \quad \Longleftrightarrow \quad x \in \mathbb{Z}.$$

Theorem 1.37 (‡)**.** *Let $c \in \mathbb{R} \setminus \mathbb{Q}$. Then*

$$\{n + mc : m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$.

Proof. Let

$$G := \{n + mc : m, n \in \mathbb{Z}\}.$$

We first show that G contains arbitrarily small positive elements.

Let $\varepsilon > 0$. By Theorem 1.28, choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon.$$

Consider the $N + 1$ fractional parts

$$\{0c\}, \{c\}, \{2c\}, \ldots, \{Nc\}.$$

They are distinct. Indeed, if $\{jc\} = \{kc\}$ for some integers $0 \le j < k \le N$, then

$$(k - j)c = kc - jc = \lfloor kc \rfloor - \lfloor jc \rfloor \in \mathbb{Z}.$$

Since $k - j \neq 0$, this implies

$$c = \frac{\lfloor kc \rfloor - \lfloor jc \rfloor}{k - j} \in \mathbb{Q},$$

a contradiction.

Divide $[0, 1)$ into the N intervals

$$\left[0, \frac{1}{N}\right), \left[\frac{1}{N}, \frac{2}{N}\right), \ldots, \left[\frac{N-1}{N}, 1\right).$$

By the pigeonhole principle, two of the $N + 1$ distinct numbers $\{0c\}, \{c\}, \ldots, \{Nc\}$ belong to the same interval. Thus there exist integers $0 \le j < k \le N$ such that

$$0 < |\{kc\} - \{jc\}| < \frac{1}{N} < \varepsilon.$$

Moreover,

$$\{kc\} - \{jc\} = (kc - \lfloor kc \rfloor) - (jc - \lfloor jc \rfloor) = (k - j)c - (\lfloor kc \rfloor - \lfloor jc \rfloor) \in G.$$

Consequently, one of $\{kc\} - \{jc\}$ and $\{jc\} - \{kc\}$ is a number $\delta \in G$ satisfying

$$0 < \delta < \varepsilon.$$

Now let $x, y \in \mathbb{R}$ satisfy $x < y$. Apply the preceding argument with $\varepsilon = y - x$ to obtain $\delta \in G$ such that

$$0 < \delta < y - x.$$

Choose an integer q such that

$$(q - 1)\delta \le x < q\delta.$$

Then

$$x < q\delta \le x + \delta < x + (y - x) = y.$$

Since $\delta \in G$ and $q \in \mathbb{Z}$, we have $q\delta \in G$. Therefore

$$x < q\delta < y.$$

Thus G is dense in $\mathbb{R}$. □

Remark 1.38 (‡). The irrationality assumption in Theorem 1.37 is necessary. If $c = p/q \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$ and $q > 0$, then

$$n + mc = \frac{nq + mp}{q} \in \frac{1}{q}\mathbb{Z}$$

for all $m, n \in \mathbb{Z}$. Hence the set $\{n + mc : m, n \in \mathbb{Z}\}$ has gaps of length $1/q$, so it is not dense in $\mathbb{R}$.

1.8 Summary

The real number system is a complete ordered field. The field axioms justify the rules of algebra, the order axioms justify inequalities, and completeness supplies the least-upper-bound principle. Completeness is the property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and will be responsible for the principal existence theorems of calculus.

1.9 Exercises

Exercise 1.39. Prove that if $ab > 0$, then a and b have the same sign.

Exercise 1.40. Prove that if $a > 0$ and $ab > a$, then $b > 1$.

Exercise 1.41. Prove that $(-a)(-b) = ab$ directly from the distributive axiom and the uniqueness of additive inverses.

Exercise 1.42. Prove that if $a < b$, then $-b < -a$.

Exercise 1.43. Solve the inequality

$$|x - 1|\,|x - 2| > 1.$$

Write the solution set as a union of intervals.

Exercise 1.44. Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Prove that $\sup S$ is unique.

Exercise 1.45. Complete the proof of Theorem 1.25 by verifying the identities

$$2 - \left(\frac{3x + 4}{2x + 3}\right)^2 = \frac{2 - x^2}{(2x + 3)^2}$$

and

$$\left(\frac{3x + 4}{2x + 3}\right)^2 - 2 = \frac{x^2 - 2}{(2x + 3)^2},$$

in their respective uses in the proof.

Chapter 2

Functions

Functions are the principal objects studied in calculus. This chapter gives a formal definition of a function, develops the standard operations on functions, and introduces graphs as a useful source of intuition. Graphs will not be used as substitutes for proofs.

2.1 Functions, domains, ranges, and codomains

Definition 2.1. A *function* is a set f of ordered pairs with the property that no first coordinate occurs in more than one pair. Thus, if

$$(x, y) \in f \qquad \text{and} \qquad (x, z) \in f,$$

then $y = z$.

If $(x, y) \in f$, we write

$$f(x) = y.$$

The variable x is an input and y is the corresponding output.

Definition 2.2. Let f be a function.

- The *domain* of f, denoted by $\mathrm{dom}(f)$, is the set of all first coordinates of pairs in f:

$$\mathrm{dom}(f) = \{x : \text{there exists } y \text{ with } (x, y) \in f\}.$$

- The *range* of f, denoted by $\mathrm{ran}(f)$, is the set of all second coordinates of pairs in f:

$$\mathrm{ran}(f) = \{y : \text{there exists } x \text{ with } (x, y) \in f\}.$$

- A *codomain* of f is any set containing $\mathrm{ran}(f)$.

If $X = \mathrm{dom}(f)$ and Y is a specified codomain for f, we write

$$f : X \to Y.$$

This notation states that every input in X has a unique output and that all outputs lie in Y. It does not assert that every element of Y is an output.

Example 2.3. The function whose domain is $\{-2, -1, 0, 1, 2\}$ and whose rule is "square the input" is the set

$$f = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}.$$

It has domain $\{-2, -1, 0, 1, 2\}$ and range $\{0, 1, 4\}$.

Remark 2.4. Different inputs may have the same output. For instance,

$$(-2)^2 = 2^2 = 4.$$

What is prohibited is one input having two distinct outputs.

Definition 2.5. A function f is a *real function* if both $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ are subsets of $\mathbb{R}$.

Unless otherwise specified, all functions in these notes are real functions.

2.2 Specifying functions

A function is often specified by its domain, codomain, and rule:

$$f : X \to Y, \qquad x \longmapsto \text{an expression involving } x.$$

For example,

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \longmapsto x^2 - 1$$

defines a function with $f(x) = x^2 - 1$ for every real x.

Definition 2.6. If a formula defining a real function is presented without an explicit domain, its *natural domain* is the largest subset of $\mathbb{R}$ on which the formula is meaningful.

Thus, the natural domain excludes values that require division by zero or the square root of a negative number.

Example 2.7. The function

$$f(x) = \frac{x^2 + 1}{x^2 - 1}$$

has natural domain

$$\mathbb{R} \setminus \{-1, 1\}.$$

The function

$$g(x) = \sqrt{\frac{x+1}{x-1}}$$

has natural domain

$$(-\infty, -1] \cup (1, \infty).$$

Definition 2.8. Two functions f and g are *equal* if they have the same domain and satisfy

$$f(x) = g(x) \qquad \text{for every } x \in \mathrm{dom}(f).$$

Example 2.9. The formulas

$$x^2 - 1 \qquad \text{and} \qquad (x+1)(x-1)$$

define the same function on $\mathbb{R}$. However,

$$\frac{x^2 - 1}{x - 1} \qquad \text{and} \qquad x + 1$$

do not define the same function on their natural domains: the first has domain $\mathbb{R} \setminus \{1\}$, while the second has domain $\mathbb{R}$.

2.2.1 Piecewise-defined functions

A function may be defined by different formulas on different parts of its domain. For example,

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Example 2.10. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

Then

$$f(-2) = 0, \qquad f(0) = 0, \qquad f(3) = 9.$$

The clauses of a piecewise definition should cover the intended domain. If two clauses overlap, they must assign the same output at every point in the overlap.

2.3 Arithmetic operations on functions

Let f and g be real functions. Their sum, difference, and product are defined by

$$\begin{aligned} (f+g)(x) &:= f(x) + g(x), \\ (f-g)(x) &:= f(x) - g(x), \\ (fg)(x) &:= f(x)g(x), \end{aligned}$$

for every $x \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$. Thus

$$\operatorname{dom}(f+g) = \operatorname{dom}(f-g) = \operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g).$$

If $c \in \mathbb{R}$, define the scalar multiple cf by

$$(cf)(x) := cf(x), \qquad x \in \operatorname{dom}(f).$$

The quotient f/g is defined by

$$\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$$

for all x in the set

$$\operatorname{dom}\left(\frac{f}{g}\right) = \{x \in \operatorname{dom}(f) \cap \operatorname{dom}(g) : g(x) \neq 0\}.$$

Lemma 2.11. *The functions $f+g$, $f-g$, fg, cf, and f/g, on the domains specified above, are functions.*

Proof. We prove the assertion for $f+g$; the remaining cases are similar. Suppose $(x, y) \in f+g$. By definition, there are unique numbers u and v such that

$$(x, u) \in f, \qquad (x, v) \in g, \qquad y = u + v.$$

The uniqueness of u and v follows because f and g are functions. Hence y is uniquely determined by x, so $f+g$ is a function. □

Lemma 2.12. *Whenever the expressions are defined, function addition and multiplication satisfy*

$$\begin{aligned} f+g &= g+f, \\ (f+g)+h &= f+(g+h), \\ fg &= gf, \\ (fg)h &= f(gh), \\ f(g+h) &= fg+fh. \end{aligned}$$

Proof. Each identity follows pointwise from the corresponding identity in $\mathbb{R}$. For example, if x belongs to the common domain of $f+g$ and $g+f$, then

$$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x).$$

The remaining identities are proved in the same way. □

2.4 Polynomial and rational functions

Definition 2.13. A *polynomial function* is a function of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $n \in \mathbb{N}_0$, $a_0, \ldots, a_n \in \mathbb{R}$, and $a_n \neq 0$ if $n \geq 1$. The degree of p is n.

By convention, a nonzero constant polynomial $p(x) = a_0$, with $a_0 \neq 0$, has degree 0. The zero polynomial $p(x) = 0$ is treated separately and is not assigned a nonnegative-integer degree; this convention matters for the division algorithm and for partial fractions, where the zero polynomial plays a special role.

Every polynomial function has domain $\mathbb{R}$. The constant functions and the identity function $x \mapsto x$ are polynomials.

Definition 2.14. A *rational function* is a quotient

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials and q is not the zero polynomial. Its natural domain is

$$\{x \in \mathbb{R} : q(x) \neq 0\}.$$

Example 2.15. The function

$$r(x) = \frac{x^2+1}{x^2-1}$$

is rational, with natural domain $\mathbb{R} \setminus \{-1, 1\}$.

2.5 Composition of functions

Definition 2.16. Let g be a function and let f be a function. The *composition of f with g*, denoted by $f \circ g$, is defined by

$$(f \circ g)(x) := f(g(x))$$

at every input x for which $x \in \operatorname{dom}(g)$ and $g(x) \in \operatorname{dom}(f)$. Thus

$$\operatorname{dom}(f \circ g) = \{x \in \operatorname{dom}(g) : g(x) \in \operatorname{dom}(f)\}.$$

The notation $f \circ g$ means "apply g first, then apply f". In general, composition is not commutative.

Example 2.17. Let $f(x) = x^2$ and $g(x) = x + 1$. Then

$$(f \circ g)(x) = (x+1)^2, \qquad (g \circ f)(x) = x^2 + 1.$$

The functions $f \circ g$ and $g \circ f$ are distinct.

Lemma 2.18 (Associativity of composition)**.** *Whenever both sides are defined,*

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Proof. For every x in the common domain,

$$\big(f \circ (g \circ h)\big)(x) = f(g(h(x))) = \big((f \circ g) \circ h\big)(x).$$

□

Example 2.19. Let

$$f(x) = \sqrt{x}, \qquad g(x) = \frac{x+1}{x-1}.$$

Then

$$(f \circ g)(x) = \sqrt{\frac{x+1}{x-1}}.$$

The domain is the set of x such that

$$x \neq 1 \qquad \text{and} \qquad \frac{x+1}{x-1} \geq 0.$$

Therefore

$$\operatorname{dom}(f \circ g) = (-\infty, -1] \cup (1, \infty).$$

2.5.1 ‡ Composition on the full domain

Proposition 2.20. *Let $g : X \to Y$ and let $f : Z \to W$. If*

$$\operatorname{ran}(g) \subseteq Z,$$

then $f \circ g$ is defined on all of X, and

$$f \circ g : X \to W.$$

In particular,

$$\operatorname{dom}(f \circ g) = X.$$

Proof. Let $x \in X$. Since $g(x) \in \operatorname{ran}(g) \subseteq Z = \operatorname{dom}(f)$, the value $f(g(x))$ is defined. Therefore $f \circ g$ is defined for every $x \in X$. Moreover, $f(g(x)) \in W$, so $f \circ g : X \to W$. □

2.6 Graphs of functions

Definition 2.21. Let $f : D \to \mathbb{R}$. The *graph* of f is the set

$$\Gamma_f = \{(x, f(x)) : x \in D\} \subseteq \mathbb{R}^2.$$

The graph records the output associated with each input. It is useful for visualization, but it does not by itself prove any assertion about a function. A graph produced by hand or by software displays only finitely much information and can conceal important local behaviour.

Theorem 2.22 (Vertical line test)**.** *A subset $G \subseteq \mathbb{R}^2$ is the graph of a function of x if and only if every vertical line intersects G in at most one point.*

Proof. Suppose first that G is the graph of a function f. If a vertical line with first coordinate x met G at two points (x, y_1) and (x, y_2), then $f(x) = y_1$ and $f(x) = y_2$. Since f is a function, $y_1 = y_2$. Thus the vertical line meets G in at most one point.

Conversely, suppose every vertical line meets G in at most one point. Let

$$D = \{x \in \mathbb{R} : \text{there exists } y \in \mathbb{R} \text{ with } (x, y) \in G\}.$$

For each $x \in D$, there exists at least one y with $(x, y) \in G$, and the vertical-line hypothesis guarantees that this y is unique. Hence G is the graph of a function on D. □

Definition 2.23. Let f be a function with domain D.

- The function f is *even* if its domain D is symmetric, in the sense that $x \in D \implies -x \in D$, and $f(-x) = f(x)$ for every $x \in D$.
- The function f is *odd* if its domain D is symmetric ($x \in D \implies -x \in D$) and $f(-x) = -f(x)$ for every $x \in D$.
- The function f is *periodic with period* $T > 0$ if $D + T = D$, that is, $x + T \in D$ for every $x \in D$ (so D is invariant under translation by T), and $f(x + T) = f(x)$ for every $x \in D$.

Remark 2.24. The graph of an even function is symmetric about the y-axis; the graph of an odd function is symmetric about the origin. For example, x^2 and $|x|$ are even, while x^3 is odd. The sine and cosine functions, studied in Chapter 9, are odd and even respectively, and are periodic.

A general periodic function need not have a least positive period (for example, a constant function has every positive number as a period), so these notes do not define a "fundamental period" without further hypotheses.

2.6.1 Linear functions

A function of the form

$$f(x) = mx + b$$

is called a *linear function*. Its graph is a line with slope m. If (x_0, y_0) and (x_1, y_1), with $x_0 \neq x_1$, lie on a line, then its slope is

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

The equation of that line is

$$y - y_0 = m(x - x_0).$$

2.6.2 Basic graph transformations

Let f be a function and let $c \neq 0$. The graphs of the following functions are obtained from the graph of f by simple transformations:

$f(x) + c$	vertical translation by c,
$f(x + c)$	horizontal translation by $-c$,
$cf(x)$	vertical scaling by c,
$f(cx)$	horizontal scaling by $1/\lvert c\rvert$.

When $c < 0$, the corresponding scaling also includes reflection in an axis.

2.7 ‡ Images and preimages

Let $f : X \to Y$. For a subset $A \subseteq X$, the *image* of A under f is

$$f(A) := \{f(x) : x \in A\} \subseteq Y.$$

For a subset $B \subseteq Y$, the *preimage* of B under f is

$$f^{-1}(B) := \{x \in X : f(x) \in B\} \subseteq X.$$

The notation $f^{-1}(B)$ here denotes a set of inputs; it does not assume that the function f has an inverse function.

Example 2.25. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. If

$$A = [-2, 1], \qquad B = [1, 4],$$

then

$$f(A) = [0, 4] \qquad \text{and} \qquad f^{-1}(B) = [-2, -1] \cup [1, 2].$$

Proposition 2.26. *Let $f : X \to Y$, let $A, C \subseteq X$, and let $B, D \subseteq Y$. Then*

$$\begin{aligned} f(A \cup C) &= f(A) \cup f(C), \\ f(A \cap C) &\subseteq f(A) \cap f(C), \\ f^{-1}(B \cup D) &= f^{-1}(B) \cup f^{-1}(D), \\ f^{-1}(B \cap D) &= f^{-1}(B) \cap f^{-1}(D). \end{aligned}$$

Proof. We prove the first and fourth identities; the other two are similar.

Let $y \in Y$. Then

$$\begin{aligned} y \in f(A \cup C) &\iff \text{there exists } x \in A \cup C \text{ such that } f(x) = y \\ &\iff \text{there exists } x \in A \text{ such that } f(x) = y, \text{ or there exists } x \in C \text{ such that } f(x) = y \\ &\iff y \in f(A) \cup f(C). \end{aligned}$$

For the fourth identity, let $x \in X$. Then

$$\begin{aligned} x \in f^{-1}(B \cap D) &\iff f(x) \in B \cap D \\ &\iff f(x) \in B \text{ and } f(x) \in D \\ &\iff x \in f^{-1}(B) \cap f^{-1}(D). \end{aligned}$$

□

Remark 2.27. The inclusion

$$f(A \cap C) \subseteq f(A) \cap f(C)$$

need not be an equality. For example, let $f(x) = x^2$, let $A = \{-1\}$, and let $C = \{1\}$. Then

$$A \cap C = \varnothing, \qquad f(A \cap C) = \varnothing, \qquad f(A) \cap f(C) = \{1\}.$$

The equality $f(A \cap C) = f(A) \cap f(C)$ does hold when f is one-to-one.

2.8 ‡ One-to-one, onto, and inverse functions

Definition 2.28. Let $f : X \to Y$.

- The function f is *one-to-one*, or *injective*, if

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2 \qquad \text{for all } x_1, x_2 \in X.$$

- The function f is *onto*, or *surjective*, if

$$\operatorname{ran}(f) = Y.$$

  Equivalently, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

- The function f is *bijective* if it is both injective and surjective.

Example 2.29. The function $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$ is surjective but not injective, because $f(-1) = f(1)$. Its restriction

$$f : [0, \infty) \to [0, \infty), \qquad f(x) = x^2,$$

is bijective.

Definition 2.30. Let $f : X \to Y$. A function $g : Y \to X$ is an *inverse function* of f if

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y,$$

where $\mathrm{id}_X : X \to X$ is the identity function defined by

$$\mathrm{id}_X(x) = x.$$

If an inverse function exists, it is denoted by f^{-1}.

Remark 2.31. Every injective function $f : X \to Y$ has an inverse defined on its range $\operatorname{ran}(f)$: for each $y \in \operatorname{ran}(f)$ there is a unique $x \in X$ with $f(x) = y$, and the rule $y \mapsto x$ is a function $\operatorname{ran}(f) \to X$. An inverse defined on the entire specified codomain Y exists exactly when f is a bijection, by Theorem 2.32. This distinction—inverse on the range versus inverse on the codomain—is used throughout the treatment of inverse functions in Chapter 8.

Theorem 2.32. *Let $f : X \to Y$. The function f has an inverse function*

$$f^{-1} : Y \to X$$

if and only if f is bijective. Moreover, this inverse function is unique.

Proof. Suppose first that f has an inverse function $g : Y \to X$. If $f(x_1) = f(x_2)$, then applying g to both sides gives

$$x_1 = (g \circ f)(x_1) = (g \circ f)(x_2) = x_2.$$

Thus f is injective. Also, if $y \in Y$, then

$$y = (f \circ g)(y) = f(g(y)),$$

so $y \in \operatorname{ran}(f)$. Hence f is surjective.

Conversely, suppose that f is bijective. Since f is surjective, for each $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Since f is injective, this x is unique. Define $g : Y \to X$ by letting $g(y)$ be the unique element of X for which

$$f(g(y)) = y.$$

Then $f \circ g = \mathrm{id}_Y$. If $x \in X$, then

$$f(g(f(x))) = f(x).$$

Since f is injective, $g(f(x)) = x$. Thus $g \circ f = \mathrm{id}_X$, so g is an inverse function of f.

Finally, suppose $g, h : Y \to X$ are both inverse functions of f. For any $y \in Y$,

$$g(y) = g(f(h(y))) = (g \circ f)(h(y)) = h(y).$$

Thus $g = h$, proving uniqueness. □

Example 2.33. For the bijection

$$f : [0, \infty) \to [0, \infty), \qquad f(x) = x^2,$$

the inverse function is

$$f^{-1}(y) = \sqrt{y}.$$

The restriction of the domain to $[0, \infty)$ is essential: the function $x \mapsto x^2$ on all of $\mathbb{R}$ is not injective and therefore has no inverse function $\mathbb{R} \to \mathbb{R}$.

2.9 ‡ Graphical cautions and pathological examples

2.9.1 Rapid oscillation

Consider

$$f(x) = \sin\left(\frac{1}{x}\right), \qquad x \neq 0.$$

The function is defined and continuous at every nonzero input. As x approaches 0, however, the function oscillates infinitely often between values close to -1 and 1. A finite-resolution plot cannot accurately represent all of these oscillations.

The related function

$$g(x) = x \sin\left(\frac{1}{x}\right), \qquad x \neq 0,$$

has oscillations of decreasing amplitude as x approaches 0. Later we will define $g(0) = 0$ and prove that the resulting function is continuous at 0, even though $\sin(1/x)$ itself has no limit there.

2.9.2 Dirichlet's function

Define

$$d(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Every interval contains both rational and irrational numbers, by Theorems 1.32 and 1.33. Thus every interval contains inputs at which d takes value 0 and inputs at which it takes value 1. The graph cannot be faithfully represented by ordinary plotting.

In Chapter 4, we will prove that Dirichlet's function is discontinuous at every real number.

2.10 Exercises

Exercise 2.34. For each function below, determine its natural domain:

$$\frac{1}{x^2-4}, \qquad \sqrt{3-x}, \qquad \frac{\sqrt{x+2}}{x-1}, \qquad \sqrt{\frac{x-2}{x+1}}.$$

Exercise 2.35. Let $f(x) = x^2 - 4x + 3$ and $g(x) = 1/(x-1)$. Determine the formulas and natural domains of $f+g$, $f-g$, fg, f/g, $f \circ g$, and $g \circ f$.

Exercise 2.36. Prove all of the identities in Lemma 2.12.

Exercise 2.37. Show that the circle

$$x^2 + y^2 = 1$$

is not the graph of a function of x. Express its upper and lower semicircles as graphs of functions.

Exercise 2.38. Let $f(x) = |x|$. Give piecewise formulas for

$$f(x-2), \qquad f(x)+2, \qquad f(2x), \qquad -f(x).$$

Sketch the resulting graphs.

Exercise 2.39. Prove that the graph of a function is unchanged if its codomain is replaced by any larger set containing its range.

Exercise 2.40 (‡)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, and let

$$A = [-3, -1], \qquad B = [1, 4].$$

Determine $f(A)$, $f^{-1}(B)$, $f(A \cap f^{-1}(B))$, and $f(A) \cap B$.

Exercise 2.41 (‡)**.** Prove that a function $f : X \to Y$ is injective if and only if

$$f(A \cap C) = f(A) \cap f(C)$$

for every pair of subsets $A, C \subseteq X$.

Exercise 2.42 (‡)**.** Determine whether each of the following functions is injective, surjective, or bijective. In each case, use the stated domain and codomain.

$$x \mapsto x^3 : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2 : \mathbb{R} \to [0, \infty), \qquad x \mapsto x^2 : [0, \infty) \to [0, \infty), \qquad x \mapsto e^x : \mathbb{R} \to (0, \infty).$$

Chapter 3

Limits

Limits describe the behaviour of a function near a point. The definition does not depend on the value of the function at the point itself; indeed, the function need not even be defined there. Limits are the basis for continuity, differentiation, and integration.

3.1 Motivating examples

Before giving a formal definition, we record several qualitatively different ways in which a function can behave near a point.

(1) The function $f(x) = x^2$ has the expected behaviour near $x = 1$: $f(x)$ approaches 1 as x approaches 1.

(2) The function $f(x) = \sqrt{x}$ is not defined in a full neighbourhood of -1, so it makes no sense to discuss a two-sided limit there.

(3) The function
$$f(x) = \frac{1}{x-1}$$
is defined near 1, except at 1, but becomes arbitrarily large in magnitude as x approaches 1.

(4) The function
$$f(x) = \frac{1}{1 + 1/(1+x)}$$
is not defined at $x = -1$, but for $x \neq -1, -2$ it can be rewritten as
$$f(x) = \frac{1+x}{2+x}.$$
Thus it has a removable hole at -1 and approaches 0 there.

(5) Altering the value of a function at one point does not alter its limiting behaviour there. Thus, if
$$g(x) = \begin{cases} (1+x)/(2+x), & x \neq -1, \\ 4, & x = -1, \end{cases}$$
then $\lim_{x \to -1} g(x) = 0$, although $g(-1) = 4$.

(6) The function
$$h(x) = \frac{x}{|x|}, \qquad x \neq 0,$$
has value -1 for $x < 0$ and 1 for $x > 0$. Its left- and right-hand behaviours at 0 differ, so it has no two-sided limit there.

(7) The function
$$x \longmapsto \sin\left(\frac{1}{x}\right), \qquad x \neq 0,$$
oscillates infinitely often near 0, and has no limit there.

(8) Dirichlet's function
$$d(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$
has no limit at any real number. Every interval contains both rational and irrational numbers.

A formal definition of limit must distinguish these behaviours.

3.2 The ε–δ definition

Definition 3.1. Let f be defined on a deleted neighbourhood of a; that is, suppose there exists $r > 0$ such that $f(x)$ is defined whenever $0 < |x - a| < r$. We say that f has *limit* L *at* a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

We write
$$\lim_{x \to a} f(x) = L, \qquad \text{or} \qquad f(x) \to L \quad \text{as } x \to a.$$

The number ε specifies the permitted error in the output. The number δ, which may depend on ε, specifies how close the input must be to a in order to guarantee that error bound.

The condition $0 < |x - a|$ excludes $x = a$. Consequently, the value $f(a)$, if defined, plays no role in determining the limit.

In the definition above the number δ is always taken to be at most the radius r on which f is defined near a, so that every x with $0 < |x - a| < \delta$ lies in the domain of f; equivalently, the condition is understood to be quantified only over points x of the domain of f. The same convention applies to all limits below.

Example 3.2. For every $a, c \in \mathbb{R}$,
$$\lim_{x \to a} c = c.$$

Proof. Let $\varepsilon > 0$. Choose any $\delta > 0$, for example $\delta = 1$. Then
$$|c - c| = 0 < \varepsilon$$
whenever $0 < |x - a| < \delta$. □

Example 3.3. For every $a \in \mathbb{R}$,
$$\lim_{x \to a} x = a.$$

Proof. Let $\varepsilon > 0$, and choose $\delta = \varepsilon$. If $0 < |x - a| < \delta$, then

$$|x - a| < \varepsilon.$$

Thus $\lim_{x \to a} x = a$. □

Theorem 3.4 (Uniqueness of limits)**.** *If*

$$\lim_{x \to a} f(x) = L \qquad \textit{and} \qquad \lim_{x \to a} f(x) = M,$$

then $L = M$*.*

Proof. Suppose, for a contradiction, that $L \neq M$. Set

$$\varepsilon = \frac{|L - M|}{3} > 0.$$

There are positive numbers δ_1, δ_2 such that

$$\begin{aligned} 0 < |x - a| < \delta_1 \quad &\Longrightarrow \quad |f(x) - L| < \varepsilon, \\ 0 < |x - a| < \delta_2 \quad &\Longrightarrow \quad |f(x) - M| < \varepsilon. \end{aligned}$$

Choose x with

$$0 < |x - a| < \min\{\delta_1, \delta_2\}.$$

Then, by the triangle inequality,

$$|L - M| \leq |L - f(x)| + |f(x) - M| < 2\varepsilon = \frac{2}{3}|L - M|,$$

a contradiction. Hence $L = M$. □

3.3 Limits from the definition

Theorem 3.5. *For every* $a \in \mathbb{R}$*,*

$$\lim_{x \to a} x^2 = a^2.$$

Proof. Let $\varepsilon > 0$. Since

$$\left|x^2 - a^2\right| = |x - a|\,|x + a|,$$

we need to control the factor $|x + a|$. First require $|x - a| < 1$. Then

$$|x| \leq |x - a| + |a| < |a| + 1,$$

and hence

$$|x + a| \leq |x| + |a| < 2|a| + 1.$$

Choose

$$\delta = \min\left\{1, \frac{\varepsilon}{2|a| + 1}\right\}.$$

If $0 < |x - a| < \delta$, then

$$\left|x^2 - a^2\right| = |x - a|\,|x + a| < \frac{\varepsilon}{2|a| + 1}(2|a| + 1) = \varepsilon.$$

□

Theorem 3.6. *For every $a \neq 0$,*

$$\lim_{x\to a} \frac{1}{x} = \frac{1}{a}.$$

Proof. Let $a \neq 0$, and let $\varepsilon > 0$. We have

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{|a|\,|x|}.$$

Choose $\delta \leq |a|/2$. If $|x-a| < \delta$, then by the reverse triangle inequality,

$$|x| \geq |a| - |x-a| > \frac{|a|}{2}.$$

Therefore,

$$\left|\frac{1}{x} - \frac{1}{a}\right| < \frac{2\,|x-a|}{|a|^2}.$$

Now choose

$$\delta = \min\left\{\frac{|a|}{2}, \frac{\varepsilon\,|a|^2}{2}\right\}.$$

Then $0 < |x-a| < \delta$ implies

$$\left|\frac{1}{x} - \frac{1}{a}\right| < \varepsilon.$$

□

3.4 Algebraic limit laws

Theorem 3.7 (Sum law)**.** *Suppose that f and g are defined near a, and that*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

Then

$$\lim_{x\to a} \big(f(x) + g(x)\big) = L + M.$$

Proof. Let $\varepsilon > 0$. There are positive δ_1 and δ_2 such that

$$\begin{aligned} 0 < |x-a| < \delta_1 \quad &\Longrightarrow \quad |f(x) - L| < \frac{\varepsilon}{2}, \\ 0 < |x-a| < \delta_2 \quad &\Longrightarrow \quad |g(x) - M| < \frac{\varepsilon}{2}. \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Lemma 3.8. *Let g be defined near a, and suppose*

$$\lim_{x\to a} g(x) = M \neq 0.$$

Then there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Longrightarrow \quad |g(x)| \geq \frac{|M|}{2} > 0.$$

Proof. Apply the definition of limit with $\varepsilon = |M|/2$. For sufficiently close x,

$$|g(x) - M| < \frac{|M|}{2}.$$

The reverse triangle inequality gives

$$|g(x)| \geq |M| - |g(x) - M| > \frac{|M|}{2}.$$

□

Theorem 3.9 (Product law)**.** *Suppose that f and g are defined near a, and that*

$$\lim_{x \to a} f(x) = L, \qquad \lim_{x \to a} g(x) = M.$$

Then

$$\lim_{x \to a} f(x)g(x) = LM.$$

Proof. Let $\varepsilon > 0$. We write

$$f(x)g(x) - LM = g(x)(f(x) - L) + L(g(x) - M).$$

Thus

$$|f(x)g(x) - LM| \leq |g(x)|\,|f(x) - L| + |L|\,|g(x) - M|\,.$$

Since $g(x) \to M$, there is $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \quad \Longrightarrow \quad |g(x) - M| < 1.$$

For such x,

$$|g(x)| \leq |M| + 1.$$

Choose $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \quad \Longrightarrow \quad |f(x) - L| < \frac{\varepsilon}{2(|M| + 1)},$$

and choose $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_3 \quad \Longrightarrow \quad |g(x) - M| < \frac{\varepsilon}{2(|L| + 1)}.$$

If $0 < |x - a| < \min\{\delta_1, \delta_2, \delta_3\}$, then

$$\begin{aligned} |f(x)g(x) - LM| &< (|M| + 1)\frac{\varepsilon}{2(|M| + 1)} + |L|\,\frac{\varepsilon}{2(|L| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Theorem 3.10 (Reciprocal law)**.** *Suppose that g is defined near a, and that*

$$\lim_{x \to a} g(x) = M \neq 0.$$

Then $1/g$ is defined near a, and

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}.$$

Proof. By Lemma 3.8, $g(x) \neq 0$ sufficiently near a, so $1/g$ is defined there. Moreover,

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|g(x) - M|}{|M|\,|g(x)|} \leq \frac{2}{|M|^2}\,|g(x) - M|\,.$$

Given $\varepsilon > 0$, choose $\delta_1 > 0$ as in Lemma 3.8, and choose $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \quad \Longrightarrow \quad |g(x) - M| < \frac{\varepsilon\,|M|^2}{2}.$$

Then the desired conclusion holds for $0 < |x - a| < \min\{\delta_1, \delta_2\}$. □

Corollary 3.11 (Quotient law)**.** *If*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M \neq 0,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Proof. Apply Theorem 3.9 to f and $1/g$, then use Theorem 3.10. □

Lemma 3.12 (Order preservation under limits)**.** *Suppose that f and g are defined near a, that*

$$f(x) \leq g(x)$$

for all x sufficiently close to a with $x \neq a$, and that the limits

$$L_f = \lim_{x\to a} f(x), \qquad L_g = \lim_{x\to a} g(x)$$

both exist. Then

$$L_f \leq L_g.$$

Proof. Suppose, for a contradiction, that $L_f > L_g$, and set

$$\varepsilon = \frac{L_f - L_g}{2} > 0.$$

For all x sufficiently close to a with $x \neq a$, we have both $f(x) \leq g(x)$ and

$$|f(x) - L_f| < \varepsilon, \qquad |g(x) - L_g| < \varepsilon.$$

Then

$$g(x) < L_g + \varepsilon = \frac{L_f + L_g}{2} = L_f - \varepsilon < f(x),$$

contradicting $f(x) \leq g(x)$. Hence $L_f \leq L_g$. □

Lemma 3.13 (Limits of nonnegative functions)**.** *Suppose that f is defined near a, that*

$$f(x) \geq 0$$

for all x sufficiently close to a with $x \neq a$, and that

$$L = \lim_{x\to a} f(x)$$

exists. Then

$$L \geq 0.$$

Proof. Apply Lemma 3.12 with $g = 0$: the constant zero function has limit 0 at a, and $f(x) \geq 0 = g(x)$ near a. □

Remark 3.14. Strict inequalities need not survive a limit: for example,

$$|x| > 0 \quad \text{for } x \neq 0, \qquad \text{yet} \qquad \lim_{x \to 0} |x| = 0.$$

Thus $f(x) < g(x)$ near a implies only $\lim f \leq \lim g$, not $\lim f < \lim g$.

Corollary 3.15. *Let p be a polynomial. Then*

$$\lim_{x \to a} p(x) = p(a) \qquad \textit{for every } a \in \mathbb{R}.$$

If $r = p/q$ is a rational function and $q(a) \neq 0$, then

$$\lim_{x \to a} r(x) = r(a).$$

Proof. The first assertion follows by repeated application of the sum and product laws, beginning with Examples 3.2 and 3.3. The second follows from the quotient law. □

3.5 Algebraic simplification and removable discontinuities

Example 3.16. For $x \neq 2$,

$$\frac{2x^2 - 4x}{x^3 - 8} = \frac{2x(x-2)}{(x-2)(x^2 + 2x + 4)} = \frac{2x}{x^2 + 2x + 4}.$$

Therefore,

$$\lim_{x \to 2} \frac{2x^2 - 4x}{x^3 - 8} = \lim_{x \to 2} \frac{2x}{x^2 + 2x + 4} = \frac{4}{12} = \frac{1}{3}.$$

The cancellation is valid only for $x \neq 2$, but this is sufficient for a limit calculation because the point $x = 2$ is excluded in the definition of $\lim_{x \to 2}$.

3.6 Nonexistence of limits

The negation of

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\big(0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon\big)$$

is

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x)\big(0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon\big).$$

Thus, to prove that f does not have limit L at a, it is enough to produce one positive output tolerance ε such that every deleted neighbourhood of a contains an input at which the function remains at least ε away from L.

Example 3.17. Let

$$f(x) = \frac{x}{|x|}, \qquad x \neq 0.$$

Then $\lim_{x \to 0} f(x)$ does not exist.

Proof. Let $L \in \mathbb{R}$ be arbitrary. If $L \geq 0$, take $\varepsilon = 1/2$. Given any $\delta > 0$, choose $x = -\delta/2$. Then

$$0 < |x| < \delta, \qquad f(x) = -1, \qquad |f(x) - L| = |-1 - L| \geq 1 > \varepsilon.$$

If $L < 0$, again take $\varepsilon = 1/2$, and choose $x = \delta/2$. Then

$$0 < |x| < \delta, \qquad f(x) = 1, \qquad |f(x) - L| = |1 - L| > 1 > \varepsilon.$$

Thus no real number L is the limit of f at 0. □

Example 3.18. The limit

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

does not exist.

Proof. Suppose that $L \in \mathbb{R}$ is proposed as a limit. At least one of $|L|$ and $|1 - L|$ is at least $1/2$.

If $|L| \geq 1/2$, take $\varepsilon = 1/4$. Given $\delta > 0$, choose $n \in \mathbb{N}$ sufficiently large that $1/(n\pi) < \delta$, and set

$$x = \frac{1}{n\pi}.$$

Then $0 < |x| < \delta$ and

$$\sin\left(\frac{1}{x}\right) = \sin(n\pi) = 0.$$

Hence

$$\left|\sin\left(\frac{1}{x}\right) - L\right| = |L| \geq \frac{1}{2} > \varepsilon.$$

If $|1 - L| \geq 1/2$, take $\varepsilon = 1/4$. Given $\delta > 0$, choose $n \in \mathbb{N}$ sufficiently large that

$$\frac{1}{2n\pi + \pi/2} < \delta,$$

and set

$$x = \frac{1}{2n\pi + \pi/2}.$$

Then $0 < |x| < \delta$ and

$$\sin\left(\frac{1}{x}\right) = 1.$$

Therefore

$$\left|\sin\left(\frac{1}{x}\right) - L\right| = |1 - L| \geq \frac{1}{2} > \varepsilon.$$

Thus the limit does not exist. □

3.7 The squeeze theorem

Theorem 3.19 (Squeeze theorem)**.** *Let f, g, and h be defined near a. Suppose that*

$$f(x) \leq g(x) \leq h(x)$$

for all x sufficiently close to a, with $x \neq a$. If

$$\lim_{x \to a} f(x) = L \qquad \text{and} \qquad \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$

Proof. Let $\varepsilon > 0$. For all sufficiently close $x \neq a$,

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon.$$

Thus $|g(x) - L| < \varepsilon$, as required. □

Example 3.20. Define

$$f(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x \to 0} f(x) = 0.$$

Proof. For $x \neq 0$,

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|.$$

Since both $-|x|$ and $|x|$ tend to 0 as $x \to 0$, the conclusion follows from Theorem 3.19. □

3.8 One-sided limits

Definition 3.21. Let f be defined on an interval $(a, a + r)$. We write

$$\lim_{x \to a^+} f(x) = L$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < x - a < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

Similarly, if f is defined on $(a - r, a)$, we write

$$\lim_{x \to a^-} f(x) = L$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$-\delta < x - a < 0 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

In both cases δ is taken to be at most the radius r of the interval on which f is defined, so that the condition is quantified only over points of the domain of f.

Theorem 3.22. *Suppose that f is defined on a deleted neighbourhood of a. Then*

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L.$$

Proof. If the two-sided limit exists, each one-sided limit follows immediately from the definition.

Conversely, let $\varepsilon > 0$. There are positive numbers δ_- and δ_+ such that

$$\begin{aligned} -\delta_- < x - a < 0 \quad &\Longrightarrow \quad |f(x) - L| < \varepsilon, \\ 0 < x - a < \delta_+ \quad &\Longrightarrow \quad |f(x) - L| < \varepsilon. \end{aligned}$$

Set $\delta = \min\{\delta_-, \delta_+\}$. If $0 < |x - a| < \delta$, then either $-\delta < x - a < 0$ or $0 < x - a < \delta$, so $|f(x) - L| < \varepsilon$. □

3.9 Infinite limits

Definition 3.23. Let f be defined near a. We write

$$\lim_{x\to a} f(x) = +\infty$$

if, for every $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) > M.$$

Similarly,

$$\lim_{x\to a} f(x) = -\infty$$

means that for every $M < 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) < M.$$

Example 3.24.

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty.$$

Proof. Let $M > 0$. Choose $\delta = 1/\sqrt{M}$. Then

$$0 < |x| < \delta \quad \Longrightarrow \quad x^2 < \frac{1}{M} \quad \Longrightarrow \quad \frac{1}{x^2} > M.$$

□

Definition 3.25. The line $x = a$ is a *vertical asymptote* of f if at least one of the one-sided limits

$$\lim_{x\to a^-} f(x), \qquad \lim_{x\to a^+} f(x)$$

is $+\infty$ or $-\infty$.

Example 3.26. The function $f(x) = 1/x^2$ has the vertical asymptote $x = 0$, because

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$

by Example 3.24.

3.10 Limits at infinity

Definition 3.27. Suppose f is defined on (A, ∞) for some $A \in \mathbb{R}$. We write

$$\lim_{x\to\infty} f(x) = L$$

if, for every $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that

$$x > M \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

The definitions of limits as $x \to -\infty$, and of infinite limits at infinity, are analogous.

Example 3.28.

$$\lim_{x\to\infty} \frac{x}{x+1} = 1.$$

Proof. Let $\varepsilon > 0$. Choose $M > \max\{1, 1/\varepsilon\}$. If $x > M$, then

$$\left|\frac{x}{x+1} - 1\right| = \frac{1}{x+1} < \frac{1}{x} < \varepsilon.$$

□

Theorem 3.29 (Rational functions at infinity)**.** *Let*

$$r(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$. Then:

(i) *if $n < m$, then $\lim_{x\to\infty} r(x) = 0$;*

(ii) *if $n = m$, then $\lim_{x\to\infty} r(x) = a_n/b_m$;*

(iii) *if $n > m$, then $r(x)$ has no finite limit as $x \to \infty$.*

Proof. For $x \neq 0$, factor x^n from the numerator and x^m from the denominator to write

$$r(x) = x^{n-m}\,\frac{a_n + a_{n-1}/x + \cdots + a_0/x^n}{b_m + b_{m-1}/x + \cdots + b_0/x^m}.$$

Since $x^{-k} \to 0$ as $x \to \infty$ for every positive integer k, the quotient tends to a_n/b_m. The algebraic limit laws carry over verbatim to limits at infinity (with $x > M$ in place of $0 < |x - a| < \delta$); together with the fact that the quotient tends to a_n/b_m, they give *(i)* and *(ii)*: when $n < m$ the factor $x^{n-m} \to 0$, and when $n = m$ it equals 1.

For *(iii)*, suppose $n > m$, so that $n - m \geq 1$ and $a_n/b_m \neq 0$. Because the quotient tends to $a_n/b_m \neq 0$, its absolute value is eventually bounded below by a positive constant: there are $c > 0$ and $A \in \mathbb{R}$ such that

$$\left|\frac{a_n + a_{n-1}/x + \cdots + a_0/x^n}{b_m + b_{m-1}/x + \cdots + b_0/x^m}\right| \geq c \qquad \text{for all } x > A.$$

Therefore, for $x > A$,

$$|r(x)| \geq c\,x^{n-m}.$$

Since $n - m \geq 1$, the right-hand side tends to $+\infty$ as $x \to \infty$. Thus $|r(x)| \to \infty$, so $r(x)$ has no finite limit as $x \to \infty$. □

Definition 3.30. The line $y = L$ is a *horizontal asymptote* of f if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

Example 3.31. The function

$$f(x) = \frac{2x+1}{x-1}$$

has the horizontal asymptote $y = 2$, since by Theorem 3.29 (with $n = m = 1$),

$$\lim_{x\to\infty} \frac{2x+1}{x-1} = \frac{2}{1} = 2.$$

3.11 Exercises

Exercise 3.32. Prove directly from the definition that

$$\lim_{x\to a}(3x-2) = 3a-2.$$

Exercise 3.33. Prove Theorem 3.9 in the special case $L = 0$.

Exercise 3.34. Compute the following limits, if they exist:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}, \qquad \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}, \qquad \lim_{x\to 0}\frac{|x|}{x}.$$

Exercise 3.35. Show that

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0,$$

assuming the standard trigonometric estimate $|1-\cos x| \le x^2/2$.

Exercise 3.36. Prove that

$$\lim_{x\to 0}\frac{1}{x}$$

does not exist. Determine its one-sided infinite limits.

Exercise 3.37. Determine the horizontal and vertical asymptotes of

$$f(x) = \frac{2x^2-3x+1}{x^2-4}.$$

Chapter 4

Continuity

Continuity expresses the principle that sufficiently small changes in the input produce sufficiently small changes in the output. The main results of this chapter, the Intermediate Value Theorem and the Extreme Value Theorem, depend essentially on the completeness of $\mathbb{R}$.

4.1 Continuity at a point

Definition 4.1. Let f be defined at and near a. The function f is *continuous at* a if

$$\lim_{x \to a} f(x) = f(a).$$

Equivalently, f is continuous at a if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon.$$

The condition $x \neq a$ is not required: when $x = a$, the conclusion is automatic.

Example 4.2. Every constant function is continuous at every point of its domain.

Proof. If $f(x) = c$, then $f(a) = c$ and

$$|f(x) - f(a)| = 0 < \varepsilon$$

for every x and every $\varepsilon > 0$. □

Example 4.3. The identity function $f(x) = x$ is continuous at every $a \in \mathbb{R}$.

Proof. Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then

$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| = |x - a| < \varepsilon.$$

□

Example 4.4 (A removable discontinuity)**.** Define

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1, \\ 0, & x = 1. \end{cases}$$

Then $\lim_{x \to 1} f(x) = 2$, whereas $f(1) = 0$. Thus f is not continuous at 1.

Example 4.5 (A jump discontinuity)**.** Define

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Then

$$\lim_{x \to 0^-} f(x) = -1, \qquad \lim_{x \to 0^+} f(x) = 1.$$

Therefore f is not continuous at 0.

4.2 Continuous functions on intervals

Definition 4.6. Let I be an interval.

- If I is open, a function $f : I \to \mathbb{R}$ is *continuous on* I if it is continuous at every point of I.
- If $I = [a, b]$, a function $f : [a, b] \to \mathbb{R}$ is *continuous on* $[a, b]$ if it is continuous at every point of (a, b), right-continuous at a, and left-continuous at b.

Definition 4.7. A function f is *right-continuous at* a if

$$\lim_{x \to a^+} f(x) = f(a),$$

and *left-continuous at* a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Theorem 4.8. *Suppose that f is defined at and near a. Then f is continuous at a if and only if it is both left-continuous and right-continuous at a.*

Proof. By definition, f is continuous at a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

By Theorem 3.22, this holds if and only if both one-sided limits exist and equal $f(a)$. □

4.3 Operations preserving continuity

Theorem 4.9. *Suppose that f and g are continuous at a. Then the functions $f + g$, $f - g$, fg, and cf, for $c \in \mathbb{R}$, are continuous at a. If $g(a) \neq 0$, then f/g is continuous at a.*

Proof. Since f and g are continuous at a,

$$\lim_{x \to a} f(x) = f(a), \qquad \lim_{x \to a} g(x) = g(a).$$

The conclusion follows from the sum, product, reciprocal, and quotient laws for limits; see Theorems 3.7, 3.9 and 3.10 and Corollary 3.11. □

Corollary 4.10. *Every polynomial is continuous on $\mathbb{R}$. Every rational function is continuous at every point in its natural domain.*

Proof. Apply Corollary 3.15. □

Theorem 4.11 (Continuity of compositions)**.** *Suppose that g is continuous at a and that f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

Proof. Let $\varepsilon > 0$. Since f is continuous at $g(a)$, there exists $\eta > 0$ such that

$$|y - g(a)| < \eta \quad \Longrightarrow \quad |f(y) - f(g(a))| < \varepsilon.$$

Since g is continuous at a, there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Longrightarrow \quad |g(x) - g(a)| < \eta.$$

Combining the implications gives

$$|x - a| < \delta \quad \Longrightarrow \quad |f(g(x)) - f(g(a))| < \varepsilon.$$

Thus $f \circ g$ is continuous at a. □

Lemma 4.12 (Sign preservation)**.** *Let f be continuous at a.*

(i) *If $f(a) > 0$, then there exists $\delta > 0$ such that*

$$|x - a| < \delta \quad \Longrightarrow \quad f(x) > \frac{f(a)}{2} > 0.$$

(ii) *If $f(a) < 0$, then there exists $\delta > 0$ such that*

$$|x - a| < \delta \quad \Longrightarrow \quad f(x) < \frac{f(a)}{2} < 0.$$

Proof. Suppose $f(a) > 0$. Apply continuity at a with $\varepsilon = f(a)/2$. Then, for sufficiently small $|x - a|$,

$$|f(x) - f(a)| < \frac{f(a)}{2}.$$

In particular,

$$f(x) > f(a) - \frac{f(a)}{2} = \frac{f(a)}{2} > 0.$$

The second assertion follows by applying the first assertion to $-f$. □

4.4 Pathological examples

The following examples show that continuity cannot be understood adequately from a naive graphical criterion. They will also be useful later in the study of differentiability and integration.

4.4.1 The floor function

Definition 4.13. For $x \in \mathbb{R}$, the *floor* of x, denoted by $\lfloor x \rfloor$, is the largest integer less than or equal to x.

Proposition 4.14. *The floor function is continuous at every non-integer and discontinuous at every integer.*

Proof. Let $a \notin \mathbb{Z}$, and let $n = \lfloor a \rfloor$. Then

$$n < a < n + 1.$$

Choose

$$\delta = \min\{a - n, n + 1 - a\} > 0.$$

If $|x - a| < \delta$, then $n < x < n + 1$, so

$$\lfloor x \rfloor = n = \lfloor a \rfloor.$$

Thus the floor function is continuous at a.

Now let $n \in \mathbb{Z}$. Then

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1, \qquad \lim_{x \to n^+} \lfloor x \rfloor = n.$$

The one-sided limits differ, so the floor function is discontinuous at n. □

4.4.2 Dirichlet's function

Definition 4.15. The *Dirichlet function* is the function $d : \mathbb{R} \to \mathbb{R}$ defined by

$$d(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Theorem 4.16. *Dirichlet's function is discontinuous at every point of* $\mathbb{R}$.

Proof. Fix $a \in \mathbb{R}$, and let $L \in \mathbb{R}$ be arbitrary. We show that $d(x) \not\to L$ as $x \to a$. At least one of the two numbers $|L|$ and $|1 - L|$ is at least $1/2$: if both were $< 1/2$, then

$$1 = |L + (1 - L)| \leq |L| + |1 - L| < 1,$$

which is impossible.

Suppose first that $|L| \geq 1/2$. Given any $\delta > 0$, density of the irrationals, Theorem 1.33, applied to the interval $(a, a + \delta)$ supplies an irrational number u with $a < u < a + \delta$, so that $0 < |u - a| < \delta$. Since $d(u) = 0$,

$$|d(u) - L| = |L| \geq \frac{1}{2}.$$

Now suppose instead that $|L| < 1/2$. Then $L < 1/2$, so $1 - L > 1/2$, and hence $|1 - L| > 1/2$. Given any $\delta > 0$, density of the rationals, Theorem 1.32, applied to the interval $(a, a + \delta)$ supplies a rational number q with $a < q < a + \delta$, so that $0 < |q - a| < \delta$. Since $d(q) = 1$,

$$|d(q) - L| = |1 - L| > \frac{1}{2}.$$

Thus, in every case, every deleted neighbourhood of a contains an input x with

$$|d(x) - L| \geq \frac{1}{2}.$$

With $\varepsilon = 1/2$ in the definition of limit, no $\delta > 0$ can satisfy the required condition, so $d(x)$ does not tend to L as $x \to a$.

Since L was arbitrary, d has no limit at a. In particular, d is not continuous at a. Since a was arbitrary, the function is discontinuous everywhere. □

4.4.3 The Stars over Babylon function

Definition 4.17 (Stars over Babylon function)**.** Define $t : (0,1) \to \mathbb{R}$ by

$$t(x) = \begin{cases} \dfrac{1}{q}, & \text{if } x = \dfrac{p}{q} \in \mathbb{Q} \text{ is expressed in lowest terms,} \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Theorem 4.18. *The function in Definition 4.17 is continuous at every irrational number in* $(0,1)$ *and discontinuous at every rational number in* $(0,1)$*.*

Proof. Let $a \in (0,1)$ be irrational. Since $t(a) = 0$, we must prove that $t(x) \to 0$ as $x \to a$. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon.$$

There are only finitely many rational numbers in $(0,1)$, expressed in lowest terms, whose denominator is at most N. Let this finite set be F. Since $a \notin F$, there exists $\delta > 0$ such that

$$(a - \delta, a + \delta) \cap F = \varnothing.$$

If $0 < |x - a| < \delta$, then either x is irrational, in which case $t(x) = 0$, or $x = p/q$ in lowest terms with $q > N$, in which case

$$0 < t(x) = \frac{1}{q} < \frac{1}{N} < \varepsilon.$$

Hence $|t(x) - t(a)| < \varepsilon$, proving continuity at a.

Now let $a = p/q \in (0,1)$ be rational in lowest terms. Then

$$t(a) = \frac{1}{q} > 0.$$

Every neighbourhood of a contains irrational numbers, at which t takes the value 0. Therefore $t(x)$ cannot tend to $t(a)$ as $x \to a$. Thus t is discontinuous at a. □

Remark 4.19. The Stars over Babylon function is also called Thomae's function or the popcorn function. It is continuous at every irrational point but discontinuous at every rational point. Thus continuity at a point does not imply that a function has an intuitively smooth-looking graph near that point.

4.4.4 An oscillatory continuous function

Example 4.20. Define

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then g is continuous at 0.

Proof. For $x \neq 0$,

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|.$$

Since both bounding functions tend to 0 as $x \to 0$, Theorem 3.19 gives

$$\lim_{x \to 0} g(x) = 0 = g(0).$$

□

4.5 Splicing continuous functions

Proposition 4.21 (Splicing lemma)**.** *Let f and g be functions defined on an interval containing c, and suppose that f and g are continuous at c and satisfy $f(c) = g(c)$. Define*

$$h(x) = \begin{cases} f(x), & x \le c, \\ g(x), & x > c. \end{cases}$$

Then h is continuous at c.

Proof. On the left of c, the function h agrees with f, so

$$\lim_{x \to c^-} h(x) = \lim_{x \to c^-} f(x) = f(c) = h(c).$$

On the right of c, the function h agrees with g, so

$$\lim_{x \to c^+} h(x) = \lim_{x \to c^+} g(x) = g(c) = h(c).$$

Thus h is both left- and right-continuous at c. By Theorem 4.8, h is continuous at c. □

4.6 The Intermediate Value Theorem

The Intermediate Value Theorem requires completeness. If we worked only with $\mathbb{Q}$, the function $x \mapsto x^2 - 2$ would be continuous on $[0, 2] \cap \mathbb{Q}$ and would take both negative and positive values, but it would have no zero in $\mathbb{Q}$.

Theorem 4.22 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If*

$$f(a) < 0 < f(b),$$

then there exists $c \in (a, b)$ such that

$$f(c) = 0.$$

Proof. Define

$$S = \{x \in [a, b] : f(t) < 0 \text{ for every } t \in [a, x]\}.$$

The set S is nonempty because $a \in S$, and it is bounded above by b. Let

$$c = \sup S.$$

We show first that $a < c < b$. Since $f(a) < 0$, Lemma 4.12 gives $\delta > 0$ such that $f(x) < 0$ for all $x \in [a, a + \delta)$. Hence $a + \delta/2 \in S$, so $c > a$.

Since $f(b) > 0$, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (b - \delta, b]$. Then $b - \delta/2$ is an upper bound for S, so $c \le b - \delta/2 < b$.

Suppose that $f(c) < 0$. By Lemma 4.12, there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in (c - \delta, c + \delta)$. If $t < c$, then $f(t) < 0$; otherwise, some such t with $f(t) \ge 0$ would make t an upper bound for S smaller than c. Therefore f is negative on $[a, c + \delta/2]$, so $c + \delta/2 \in S$, contradicting that c is an upper bound for S.

Suppose that $f(c) > 0$. Then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c-\delta, c+\delta)$. Consequently, $c - \delta/2$ is an upper bound for S, contradicting that c is the least upper bound of S.

Neither $f(c) < 0$ nor $f(c) > 0$ is possible. By trichotomy, $f(c) = 0$. □

Corollary 4.23 (General Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If y lies between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that*

$$f(c) = y.$$

Proof. If $y = f(a)$ or $y = f(b)$, then $c = a$ or $c = b$ already satisfies $f(c) = y$, and there is nothing to prove. Otherwise y lies strictly between $f(a)$ and $f(b)$. If $f(a) < y < f(b)$, then the continuous function $f - y$ satisfies $(f - y)(a) < 0 < (f - y)(b)$, so Theorem 4.22 gives $c \in (a, b)$ with $(f - y)(c) = 0$, that is, $f(c) = y$. If instead $f(b) < y < f(a)$, apply the same argument to $y - f$ (equivalently, to $-f$), since $(y - f)(a) < 0 < (y - f)(b)$. □

Theorem 4.24 (Existence and uniqueness of roots)**.** *Let $n \in \mathbb{N}$, and let $A > 0$. There exists a unique positive real number x such that*

$$x^n = A.$$

Proof. Consider

$$f(x) = x^n - A$$

on $[0, A + 1]$. The function f is continuous. Moreover,

$$f(0) = -A < 0,$$

while

$$f(A + 1) = (A + 1)^n - A > 0.$$

By Theorem 4.22, there exists $x \in (0, A + 1)$ with $x^n = A$.

For uniqueness, suppose $0 < x < y$. Then

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + \cdots + x^{n-1}) > 0.$$

Thus $x^n < y^n$, so two distinct positive numbers cannot have the same n-th power. □

Corollary 4.25. *Every polynomial of odd degree has at least one real root.*

Proof. Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

where n is odd and $a_n \neq 0$. For sufficiently large positive x, the leading term $a_n x^n$ determines the sign of $p(x)$. Thus p takes opposite signs at two sufficiently large real numbers of opposite sign. The conclusion follows from Theorem 4.22. □

Theorem 4.26. *If $f : [0, 1] \to [0, 1]$ is continuous, then there exists $c \in [0, 1]$ such that*

$$f(c) = c.$$

Proof. Define $g(x) = f(x) - x$, which is continuous on $[0, 1]$. Since f takes values in $[0, 1]$,

$$g(0) = f(0) \geq 0 \qquad \text{and} \qquad g(1) = f(1) - 1 \leq 0.$$

If $g(0) = 0$, then $f(0) = 0$, so $c = 0$ is a fixed point. If $g(1) = 0$, then $f(1) = 1$, so $c = 1$ is a fixed point. Otherwise $g(0) > 0 > g(1)$, so the continuous function $-g$ satisfies $(-g)(0) < 0 < (-g)(1)$. By Theorem 4.22, there exists $c \in (0, 1)$ with $(-g)(c) = 0$, that is, $g(c) = 0$, i.e. $f(c) = c$. □

Remark 4.27. The conclusion fails on open intervals: the function $f(x) = x^2$ maps $(0, 1)$ into $(0, 1)$, but its fixed points are only 0 and 1, neither of which lies in $(0, 1)$.

4.7 The Extreme Value Theorem

Definition 4.28. Let f be defined on $S \subseteq \mathbb{R}$. The function f *attains its maximum on* S if there exists $x_0 \in S$ such that

$$f(x) \leq f(x_0) \qquad \text{for every } x \in S.$$

It *attains its minimum on* S if there exists $x_1 \in S$ such that

$$f(x_1) \leq f(x) \qquad \text{for every } x \in S.$$

Theorem 4.29 (Extreme Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded on $[a, b]$ and attains both a maximum and a minimum on $[a, b]$.*

Proof. We first prove that f is bounded above. Define

$$S = \{x \in [a, b] : f \text{ is bounded above on } [a, x]\}.$$

The set S is nonempty: continuity at a provides $\delta > 0$ such that f is bounded on $[a, a+\delta] \cap [a, b]$. The set S is bounded above by b, so let $c = \sup S$.

Suppose $c < b$. Since $c \in [a, b]$, continuity at c gives $\delta > 0$ such that f is bounded on $(c - \delta, c + \delta) \cap [a, b]$; shrinking δ if necessary, we may assume that $c + \delta < b$. Since $c = \sup S$, there is some $x \in S$ with $c - \delta/2 < x \leq c$. The function is bounded above on $[a, x]$, and also on $[x, c + \delta/2]$, because $[x, c + \delta/2] \subseteq (c - \delta, c + \delta)$. Hence it is bounded above on $[a, c + \delta/2]$. This says $c + \delta/2 \in S$, contradicting that c is an upper bound for S. Therefore $c = b$.

We still have to show that $b \in S$; the equality $\sup S = b$ alone does not give this. Since f is left-continuous at b, there is $\eta > 0$ such that f is bounded on $(b - \eta, b]$. Because $b = \sup S$, there exists $x \in S$ with $b - \eta < x \leq b$. Then f is bounded above on $[a, x]$ (as $x \in S$) and on $[x, b]$ (as $[x, b] \subseteq (b - \eta, b]$). Therefore f is bounded above on $[a, b] = [a, x] \cup [x, b]$. Hence $b \in S$, and f is bounded above on $[a, b]$.

Applying the same argument to $-f$ proves that f is bounded below.

Let

$$M = \sup \{f(x) : x \in [a, b]\}.$$

Suppose no $x \in [a, b]$ satisfies $f(x) = M$. Then

$$g(x) = \frac{1}{M - f(x)}$$

is a continuous function on $[a, b]$. By the boundedness result already proved, there is $K > 0$ with $g(x) \leq K$ for every $x \in [a, b]$. Hence

$$M - f(x) \geq \frac{1}{K},$$

or

$$f(x) \leq M - \frac{1}{K} \qquad \text{for every } x \in [a, b].$$

This contradicts the definition of M as the least upper bound of the range of f. Therefore f attains its maximum. Applying the same argument to $-f$ proves that f attains its minimum. □

Example 4.30. The hypothesis that the interval be closed cannot be removed. The function $f(x) = x$ is continuous on $(0, 1)$, but has neither a maximum nor a minimum on $(0, 1)$.

4.8 Uniform continuity

Definition 4.31. Let $f : I \to \mathbb{R}$, where I is an interval. The function f is *uniformly continuous on I* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in I$,

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Theorem 4.32 (Heine–Cantor Theorem)**.** *Every continuous function on a closed interval $[a, b]$ is uniformly continuous on $[a, b]$.*

Proof. Fix $\varepsilon > 0$. Let

$$S = \{x \in [a, b] : f \text{ is uniformly } \varepsilon\text{-continuous on } [a, x]\},$$

meaning that there exists $\delta > 0$ such that for all $u, v \in [a, x]$,

$$|u - v| < \delta \quad \Longrightarrow \quad |f(u) - f(v)| < \varepsilon.$$

The set S is nonempty, since the assertion is trivially true on $[a, a]$. It is bounded above by b, so let $c = \sup S$.

Suppose $c < b$. Since $c \in [a, b]$, continuity at c gives $\eta > 0$ such that

$$|t - c| < \eta \quad \Longrightarrow \quad |f(t) - f(c)| < \frac{\varepsilon}{2} \qquad (t \in [a, b]).$$

Shrinking η if necessary, we may assume that $c + \eta < b$. Consequently, whenever $u, v \in (c - \eta, c + \eta) \cap [a, b]$,

$$|f(u) - f(v)| \le |f(u) - f(c)| + |f(v) - f(c)| < \varepsilon.$$

Since $c = \sup S$, there is $x \in S$ with $c - \eta/2 < x \le c$. Because $x \in S$, there exists $\delta_{\text{old}} > 0$ such that, for all $u, v \in [a, x]$,

$$|u - v| < \delta_{\text{old}} \quad \Longrightarrow \quad |f(u) - f(v)| < \varepsilon.$$

Set

$$\delta = \min\{\delta_{\text{old}},\, \eta/2\}.$$

We claim that this δ witnesses uniform ε-continuity of f on $[a, c + \eta/2]$. Let $u, v \in [a, c + \eta/2]$ with $|u - v| < \delta$.

If $u, v \le x$, then $u, v \in [a, x]$ and $|u - v| < \delta \le \delta_{\text{old}}$, so $|f(u) - f(v)| < \varepsilon$.

If $u, v \ge x$, then $u, v \in [x, c + \eta/2]$. Since $x > c - \eta/2 > c - \eta$ and $c + \eta/2 < c + \eta$, we have $[x, c + \eta/2] \subseteq (c - \eta, c + \eta)$, so the local estimate above gives $|f(u) - f(v)| < \varepsilon$.

It remains to treat the case in which u and v straddle x; suppose $u < x < v$. Then $x - u \le |u - v| < \delta \le \eta/2$ and $v - x < \delta \le \eta/2$, so

$$u > x - \eta/2 > c - \eta \qquad \text{and} \qquad v < x + \eta/2 \le c + \eta/2 < c + \eta.$$

Thus $u, v \in (c - \eta, c + \eta) \cap [a, b]$, and again the local estimate gives $|f(u) - f(v)| < \varepsilon$.

Hence f is uniformly ε-continuous on $[a, c + \eta/2]$, i.e. $c + \eta/2 \in S$, contradicting that c is an upper bound for S. Therefore $c = b$.

It remains to show that $b \in S$; the equality $\sup S = b$ alone does not give this. Since f is left-continuous at b, there is $\eta > 0$ such that

$$|t - b| < \eta \quad \Longrightarrow \quad |f(t) - f(b)| < \frac{\varepsilon}{2} \qquad (t \in [a, b]).$$

Thus, whenever $u, v \in (b-\eta, b] \cap [a, b]$,

$$|f(u) - f(v)| \le |f(u) - f(b)| + |f(v) - f(b)| < \varepsilon.$$

Since $b = \sup S$, there is $x \in S$ with $b - \eta/2 < x \le b$. Let $\delta_{\text{old}} > 0$ witness uniform ε-continuity of f on $[a, x]$, and set

$$\delta = \min\{\delta_{\text{old}},\, \eta/2\}.$$

Let $u, v \in [a, b]$ with $|u - v| < \delta$. If $u, v \le x$, then $|u - v| < \delta_{\text{old}}$ gives $|f(u) - f(v)| < \varepsilon$. If $u, v \ge x$, then $u, v \in [x, b] \subseteq (b - \eta, b]$, so the estimate above gives $|f(u) - f(v)| < \varepsilon$. If $u < x < v$, then

$$u > x - \eta/2 > b - \eta \qquad \text{and} \qquad v \le b,$$

so again $u, v \in (b - \eta, b] \cap [a, b]$, and $|f(u) - f(v)| < \varepsilon$.

Thus f is uniformly ε-continuous on $[a, b]$, i.e. $b \in S$. Since $\varepsilon > 0$ was arbitrary, f is uniformly continuous on $[a, b]$. □

4.9 Exercises

Exercise 4.33. Determine all points of continuity of

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

Can the function be redefined at one point to become continuous on $\mathbb{R}$?

Exercise 4.34. Prove directly from the ε–δ definition that $f(x) = |x|$ is continuous at every $a \in \mathbb{R}$.

Exercise 4.35. Use Theorem 4.22 to prove that

$$x^3 + x - 1 = 0$$

has a real solution.

Exercise 4.36. Give an example of a function that is continuous and unbounded on $(0, 1)$.

Exercise 4.37. Prove that the function

$$f(x) = \begin{cases} x^2, & x \le 1, \\ 2x - 1, & x > 1 \end{cases}$$

is continuous on $\mathbb{R}$.

Exercise 4.38. Show that the Stars over Babylon function is bounded on $(0, 1)$, but does not attain a positive minimum there.

Part II

Differential Calculus

Chapter 5

The Derivative

The derivative measures instantaneous rate of change. It arises naturally in problems concerning velocity and tangent lines, and it provides the principal local tool of differential calculus.

5.1 Motivation: velocity and tangent lines

Suppose that a particle moves along a line and that its position at time t is $s(t)$. The average velocity over the interval from $t = a$ to $t = b$, where $a \neq b$, is

$$\frac{s(b) - s(a)}{b - a}.$$

To describe the velocity at a single instant a, consider the average velocity over the shorter interval from a to $a + h$:

$$\frac{s(a + h) - s(a)}{h}.$$

If this expression has a limit as $h \to 0$, that limit is the instantaneous velocity at time a.

The same expression appears in geometry. Let f be a function, and consider the graph of f. The line through the two graph points

$$(a, f(a)) \qquad \text{and} \qquad (a + h, f(a + h))$$

is a secant line. Its slope is

$$\frac{f(a + h) - f(a)}{h}.$$

If these secant slopes approach a limit as $h \to 0$, the limit is the slope of the tangent line to the graph at $(a, f(a))$.

5.2 Definition of the derivative

Definition 5.1. Let f be defined at and near a. The function f is *differentiable at* a if the limit

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

exists as a finite real number. In this case, the *derivative of* f *at* a is

$$f'(a) := \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

If f is differentiable at every point of a set D, the *derivative function* f' is the function with domain

$$\mathrm{dom}(f') = \{a \in D : f \text{ is differentiable at } a\}$$

and values $f'(a)$.

The notation

$$f'(a), \qquad \frac{d}{dx}f(x)\Big|_{x=a}, \qquad \frac{df}{dx}(a)$$

is commonly used for the derivative at a. We use prime notation whenever possible.

Remark 5.2. The quotient

$$\frac{f(a+h)-f(a)}{h}$$

is not defined at $h = 0$. This does not prevent its limit as $h \to 0$ from existing. In derivative calculations, algebraic simplification is performed only for $h \neq 0$, which is sufficient for evaluating the limit.

5.3 Interpretations of the derivative

If $s(t)$ is the position of a particle at time t, then $s'(a)$ is its instantaneous velocity at time a.

If $f'(a)$ exists, then the tangent line to the graph of f at $(a, f(a))$ has slope $f'(a)$. Its equation is

$$y - f(a) = f'(a)(x-a),$$

or equivalently,

$$y = f(a) + f'(a)(x-a).$$

Definition 5.3. If f is differentiable at a, the *linearization* of f at a is the linear function

$$L_{f,a}(x) = f(a) + f'(a)(x-a).$$

The linearization agrees with the function and its first derivative at a:

$$L_{f,a}(a) = f(a), \qquad L'_{f,a}(a) = f'(a).$$

It is often a useful approximation to $f(x)$ when x is close to a.

5.4 Derivatives from the definition

Theorem 5.4. *If $f(x) = c$ is constant, then $f'(a) = 0$ for every $a \in \mathbb{R}$.*

Proof. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{c-c}{h} = 0.$$

Therefore

$$f'(a) = \lim_{h\to 0} 0 = 0.$$

□

Theorem 5.5. *Let $f(x) = mx + b$, where $m, b \in \mathbb{R}$. Then $f'(a) = m$ for every $a \in \mathbb{R}$.*

Proof. For $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{m(a+h)+b-(ma+b)}{h}$$
$$= \frac{mh}{h} = m.$$

Taking the limit as $h \to 0$ gives $f'(a) = m$. □

Theorem 5.6 (Power rule for nonnegative integer powers)**.** *Let $n \in \mathbb{N}$. If $f(x) = x^n$, then*

$$f'(a) = na^{n-1} \qquad \text{for every } a \in \mathbb{R}.$$

Proof. For $h \neq 0$, the binomial theorem gives

$$\frac{(a+h)^n - a^n}{h} = \frac{a^n + na^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots + h^n - a^n}{h}$$
$$= na^{n-1} + \binom{n}{2}a^{n-2}h + \cdots + h^{n-1}.$$

As $h \to 0$, all terms except the first tend to 0. Therefore

$$\lim_{h\to 0} \frac{(a+h)^n - a^n}{h} = na^{n-1}.$$

□

Example 5.7. Let $f(x) = x^{-2} = 1/x^2$, with domain $\mathbb{R} \setminus \{0\}$. Then

$$f'(a) = -\frac{2}{a^3} \qquad \text{for every } a \neq 0.$$

Proof. For $a \neq 0$ and sufficiently small $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = \frac{1/(a+h)^2 - 1/a^2}{h}$$
$$= \frac{a^2-(a+h)^2}{ha^2(a+h)^2}$$
$$= -\frac{2a+h}{a^2(a+h)^2}.$$

Taking the limit as $h \to 0$ gives

$$f'(a) = -\frac{2a}{a^4} = -\frac{2}{a^3}.$$

□

5.5 One-sided derivatives

Definition 5.8. Let f be defined at and to the right of a. The *right-hand derivative* of f at a, if it exists, is

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}.$$

Similarly, if f is defined at and to the left of a, the *left-hand derivative* of f at a, if it exists, is

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}.$$

Proposition 5.9. *A function f is differentiable at a if and only if both one-sided derivatives exist and satisfy*

$$f'_-(a) = f'_+(a).$$

Proof. Apply Theorem 3.22 to the difference quotient

$$h \longmapsto \frac{f(a+h) - f(a)}{h}.$$

□

Example 5.10 (The absolute-value function)**.** The function $f(x) = |x|$ is differentiable at every $a \neq 0$, with

$$f'(a) = \begin{cases} 1, & a > 0, \\ -1, & a < 0. \end{cases}$$

It is not differentiable at 0.

Proof. If $a > 0$, then $|a + h| = a + h$ for all sufficiently small h, so

$$\frac{|a+h| - |a|}{h} = 1$$

near $h = 0$. Thus $f'(a) = 1$. The case $a < 0$ is similar.

At $a = 0$,

$$\frac{|h| - |0|}{h} = \frac{|h|}{h}.$$

The right-hand derivative is 1, while the left-hand derivative is -1. By Proposition 5.9, the derivative at 0 does not exist. □

5.6 Differentiability implies continuity

Theorem 5.11. *If f is differentiable at a, then f is continuous at a.*

Proof. Since f is differentiable at a,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Since $\lim_{h \to 0} h = 0$, the product law for limits gives

$$\begin{aligned} \lim_{h \to 0} \big(f(a+h) - f(a)\big) &= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right) h \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

Therefore

$$\lim_{h \to 0} f(a+h) = f(a),$$

which is continuity at a. □

Remark 5.12. The converse of Theorem 5.11 is false. The function $f(x) = |x|$ is continuous at 0, but Example 5.10 shows that it is not differentiable there.

5.7 Basic differentiation rules

Theorem 5.13 (Sum rule and constant-multiple rule)**.** *Suppose that f and g are differentiable at a, and let $c \in \mathbb{R}$. Then $f+g$ and cf are differentiable at a, with*

$$(f+g)'(a) = f'(a) + g'(a), \qquad (cf)'(a) = cf'(a).$$

Proof. Using the sum law for limits,

$$\begin{aligned}(f+g)'(a) &= \lim_{h\to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h} \right) \\ &= f'(a) + g'(a).\end{aligned}$$

Similarly,

$$\begin{aligned}(cf)'(a) &= \lim_{h\to 0} \frac{cf(a+h) - cf(a)}{h} \\ &= c \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= cf'(a).\end{aligned}$$

□

Theorem 5.14 (Product rule)**.** *Suppose that f and g are differentiable at a. Then fg is differentiable at a, and*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof. Since differentiability implies continuity,

$$\lim_{h\to 0} f(a+h) = f(a).$$

For $h \neq 0$, add and subtract $f(a+h)g(a)$ in the numerator:

$$\begin{aligned}\frac{f(a+h)g(a+h) - f(a)g(a)}{h} &= \frac{f(a+h)\big(g(a+h) - g(a)\big)}{h} \\ &\quad + \frac{\big(f(a+h) - f(a)\big)g(a)}{h} \\ &= f(a+h)\frac{g(a+h)-g(a)}{h} + \frac{f(a+h)-f(a)}{h} g(a).\end{aligned}$$

Taking limits and applying the product and sum laws gives

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

□

Theorem 5.15 (Reciprocal rule)**.** *Suppose that g is differentiable at a and $g(a) \neq 0$. Then $1/g$ is differentiable at a, with*

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}.$$

Proof. By continuity of g at a, $g(x) \neq 0$ in a sufficiently small neighbourhood of a. For $h \neq 0$,

$$\begin{aligned}\frac{1/g(a+h) - 1/g(a)}{h} &= \frac{g(a) - g(a+h)}{h\, g(a+h)g(a)} \\ &= -\frac{g(a+h) - g(a)}{h}\frac{1}{g(a+h)g(a)}.\end{aligned}$$

Taking limits gives

$$\left(\frac{1}{g}\right)'(a) = -g'(a)\frac{1}{g(a)^2}.$$

□

Theorem 5.16 (Quotient rule)**.** *Suppose that f and g are differentiable at a and $g(a) \neq 0$. Then f/g is differentiable at a, with*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof. Write

$$\frac{f}{g} = f \cdot \frac{1}{g}.$$

The conclusion follows from Theorems 5.14 and 5.15. □

Corollary 5.17 (Power rule for integer powers)**.** *For every integer n, the function $x \mapsto x^n$ is differentiable at every point in its domain. For $n \neq 0$,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

For $n = 0$, the function $x \mapsto x^0 = 1$ is constant, hence differentiable everywhere with derivative 0. The displayed expression nx^{n-1} is not literally defined when $n = 0$ and $x = 0$, so it is understood only for $n \neq 0$.

Proof. For $n = 0$, the function is the constant 1, and Theorem 5.4 gives derivative 0 at every point. For $n \geq 1$, the result is Theorem 5.6. For $n < 0$, write $n = -m$, where $m \in \mathbb{N}$, and apply Theorem 5.15 to $x^{-m} = 1/x^m$. □

5.8 The chain rule

Theorem 5.18 (Chain rule)**.** *Suppose that g is differentiable at a and that f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a, and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. Define

$$E(u) = \begin{cases} \dfrac{f(u) - f(g(a))}{u - g(a)}, & u \neq g(a), \\ f'(g(a)), & u = g(a). \end{cases}$$

Since f is differentiable at $g(a)$, the function E is continuous at $g(a)$. For $h \neq 0$,

$$f(g(a+h)) - f(g(a)) = E(g(a+h))(g(a+h) - g(a)).$$

Therefore

$$\frac{f(g(a+h))-f(g(a))}{h} = E(g(a+h))\frac{g(a+h)-g(a)}{h}.$$

As $h \to 0$, continuity of g at a gives

$$E(g(a+h)) \to E(g(a)) = f'(g(a)),$$

while the second factor tends to $g'(a)$. The product law gives the result. □

Example 5.19. The chain rule gives

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2),$$

and

$$\frac{d}{dx}(\sin x)^2 = 2\sin x\cos x.$$

The two functions are different because composition is not commutative.

5.9 Derivatives of elementary functions

The following derivatives will be used repeatedly:

$$\begin{aligned}
\frac{d}{dx}(c) &= 0,\\
\frac{d}{dx}(x^n) &= nx^{n-1} \qquad (n \in \mathbb{Z}),\\
\frac{d}{dx}(\sin x) &= \cos x,\\
\frac{d}{dx}(\cos x) &= -\sin x,\\
\frac{d}{dx}(e^x) &= e^x.
\end{aligned}$$

The trigonometric and exponential derivatives will be justified formally in Chapter 9. Until then, they may be used when needed.

5.10 Higher derivatives

Definition 5.20. If f' is differentiable at a, then the derivative of f' at a is the *second derivative* of f, denoted by

$$f''(a) = f^{(2)}(a).$$

More generally, the k-th derivative is denoted by $f^{(k)}$, with $f^{(0)} = f$.

If $s(t)$ denotes position as a function of time, then

$$s'(t) = \text{velocity}, \qquad s''(t) = \text{acceleration}.$$

Example 5.21. For $f(x) = x^n$, repeated differentiation gives

$$f^{(k)}(x) = n(n-1)\cdots(n-k+1)x^{n-k}$$

for $0 \le k \le n$, and $f^{(k)} = 0$ for $k > n$.

Remark 5.22. Saying that f is n *times differentiable at* a means that $f^{(n-1)}$ exists on some neighbourhood of a and is differentiable at a, so that $f^{(n)}(a)$ exists. Iterating, this is equivalent to: $f^{(k)}$ exists on a neighbourhood of a for every $0 \le k < n$, and $f^{(n)}(a)$ exists. This is the sense in which "n times differentiable at a" is used below.

Theorem 5.23 (Leibniz rule)**.** *If f and g are n times differentiable at a, then fg is n times differentiable at a, and*

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a).$$

Proof. We proceed by induction on n. The case $n = 0$ is the definition, and $n = 1$ is the product rule, Theorem 5.14. Suppose the formula holds for n, and assume that f and g are $(n+1)$ times differentiable at a. By the preceding remark, $f^{(n)}$ and $g^{(n)}$ exist on some neighbourhood U of a. Hence $f^{(k)}$ and $g^{(j)}$ exist and are differentiable on U for every $0 \le k, j \le n$. In particular, f and g are n times differentiable at every point $x \in U$, so the induction hypothesis applies at each such x. Therefore the order-n formula holds as an identity of functions on U:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \qquad \text{for all } x \in U.$$

Moreover, each $f^{(k)}$ and $g^{(n-k)}$, for $0 \le k \le n$, is differentiable at a. Differentiating the identity at $x = a$ and applying the product rule term by term gives

$$\begin{aligned}
(fg)^{(n+1)}(a) &= \sum_{k=0}^{n} \binom{n}{k} (f^{(k)} g^{(n-k)})'(a) \\
&= \sum_{k=0}^{n} \binom{n}{k} (f^{(k+1)}(a) g^{(n-k)}(a) + f^{(k)}(a) g^{(n-k+1)}(a)) \\
&= \sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right) f^{(k)}(a) g^{(n+1-k)}(a) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(a) g^{(n+1-k)}(a),
\end{aligned}$$

using Pascal's identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

with the convention $\binom{n}{-1} = \binom{n}{n+1} = 0$. □

Example 5.24. For $f(x) = x^2 e^x$, the Leibniz rule gives

$$\begin{aligned}
(x^2 e^x)^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} (x^2)^{(k)} (e^x)^{(n-k)} \\
&= \binom{n}{2} \cdot 2 \cdot e^x + \binom{n}{1} \cdot 2x \cdot e^x + x^2 e^x \\
&= (x^2 + 2nx + n(n-1)) e^x,
\end{aligned}$$

since $(x^2)^{(k)} = 0$ for $k \ge 3$.

5.11 Linear approximation

The linearization of a differentiable function gives a local approximation:

$$f(x) \approx f(a) + f'(a)(x-a) \qquad \text{when } x \text{ is close to } a.$$

Example 5.25. Let $f(x) = \sqrt{x}$, and linearize at $a = 16$. Since

$$f(16) = 4, \qquad f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(16) = \frac{1}{8},$$

the linearization is

$$L_{f,16}(x) = 4 + \frac{1}{8}(x-16).$$

Thus

$$\sqrt{14} \approx L_{f,16}(14) = 4 - \frac{2}{8} = \frac{15}{4} = 3.75.$$

The error in linear approximation is smaller than the displacement from the base point in the following precise sense:

$$\lim_{x\to a} \frac{f(x) - L_{f,a}(x)}{x-a} = 0.$$

Indeed,

$$\frac{f(x) - L_{f,a}(x)}{x-a} = \frac{f(x)-f(a)}{x-a} - f'(a),$$

which tends to 0 by the definition of the derivative.

5.12 Exercises

Exercise 5.26. Use the definition of derivative to prove that

$$\frac{d}{dx}(x^3) = 3x^2.$$

Exercise 5.27. Use the definition of derivative to show that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

for $x > 0$.

Exercise 5.28. Determine the values of a and b for which the function

$$f(x) = \begin{cases} x^2, & x < 1, \\ ax + b, & x \geq 1 \end{cases}$$

is differentiable at $x = 1$.

Exercise 5.29. Show that

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is differentiable at 0, and compute $f'(0)$.

Exercise 5.30. Use the product and chain rules to differentiate

$$f(x) = x^3 \sin(x^2).$$

Exercise 5.31. Find the tangent-line equation to the graph of

$$f(x) = \frac{1}{x}$$

at $x = 2$.

Chapter 6

Applications of the Derivative

The derivative provides local information about a function. In this chapter we use that information to locate extrema, establish monotonicity, analyze graphs, evaluate certain limits, locate roots numerically, study convexity, and derive some classical inequalities.

6.1 Maximum and minimum points

Definition 6.1. Let f be defined on a set $S \subseteq \mathbb{R}$. A point $x_0 \in S$ is a *maximum point* of f on S if

$$f(x) \leq f(x_0) \qquad \text{for every } x \in S.$$

A point $x_1 \in S$ is a *minimum point* of f on S if

$$f(x_1) \leq f(x) \qquad \text{for every } x \in S.$$

Definition 6.2. Let $x_0 \in S$. We say that x_0 is a *local maximum point* of f on S if there exists $\delta > 0$ such that

$$f(x) \leq f(x_0) \qquad \text{for every } x \in S \cap (x_0 - \delta, x_0 + \delta).$$

A *local minimum point* is defined similarly, with the inequality reversed.

A global maximum or minimum point is automatically a local maximum or minimum point, but not conversely.

Theorem 6.3 (Fermat's theorem)**.** *Let $f : (a, b) \to \mathbb{R}$, and let $c \in (a, b)$. If f has a local maximum or a local minimum at c, and if f is differentiable at c, then*

$$f'(c) = 0.$$

Proof. Suppose first that f has a local maximum at c. There exists $\delta > 0$ such that

$$f(c + h) \leq f(c) \qquad \text{whenever } |h| < \delta.$$

If $0 < h < \delta$, then

$$\frac{f(c + h) - f(c)}{h} \leq 0.$$

Hence

$$f'_+(c) \leq 0.$$

If $-\delta < h < 0$, then the numerator $f(c+h) - f(c)$ is nonpositive and the denominator is negative, so

$$\frac{f(c+h) - f(c)}{h} \geq 0.$$

Hence

$$f'_-(c) \geq 0.$$

Since f is differentiable at c, its left- and right-hand derivatives are equal. Thus

$$0 \leq f'_-(c) = f'_+(c) \leq 0,$$

so $f'(c) = 0$.

The proof for a local minimum is analogous: the inequalities reverse, but the same conclusion follows. □

Remark 6.4. The converse of Theorem 6.3 is false. The function $f(x) = x^3$ satisfies $f'(0) = 0$, but has neither a local maximum nor a local minimum at 0.

Definition 6.5. Let f be differentiable at c. The point c is a *critical point* of f if

$$f'(c) = 0.$$

Corollary 6.6. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. If f attains a maximum or minimum at $c \in [a, b]$, then at least one of the following holds:*

(i) *$c = a$ or $c = b$;*

(ii) *$c \in (a, b)$ is a critical point of f;*

(iii) *$c \in (a, b)$ is a point at which f is not differentiable.*

Proof. If c is not an endpoint and f is differentiable at c, then Theorem 6.3 gives $f'(c) = 0$. □

Example 6.7. Among all rectangles with a fixed perimeter P, find the one of largest area.

Let the two side lengths be x and $P/2 - x$, where $0 \leq x \leq P/2$. The area is

$$A(x) = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2.$$

The continuous function A attains a maximum on $[0, P/2]$ by the Extreme Value Theorem. By Corollary 6.6, a maximum point is an endpoint, a critical point, or a point of nondifferentiability. Since

$$A'(x) = \frac{P}{2} - 2x,$$

the only critical point is $x = P/4$, and A is differentiable everywhere. The endpoints give

$$A(0) = A(P/2) = 0, \qquad \text{while} \qquad A(P/4) = \frac{P^2}{16} > 0.$$

Hence the maximum occurs at $x = P/4$, where both side lengths equal $P/4$: among all rectangles with a given perimeter, the square has the largest area.

6.2 The Mean Value Theorem

Theorem 6.8 (Rolle's theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that*

$$f'(c) = 0.$$

Proof. By the Extreme Value Theorem, f attains both a maximum and a minimum on $[a, b]$. If either extremum occurs at an interior point, then Theorem 6.3 gives the desired conclusion.

Suppose that neither an interior maximum nor an interior minimum exists. Then both the maximum and the minimum occur at the endpoints. Since $f(a) = f(b)$, maximum and minimum have the same value. Therefore f is constant on $[a, b]$, and hence $f'(c) = 0$ for every $c \in (a, b)$. □

Theorem 6.9 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define the affine function

$$\ell(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

and set

$$h(x) = f(x) - \ell(x).$$

The function h is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$h(a) = f(a) - \ell(a) = 0,$$

and

$$h(b) = f(b) - \ell(b) = 0.$$

By Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. Since

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

the result follows. □

Corollary 6.10. *Let $f : (a, b) \to \mathbb{R}$ be differentiable. If*

$$f'(x) = 0 \qquad \text{for every } x \in (a, b),$$

then f is constant on (a, b).

Proof. Let $u, v \in (a, b)$ with $u < v$. Applying the Mean Value Theorem to $[u, v]$, there is $c \in (u, v)$ such that

$$f(v) - f(u) = f'(c)(v - u) = 0.$$

Thus $f(u) = f(v)$. Since u, v were arbitrary, f is constant. □

Corollary 6.11. *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable. If $f' = g'$ on (a, b), then there exists $C \in \mathbb{R}$ such that*

$$f(x) = g(x) + C \qquad \text{for every } x \in (a, b).$$

Proof. Apply Corollary 6.10 to $f - g$. □

Theorem 6.12 (Cauchy Mean Value Theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Proof. Define

$$h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x).$$

Then h is continuous on $[a, b]$ and differentiable on (a, b), and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

By Theorem 6.8, there exists $c \in (a, b)$ with $h'(c) = 0$, which is exactly the desired identity. □

Remark 6.13. Taking $g(x) = x$ recovers Theorem 6.9. The Cauchy form is the version used in the proof of L'Hôpital's rule in Section 6.6.

Corollary 6.14. *Let f be differentiable on an interval I, and suppose that*

$$|f'(x)| \le M \qquad \text{for every } x \in I.$$

Then

$$|f(x) - f(y)| \le M\,|x - y| \qquad \text{for all } x, y \in I.$$

Proof. For $x < y$ in I, the Mean Value Theorem provides $c \in (x, y)$ such that

$$f(y) - f(x) = f'(c)(y - x).$$

Taking absolute values gives the result. □

Theorem 6.15 (Darboux's theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and suppose that the one-sided derivatives at the endpoints exist. If*

$$f'(a) < \gamma < f'(b) \qquad \text{or} \qquad f'(b) < \gamma < f'(a),$$

then there exists $c \in (a, b)$ such that

$$f'(c) = \gamma.$$

(Here $f'(a)$ and $f'(b)$ denote the one-sided derivatives.)

Proof. Consider the first case, and set $g(x) = f(x) - \gamma x$. Then

$$g'(a) = f'(a) - \gamma < 0 \qquad \text{and} \qquad g'(b) = f'(b) - \gamma > 0.$$

Since

$$\lim_{h \to 0^+} \frac{g(a + h) - g(a)}{h} = g'(a) < 0,$$

we have $g(a + h) < g(a)$ for all sufficiently small $h > 0$, so the endpoint a is not a minimum point of g on $[a, b]$. Similarly,

$$\lim_{h \to 0^-} \frac{g(b + h) - g(b)}{h} = g'(b) > 0$$

gives $g(b + h) < g(b)$ for all sufficiently small $h < 0$, so b is not a minimum point either.

The continuous function g attains a minimum on $[a, b]$ by the Extreme Value Theorem. By the preceding paragraph, the minimum occurs at some $c \in (a, b)$. Theorem 6.3 gives $g'(c) = 0$, hence

$$f'(c) = \gamma.$$

The second case follows by applying the first to $-f$. □

Remark 6.16. Darboux's theorem says that derivatives have the intermediate value property even though a derivative need not be continuous. In particular, a derivative can never have a jump discontinuity: it cannot skip any intermediate value.

6.3 Monotonicity

Definition 6.17. Let I be an interval. A function $f : I \to \mathbb{R}$ is *increasing* on I if

$$x < y \quad \Longrightarrow \quad f(x) \leq f(y) \qquad \text{for all } x, y \in I.$$

It is *strictly increasing* if the conclusion is $f(x) < f(y)$.

A function is *decreasing* or *strictly decreasing* if the respective inequalities are reversed.

Theorem 6.18. *Let $f : I \to \mathbb{R}$ be differentiable on an interval I.*

(i) *If $f'(x) > 0$ for every $x \in I$, then f is strictly increasing on I.*

(ii) *If $f'(x) < 0$ for every $x \in I$, then f is strictly decreasing on I.*

(iii) *If $f'(x) \geq 0$ for every $x \in I$, then f is increasing on I.*

(iv) *If $f'(x) \leq 0$ for every $x \in I$, then f is decreasing on I.*

Proof. We prove *(i)*. Let $u < v$ be points of I. The Mean Value Theorem provides $c \in (u, v)$ such that

$$f(v) - f(u) = f'(c)(v - u).$$

Both factors on the right are positive, so $f(v) > f(u)$. Thus f is strictly increasing. The other assertions are proved similarly. □

Proposition 6.19. *Let $f : I \to \mathbb{R}$ be differentiable on I. If f is increasing on I, then*

$$f'(x) \geq 0 \qquad \textit{for every } x \in I.$$

Similarly, if f is decreasing, then $f'(x) \leq 0$.

Proof. Fix an interior point $x \in I$. For $h > 0$ sufficiently small, $f(x + h) \geq f(x)$, so

$$\frac{f(x+h) - f(x)}{h} \geq 0.$$

For $h < 0$ sufficiently small, $f(x + h) \leq f(x)$, and division by the negative number h again gives

$$\frac{f(x+h) - f(x)}{h} \geq 0.$$

Taking the limit as $h \to 0$ gives $f'(x) \geq 0$. □

Proposition 6.20. *Let $f : I \to \mathbb{R}$ be differentiable on the interval I, and suppose that*

$$f'(x) \geq 0 \qquad \textit{for every } x \in I.$$

If the set

$$\{x \in I : f'(x) = 0\}$$

contains no nondegenerate interval, then f is strictly increasing on I. Similarly, if $f' \leq 0$ on I and the zero set of f' contains no nondegenerate interval, then f is strictly decreasing on I.

Proof. By Theorem 6.18, f is increasing. Suppose that $u < v$ are points of I with $f(u) = f(v)$. For every $x \in (u, v)$,

$$f(u) \le f(x) \le f(v) = f(u),$$

so f is constant on $[u, v]$. Hence $f' = 0$ on (u, v), a nondegenerate interval on which f' vanishes, contradicting the hypothesis. Thus $f(u) < f(v)$ whenever $u < v$, so f is strictly increasing. The strictly decreasing case is analogous. □

6.4 First- and second-derivative tests

Theorem 6.21 (First-derivative test)**.** *Let f be continuous on an open interval containing c, and differentiable there except possibly at c.*

(i) *If f' is negative immediately to the left of c and positive immediately to the right of c, then c is a strict local minimum point.*

(ii) *If f' is positive immediately to the left of c and negative immediately to the right of c, then c is a strict local maximum point.*

Proof. We prove *(i)*. There exists $\delta > 0$ such that

$$f'(x) < 0 \quad \text{on } (c-\delta, c), \qquad f'(x) > 0 \quad \text{on } (c, c+\delta).$$

For $c - \delta < x < c$, the Mean Value Theorem applied to f on $[x, c]$ gives $\xi \in (x, c)$ with

$$f(c) - f(x) = f'(\xi)(c - x) < 0,$$

so $f(x) > f(c)$. Similarly, for $c < x < c + \delta$, the Mean Value Theorem applied on $[c, x]$ gives

$$f(x) - f(c) = f'(\xi)(x - c) > 0,$$

so $f(x) > f(c)$. Hence $f(x) > f(c)$ for every $x \in (c-\delta, c+\delta) \setminus \{c\}$, and c is a strict local minimum point. The second assertion is analogous. □

Remark 6.22. The continuity hypothesis is essential: the derivative signs on the two punctured sides say nothing about an arbitrarily assigned value at c. For example,

$$f(x) = \begin{cases} x^2, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

has negative derivative on $(-\delta, 0)$ and positive derivative on $(0, \delta)$, yet 0 is a strict local maximum.

Theorem 6.23 (Second-derivative test)**.** *Let f be twice differentiable near c, and suppose $f'(c) = 0$.*

(i) *If $f''(c) > 0$, then c is a strict local minimum point.*

(ii) *If $f''(c) < 0$, then c is a strict local maximum point.*

(iii) *If $f''(c) = 0$, no conclusion follows in general.*

Proof. Suppose $f''(c) > 0$. By definition,

$$f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h} > 0.$$

For sufficiently small positive h, the quotient is positive, so $f'(c+h) > 0$. For sufficiently small negative h, the quotient is still positive, so $f'(c+h) < 0$. The first-derivative test gives a strict local minimum at c. The case $f''(c) < 0$ is similar.

For $f''(c) = 0$, consider x^3, x^4, and $-x^4$ at $c = 0$. Their first and second derivatives vanish at 0, but the first has neither a local maximum nor a local minimum, while the other two have a local minimum and a local maximum, respectively. □

The second-derivative test is the case $n = 2$ of the following general criterion, which settles the matter whenever some derivative of higher order is the first not to vanish.

Theorem 6.24 (Higher-order derivative test)**.** *Let f be n times differentiable at c, meaning that $f^{(n-1)}$ exists near c and $f^{(n)}(c)$ exists. Suppose that*

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \qquad f^{(n)}(c) \neq 0.$$

(i) *If n is even and $f^{(n)}(c) > 0$, then c is a strict local minimum point.*

(ii) *If n is even and $f^{(n)}(c) < 0$, then c is a strict local maximum point.*

(iii) *If n is odd, then c is neither a local maximum nor a local minimum point.*

Proof. Replacing f by $F = f - f(c)$ does not change any derivative of order at least 1, and c is a local extremum point of f if and only if it is one of F. We prove the result for F, assuming first that $F^{(n)}(c) > 0$; the case $F^{(n)}(c) < 0$ follows by applying the result to $-F$.

The case $n = 1$ is immediate: since

$$F'(c) = \lim_{x \to c} \frac{F(x) - F(c)}{x - c} > 0,$$

the quotient is positive for x near c, so $F(x) < F(c) = 0$ for $x < c$ and $F(x) > 0$ for $x > c$, and c is not a local extremum.

Assume now that $n \geq 2$. Since

$$(F^{(n-2)})'(c) = F^{(n-1)}(c) = 0 \qquad \text{and} \qquad (F^{(n-2)})''(c) = F^{(n)}(c) > 0,$$

Theorem 6.23 applied to $F^{(n-2)}$ shows that $F^{(n-2)}$ has a strict local minimum at c. For $n = 2$ this means that F itself has a strict local minimum at c, proving *(i)*. For $n \geq 3$ we have $F^{(n-2)}(c) = 0$, so there exists $\delta > 0$ with

$$F^{(n-2)}(x) > 0 \qquad \text{for } 0 < |x - c| < \delta.$$

We now descend two derivatives at a time. Suppose that for some $j \geq 2$ the function $F^{(j)}$ is strictly positive on a deleted neighbourhood of c and satisfies $F^{(j)}(c) = 0$. Then $(F^{(j-1)})' = F^{(j)} \geq 0$ near c, with equality only at c, so by Proposition 6.20 the function $F^{(j-1)}$ is strictly increasing near c. Since $F^{(j-1)}(c) = 0$, it follows that $F^{(j-1)} < 0$ immediately to the left of c and $F^{(j-1)} > 0$ immediately to the right. Hence $(F^{(j-2)})' = F^{(j-1)}$ changes sign from negative to positive at c, and Theorem 6.21 shows that $F^{(j-2)}$ has a strict local minimum at c. If $j - 2 \geq 1$, then $F^{(j-2)}(c) = 0$ by the hypotheses, so $F^{(j-2)}$ is again strictly positive on a deleted neighbourhood of c.

Starting from $j = n - 2 \geq 1$ and descending by 2 while $j \geq 2$, this process reaches $j = 0$ when n is even, proving *(i)*: F, and hence f, has a strict local minimum at c. When n is odd, the descent reaches $j = 1$: the derivative F' has a strict local minimum at c and vanishes there, so $F' > 0$ on a deleted neighbourhood of c. Hence F is strictly increasing near c, and

$$F(x) < F(c) = 0 \quad \text{for } x < c, \qquad F(x) > 0 \quad \text{for } x > c$$

for x sufficiently close to c. Thus c is neither a local maximum nor a local minimum point of f, proving *(iii)*. □

6.5 Curve sketching

A systematic derivative-based analysis of a graph usually proceeds as follows:

1. Determine the domain.
2. Find intercepts, symmetries, and limiting behaviour at excluded points and at infinity.
3. Compute the first derivative and identify critical points and intervals of increase or decrease.
4. Use the first-derivative test to identify local extrema.
5. Compute the second derivative and identify intervals of convexity and concavity, as well as possible inflection points.
6. Combine the preceding information to sketch a graph consistent with all established facts.

Example 6.25. Consider

$$f(x) = x^3 + 3x^2 - 9x + 12.$$

We have

$$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1).$$

Therefore f is strictly increasing on $(-\infty, -3)$ and $(1, \infty)$, and strictly decreasing on $(-3, 1)$. Thus f has a local maximum at $x = -3$ and a local minimum at $x = 1$. Further,

$$f''(x) = 6x + 6,$$

so the graph is concave on $(-\infty, -1)$, convex on $(-1, \infty)$, and has an inflection point at $x = -1$.

6.6 L'Hôpital's rule

Theorem 6.26 (L'Hôpital's rule, 0/0 form)**.** *Let f and g be differentiable on a deleted neighbourhood of a. Suppose that*

$$\lim_{x \to a} f(x) = 0, \qquad \lim_{x \to a} g(x) = 0,$$

and that

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

exists, where $g'(x) \neq 0$ sufficiently near a. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. We give the proof for the right-hand limit; the left-hand argument is analogous. Redefine $f(a) = g(a) = 0$, which does not alter any deleted neighbourhood limit. Note that $g(x) \neq g(a)$ for every $x > a$ sufficiently close to a: otherwise Rolle's theorem applied to g on $[a, x]$ would give a point of (a, x) at which g' vanishes, contradicting $g'(x) \neq 0$ near a. For each such x, the Cauchy Mean Value Theorem (Theorem 6.12) applied on $[a, x]$ gives a point $c_x \in (a, x)$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c_x)}{g'(c_x)}.$$

Hence

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}.$$

As $x \to a^+$, we have $c_x \to a^+$. Therefore the right-hand side tends to L, proving the result. □

Remark 6.27. The proof invokes the Cauchy Mean Value Theorem, Theorem 6.12. L'Hôpital's rule is a theorem about a special class of indeterminate quotients, not a substitute for checking hypotheses.

Theorem 6.28 (L'Hôpital's rule, ∞/∞ form)**.** *Let f and g be differentiable on a deleted neighbourhood of a. Suppose that*

$$\lim_{x\to a} f(x) = \infty, \qquad \lim_{x\to a} g(x) = \infty,$$

and that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

exists as a finite real number, where $g'(x) \neq 0$ sufficiently near a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Proof. We give the proof for the right-hand limit; the left-hand argument is analogous. Let $\varepsilon > 0$, and choose $y > a$ sufficiently close to a so that g is defined and nonzero on (a, y) and

$$\left|\frac{f'(t)}{g'(t)} - L\right| < \varepsilon \qquad \text{for all } t \in (a, y).$$

For $a < x < y$, the Cauchy Mean Value Theorem Theorem 6.12 applied on $[x, y]$ gives $c_x \in (x, y)$ with

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c_x)}{g'(c_x)}.$$

(If $g(x) = g(y)$, Rolle's theorem would give a zero of g' in (x, y), which is impossible.) Setting $q_x = f'(c_x)/g'(c_x)$, we have

$$f(x) - f(y) = q_x(g(x) - g(y)),$$

hence

$$\frac{f(x)}{g(x)} = q_x + \frac{f(y) - q_x g(y)}{g(x)}.$$

Since $|q_x - L| < \varepsilon$, the numbers q_x are bounded, while $f(y)$ and $g(y)$ are fixed. As $x \to a^+$, we have $g(x) \to \infty$, so the last term tends to 0. For x sufficiently close to a,

$$\left|\frac{f(x)}{g(x)} - q_x\right| < \varepsilon,$$

and therefore

$$\left|\frac{f(x)}{g(x)} - L\right| < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

Remark 6.29. Both forms of L'Hôpital's rule also hold in the following variants, with essentially the same proofs:

(i) *One-sided limits.* The proofs above are already one-sided; Theorems 6.26 and 6.28 hold verbatim for $x \to a^+$ and $x \to a^-$.

(ii) *Infinite L.* If $\lim f'/g' = \infty$ or $-\infty$, the conclusion becomes $\lim f/g = \infty$ or $-\infty$ respectively. The arguments above adapt directly.

(iii) *Limits at infinity.* To evaluate $\lim_{x\to\infty} f/g$, apply the rule to $F(t) = f(1/t)$ and $G(t) = g(1/t)$ as $t \to 0^+$. The chain rule gives

$$\frac{F'(t)}{G'(t)} = \frac{f'(1/t)}{g'(1/t)},$$

so the hypotheses transfer, and $\lim_{x\to\infty} f/g = \lim_{t\to 0^+} F/G$. The case $x \to -\infty$ is handled with $t \to 0^-$.

Example 6.30.

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

The quotient has the form ∞/∞, and the derivative quotient tends to 0. (The derivative of e^x is justified in Chapter 9.)

Example 6.31.

$$\lim_{x\to 1} \frac{x^2 - 1}{x^3 - 1} = \lim_{x\to 1} \frac{2x}{3x^2} = \frac{2}{3}.$$

The same answer follows by factoring numerator and denominator.

6.7 Convexity and concavity

Definition 6.32. Let $f : I \to \mathbb{R}$, where I is an interval. The function f is *convex* on I if, for all $a, b \in I$ and all $t \in (0, 1)$,

$$f((1-t)a + tb) \le (1-t)f(a) + tf(b).$$

It is *concave* on I if the reverse inequality holds for all such a, b, t. It is *strictly convex* if the inequality is strict whenever $a \neq b$, and *strictly concave* if the reverse inequality is strict whenever $a \neq b$.

As t runs through $(0, 1)$, the point $(1-t)a + tb$ runs through the open interval between a and b, while $(1-t)f(a) + tf(b)$ runs through the chord joining the graph points $(a, f(a))$ and $(b, f(b))$. A convex function therefore lies on or below each of its chords, while a concave function lies on or above each of its chords.

Example 6.33. The function $f(x) = x^2$ is strictly convex on $\mathbb{R}$: for $a \neq b$ and $t \in (0, 1)$,

$$(1-t)a^2 + tb^2 - ((1-t)a + tb)^2 = t(1-t)(a-b)^2 > 0.$$

The function $f(x) = |x|$ is convex on $\mathbb{R}$ by the triangle inequality, but not strictly convex, because a, b of the same sign give equality. Note that $|x|$ is not differentiable at 0: Definition 6.32 itself requires no differentiability.

Proposition 6.34. ‡ *Let $f : I \to \mathbb{R}$ be continuous on the interval I. If*

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a) + f(b)}{2} \qquad \text{for all } a, b \in I,$$

then f is convex on I.

Proof. Fix $a, b \in I$. We first prove the convexity inequality for every dyadic parameter $t = k/2^m$, where $m \in \mathbb{N}_0$ and $0 \le k \le 2^m$, by induction on m. The cases $m = 0$ and $m = 1$ are trivial and the hypothesis, respectively. If $t = k/2^{m+1}$ with k even, then t is already of the form $j/2^m$. If $k = 2j + 1$ is odd, then t is the midpoint of $j/2^m$ and $(j + 1)/2^m$. Writing

$$u = \left(1 - \frac{j}{2^m}\right) a + \frac{j}{2^m} b, \qquad v = \left(1 - \frac{j+1}{2^m}\right) a + \frac{j+1}{2^m} b,$$

we have $(1 - t)a + tb = (u + v)/2$. The midpoint hypothesis and the induction hypothesis therefore give

$$\begin{aligned} f((1-t)a + tb) &\le \frac{f(u) + f(v)}{2} \\ &\le (1-t)f(a) + tf(b). \end{aligned}$$

Now let $t \in [0, 1]$ be arbitrary, and choose dyadic numbers $t_m \in [0, 1]$ with $t_m \to t$; such numbers exist by repeated bisection of $[0, 1]$. The inequality just proved gives

$$f((1 - t_m)a + t_m b) \le (1 - t_m)f(a) + t_m f(b)$$

for every m. Letting $m \to \infty$, continuity of f shows that the left-hand side tends to $f((1 - t)a + tb)$, and the result follows. □

Lemma 6.35 (The three-slope inequality)**.** *Let $f : I \to \mathbb{R}$ be convex on the interval I, and let $a < x < b$ be points of I. Then*

$$\frac{f(x) - f(a)}{x - a} \le \frac{f(b) - f(a)}{b - a} \le \frac{f(b) - f(x)}{b - x}.$$

If f is concave, both inequalities are reversed; if f is strictly convex, both are strict.

Proof. Since $x = (1 - t)a + tb$ with

$$t = \frac{x - a}{b - a} \in (0, 1),$$

convexity gives

$$f(x) \le (1 - t)f(a) + tf(b).$$

Subtracting $f(a)$ from both sides and dividing by $x - a = t(b - a) > 0$ yields the first inequality. Similarly,

$$f(b) - f(x) \ge (1 - t)(f(b) - f(a)),$$

and division by $b - x = (1 - t)(b - a) > 0$ yields the second. □

In words, among the three chords determined by $a < x < b$, the slope over $[a, x]$ is the smallest and the slope over $[x, b]$ is the largest. Fixing a, it follows that the difference quotient

$$x \longmapsto \frac{f(x) - f(a)}{x - a}$$

is increasing on $I \setminus \{a\}$ when f is convex, and strictly increasing when f is strictly convex.

Proposition 6.36. *Let $f : I \to \mathbb{R}$ be differentiable on the interval I.*

(i) *f is convex on I if and only if f' is increasing on I.*

(ii) *f is concave on I if and only if f' is decreasing on I.*

With "convex" and "concave" replaced by "strictly convex" and "strictly concave", the same statements hold with "increasing" and "decreasing" replaced by "strictly increasing" and "strictly decreasing".

Proof. We prove *(i)*; *(ii)* follows by applying *(i)* to $-f$.

Suppose first that f' is increasing, and let $a < x < b$ be points of I. The Mean Value Theorem provides $u \in (a, x)$ and $v \in (x, b)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(u) \qquad \text{and} \qquad \frac{f(b) - f(x)}{b - x} = f'(v).$$

Since $u < v$, we have $f'(u) \le f'(v)$. Thus

$$(b - x)(f(x) - f(a)) \le (x - a)(f(b) - f(x)),$$

which rearranges to

$$f(x) \le \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b).$$

Since a, x, b were arbitrary, f is convex. If f' is strictly increasing, the same argument with a strict inequality shows that f is strictly convex.

Conversely, suppose that f is convex, and let $a < b$ be points of I. By Lemma 6.35,

$$\frac{f(x) - f(a)}{x - a} \le \frac{f(b) - f(a)}{b - a} \le \frac{f(b) - f(x)}{b - x}$$

for every $x \in (a, b)$. Letting $x \to a^+$ in the first inequality and $x \to b^-$ in the last gives

$$f'(a) \le \frac{f(b) - f(a)}{b - a} \le f'(b).$$

(At an endpoint of I these limits are one-sided derivatives.) Hence f' is increasing.

If f is strictly convex, the three slopes are strictly ordered, so for $a < y < b$,

$$f'(a) \le \frac{f(y) - f(a)}{y - a} < \frac{f(b) - f(a)}{b - a} < \frac{f(b) - f(y)}{b - y} \le f'(b).$$

Thus $f'(a) < f'(b)$, and f' is strictly increasing. □

Theorem 6.37 (Second-derivative criterion for convexity)**.** *Let f be twice differentiable on an interval I.*

(i) *f is convex on I if and only if $f''(x) \ge 0$ for every $x \in I$.*

(ii) *f is concave on I if and only if $f''(x) \le 0$ for every $x \in I$.*

Proof. By Proposition 6.36, convexity of f is equivalent to increase of f'. By Theorem 6.18 and Proposition 6.19 applied to f', the latter is equivalent to $f'' \ge 0$ on I. The concave case follows from $-f$. □

Corollary 6.38. *If $f''(x) > 0$ for every $x \in I$, then f is strictly convex on I. Similarly, if $f''(x) < 0$ for every $x \in I$, then f is strictly concave on I.*

Proof. Then f' is strictly increasing on I by Theorem 6.18, and the strict form of Proposition 6.36 applies. □

Remark 6.39. The converse of Corollary 6.38 is false. The function $f(x) = x^4$ is strictly convex on $\mathbb{R}$, because its derivative $4x^3$ is strictly increasing, yet $f''(0) = 0$. Strict convexity is therefore detected most reliably through the derivative criterion of Proposition 6.36 rather than through f''.

Proposition 6.40 (Supporting tangents)**.** *Let $f : I \to \mathbb{R}$ be convex on the interval I, and suppose that f is differentiable at $a \in I$. Then*

$$f(x) \geq f(a) + f'(a)(x - a) \qquad \text{for every } x \in I.$$

If f is concave, the inequality is reversed. If f is strictly convex or strictly concave, equality holds only at $x = a$.

Proof. For $x > a$ in I, Lemma 6.35 applied to the triple $a < u < x$ gives

$$\frac{f(u) - f(a)}{u - a} \leq \frac{f(x) - f(a)}{x - a}$$

for every $u \in (a, x)$. Letting $u \to a^+$ gives

$$f'(a) \leq \frac{f(x) - f(a)}{x - a},$$

hence $f(x) \geq f(a) + f'(a)(x - a)$. For $x < a$, apply the lemma to the triple $x < u < a$:

$$\frac{f(a) - f(x)}{a - x} \leq \frac{f(a) - f(u)}{a - u}.$$

Letting $u \to a^-$ gives

$$\frac{f(a) - f(x)}{a - x} \leq f'(a),$$

hence $f(a) - f(x) \leq f'(a)(a - x)$, which is again $f(x) \geq f(a) + f'(a)(x - a)$.

It remains to see that for strictly convex f equality holds only at $x = a$. Strict inequalities do not in general survive a limit, so argue through an intermediate point. For $x > a$, choose $y \in (a, x)$. By the strict form of Lemma 6.35,

$$\frac{f(u) - f(a)}{u - a} < \frac{f(y) - f(a)}{y - a} < \frac{f(x) - f(a)}{x - a} \qquad (a < u < y).$$

Letting $u \to a^+$ gives

$$f'(a) \leq \frac{f(y) - f(a)}{y - a} < \frac{f(x) - f(a)}{x - a},$$

so $f'(a) < (f(x) - f(a))/(x - a)$, and $f(x) > f(a) + f'(a)(x - a)$. The case $x < a$ is analogous: with $y \in (x, a)$, the strict form of the lemma applied to the triple $x < y < a$ gives

$$\frac{f(a) - f(x)}{a - x} < \frac{f(a) - f(y)}{a - y} \leq f'(a),$$

where the last inequality follows because $(f(a) - f(u))/(a - u) > (f(a) - f(y))/(a - y)$ for every $u \in (y, a)$, and letting $u \to a^-$ preserves the lower bound for the limit $f'(a)$. Hence $f(a) - f(x) > f'(a)(a - x)$, that is, $f(x) > f(a) + f'(a)(x - a)$ again. □

Remark 6.41. Proposition 6.40 completes the geometric picture: the graph of a convex differentiable function lies below every chord and above every tangent line; the graph of a concave differentiable function lies above every chord and below every tangent line.

Theorem 6.42 (Jensen's inequality)**.** *Let $f : I \to \mathbb{R}$ be convex on the interval I. For any points $x_1, \dots, x_n \in I$ and any weights $t_1, \dots, t_n \geq 0$ with*

$$t_1 + \cdots + t_n = 1,$$

the weighted average $t_1x_1 + \cdots + t_nx_n$ lies in I, and

$$f(t_1x_1 + \cdots + t_nx_n) \leq t_1f(x_1) + \cdots + t_nf(x_n).$$

If f is concave, the inequality is reversed. If f is strictly convex and all weights are positive, equality holds only when $x_1 = \cdots = x_n$.

Proof. A weighted average of points of an interval lies between the smallest and the largest of those points, hence in I. Proceed by induction on n. For $n = 1$ there is nothing to prove, and for $n = 2$ the assertion is the definition of convexity. Suppose $n \geq 3$, and assume first that $t_n < 1$ (if $t_n = 1$, the remaining weights vanish and the claim is trivial). Set

$$y = \frac{t_1x_1 + \cdots + t_{n-1}x_{n-1}}{t_1 + \cdots + t_{n-1}} \in I.$$

Since

$$t_1x_1 + \cdots + t_nx_n = (1 - t_n)y + t_nx_n,$$

convexity gives

$$f(t_1x_1 + \cdots + t_nx_n) \leq (1 - t_n)f(y) + t_nf(x_n).$$

By the induction hypothesis applied with the normalized weights

$$\frac{t_1}{t_1 + \cdots + t_{n-1}}, \dots, \frac{t_{n-1}}{t_1 + \cdots + t_{n-1}},$$

we have

$$f(y) \leq \frac{t_1f(x_1) + \cdots + t_{n-1}f(x_{n-1})}{t_1 + \cdots + t_{n-1}}.$$

Combining the two inequalities completes the induction. □

Example 6.43. Applying Theorem 6.42 to the strictly convex function $f(x) = x^2$ with equal weights $t_1 = \cdots = t_n = 1/n$ gives

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^2 \leq \frac{x_1^2 + \cdots + x_n^2}{n},$$

so the arithmetic mean never exceeds the quadratic (root-mean-square) mean, with equality only when all x_i coincide. Jensen's inequality is the general mechanism behind classical mean inequalities of this type.

Remark 6.44. In Chapter 9 it is shown that log is strictly concave on $(0, \infty)$. Applying Theorem 6.42 to $f(x) = -\log x$ there yields the arithmetic–geometric mean inequality: for positive $x_1, \dots, x_n$,

$$(x_1x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

Remark 6.45. A convex function is automatically continuous at every interior point of its domain. Indeed, if $x_0 \in I$ is interior, choose $a, b \in I$ with $a < x_0 < b$. For $x_0 < x < b$, Lemma 6.35 applied twice gives

$$\frac{f(x_0) - f(a)}{x_0 - a} \leq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(b) - f(x_0)}{b - x_0},$$

so $|f(x) - f(x_0)| \leq C(x - x_0)$ for a constant C independent of x, and $f(x) \to f(x_0)$ as $x \to x_0^+$. The left-hand side is handled the same way.

Definition 6.46. Let $f : I \to \mathbb{R}$ be continuous, and let c be an interior point of the interval I. The point c is an *inflection point* of f if there exists $\delta > 0$ such that f is strictly convex on $(c-\delta, c)$ and strictly concave on $(c, c+\delta)$, or strictly concave on $(c-\delta, c)$ and strictly convex on $(c, c+\delta)$.

The strictness requirement makes the change of convexity genuine: an affine function satisfies $f'' \equiv 0$ and is both convex and concave everywhere, but has no inflection point, because it changes neither from strict convexity to strict concavity nor in the reverse direction. Continuity of f at c is included so that the graph passes through the single point $(c, f(c))$ where the curvature changes direction.

Proposition 6.47. *Suppose that c is an inflection point of f and that $f''(c)$ exists. Then*

$$f''(c) = 0.$$

Proof. Suppose first that f is strictly convex to the left of c and strictly concave to the right of c. By Proposition 6.36, f' is strictly increasing on a left neighbourhood of c and strictly decreasing on a right neighbourhood of c. Since f' is differentiable at c, it is continuous there, and it follows that f' has a local maximum at c. Theorem 6.3 applied to f' gives

$$f''(c) = (f')'(c) = 0.$$

The other case is analogous. □

Remark 6.48. Conversely, suppose that f'' exists near c, that $f''(c) = 0$, and that f'' is strictly positive on one side of c and strictly negative on the other. Then Corollary 6.38 shows that f is strictly convex on one side and strictly concave on the other, so c is an inflection point. In practice, inflection points of a twice differentiable function are located by looking for sign changes of f''.

Warning 6.49. The condition $f''(c) = 0$ alone does not guarantee an inflection point. For $f(x) = x^4$,

$$f''(x) = 12x^2 \geq 0,$$

so f is convex on all of $\mathbb{R}$, and 0 is not an inflection point even though $f''(0) = 0$.

Example 6.50. For $f(x) = x^3$,

$$f'(x) = 3x^2, \qquad f''(x) = 6x.$$

Since $f'' < 0$ on $(-\infty, 0)$ and $f'' > 0$ on $(0, \infty)$, the function is strictly concave on $(-\infty, 0)$, strictly convex on $(0, \infty)$, and has an inflection point at 0.

Example 6.51. An inflection point can also occur where the second derivative fails to exist. For $f(x) = \sqrt[3]{x}$,

$$f'(x) = \frac{1}{3x^{2/3}}, \qquad f''(x) = -\frac{2}{9x^{5/3}} \qquad (x \neq 0).$$

Thus $f'' > 0$ on $(-\infty, 0)$ and $f'' < 0$ on $(0, \infty)$, so f is strictly convex on $(-\infty, 0)$, strictly concave on $(0, \infty)$, and has an inflection point at 0, where $f''(0)$ does not exist.

6.8 Newton's method

Suppose that we want to solve $f(x) = 0$. If x_n is an approximation to a root, the tangent line to the graph of f at $(x_n, f(x_n))$ has equation

$$y = f(x_n) + f'(x_n)(x - x_n).$$

Its intersection with the x-axis is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Iterating this recipe is *Newton's method*. Provided the starting point is close enough to a simple root, the approximations converge extremely rapidly.

Theorem 6.52. *Let f be twice differentiable on an interval containing r, with f'' bounded in a neighbourhood of r. Suppose that*

$$f(r) = 0, \qquad f'(r) \neq 0.$$

Then there exist $\delta > 0$ and $C > 0$ such that for every $x_0 \in (r - \delta, r + \delta)$ the Newton iterates are defined, converge to r, and satisfy

$$|x_{n+1} - r| \leq C\,|x_n - r|^2 \qquad \text{for every } n \geq 0.$$

The proof uses the convergence of sequences, which is developed in Chapter 12; the present discussion is a preview of the machinery established there.

Proof. Since f' is differentiable at r, it is continuous there. Because $f'(r) \neq 0$, there exist $\rho > 0$ and $m > 0$ with

$$|f'(x)| \geq m \qquad \text{for } |x - r| \leq \rho.$$

Choose $M \geq 1$ with $|f''(x)| \leq M$ for $|x - r| \leq \rho$. By Corollary 6.14 applied to f',

$$|f'(u) - f'(v)| \leq M\,|u - v| \qquad \text{for } u, v \in [r - \rho, r + \rho].$$

If $x_n = r$, then $f(x_n) = 0$, so the Newton recipe gives $x_{n+1} = x_n = r$: once an iterate hits the root, all later iterates stay there. Suppose now that $x_n \in [r - \rho, r + \rho]$ with $x_n \neq r$, so that $f'(x_n) \neq 0$ and the next iterate is defined. By the Mean Value Theorem there exists ξ_n between x_n and r with

$$f(x_n) = f(x_n) - f(r) = f'(\xi_n)(x_n - r).$$

Therefore

$$\begin{aligned} x_{n+1} - r &= x_n - r - \frac{f(x_n)}{f'(x_n)} \\ &= \frac{f'(x_n)(x_n - r) - f(x_n)}{f'(x_n)} \\ &= \frac{(f'(x_n) - f'(\xi_n))(x_n - r)}{f'(x_n)}. \end{aligned}$$

Taking absolute values,

$$|x_{n+1} - r| \leq \frac{M}{m}\,|x_n - r|^2.$$

Set $C = M/m$, and choose $\delta > 0$ with $\delta \le \rho$ and $C\delta < 1$. If $|x_0 - r| \le \delta$, then

$$|x_1 - r| \le C\,|x_0 - r|^2 \le C\delta\,|x_0 - r| < |x_0 - r| \le \delta.$$

By induction, every x_n satisfies $|x_n - r| \le \delta$, the iterates are all defined, and

$$|x_n - r| \le (C\delta)^n\,|x_0 - r| \to 0.$$

Thus $x_n \to r$, and the quadratic estimate

$$|x_{n+1} - r| \le C\,|x_n - r|^2$$

holds at every step. □

Example 6.53. Take $f(x) = x^2 - 2$ and $x_0 = 1$. Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Thus

$$x_1 = \frac{3}{2}, \qquad x_2 = \frac{17}{12} \approx 1.41667, \qquad x_3 = \frac{577}{408} \approx 1.414216,$$

while $\sqrt{2} \approx 1.414214$. Each step roughly doubles the number of correct decimal digits, as predicted by the quadratic estimate in Theorem 6.52.

Remark 6.54. Newton's method can fail if $f'(x_n) = 0$, or if the starting point is not close enough to the root: the sequence may leave the good region, oscillate, or diverge. Theorem 6.52 guarantees convergence only locally.

6.9 Exercises

Exercise 6.55. Use Theorem 6.3 to show that a differentiable function cannot have a local extremum at an interior point where its derivative is nonzero.

Exercise 6.56. Use the Mean Value Theorem to prove that

$$|\sin x - \sin y| \le |x - y|$$

for all $x, y \in \mathbb{R}$.

Exercise 6.57. Determine the absolute maximum and minimum of

$$f(x) = x^3 - 3x$$

on $[-2, 3]$.

Exercise 6.58. Sketch the graph of

$$f(x) = \frac{x}{1 + x^2}.$$

Identify all critical points, local extrema, intervals of increase and decrease, intervals of convexity and concavity, and inflection points.

Exercise 6.59. Let f be differentiable on an interval I, and let $c \in I$ be an interior point at which both one-sided limits of f' exist. Use Theorem 6.15 to show that both one-sided limits equal $f'(c)$. In particular, a derivative can never have a jump discontinuity.

Exercise 6.60. Use L'Hôpital's rule, after verifying its hypotheses, to compute

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

This is a subsequent application of the rule: the limit itself is established independently in Chapter 9 (where it follows from the derivative of sine), so no circularity is involved.

Exercise 6.61. Apply Newton's method to $f(x) = x^2 - 2$ with $x_0 = 1$, and compute x_1, x_2, x_3. Compare with $\sqrt{2} \approx 1.41421356$.

Exercise 6.62. Show that the function $f(x) = x^4$ is strictly convex on $\mathbb{R}$ directly from Proposition 6.36, without invoking its second derivative at 0.

Exercise 6.63. Use Jensen's inequality with $f(x) = 1/x$ on $(0, \infty)$ to prove that for positive $x_1, \dots, x_n$,

$$\frac{n}{x_1 + \cdots + x_n} \leq \frac{1}{n} \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right),$$

that is, the harmonic mean never exceeds the arithmetic mean.

Exercise 6.64. Let $f : I \to \mathbb{R}$ be convex, and suppose that f has a local minimum at $c \in I$. Prove that c is a global minimum point of f on I.

Exercise 6.65. Let $f : I \to \mathbb{R}$ be differentiable, and suppose that

$$f(x) \geq f(a) + f'(a)(x - a) \qquad \text{for all } a, x \in I.$$

Prove that f is convex on I. Thus the supporting-tangent property Proposition 6.40 characterizes convexity among differentiable functions.

Part III

Integral Calculus

Chapter 7

The Darboux Integral

The integral will be developed from first principles. The central idea is to approximate the area under a bounded function by rectangles. The Darboux formulation of the integral uses lower and upper rectangular approximations; it is equivalent to the more familiar Riemann integral, but is particularly well suited to rigorous proofs.

7.1 Motivation: area and rectangular approximation

Let $f : [a, b] \to \mathbb{R}$ be nonnegative. Informally, one would like to define the area of the region bounded by the graph $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. For a general function, this geometric description is not itself a definition.

The basic approximation is to partition $[a, b]$ into short intervals and, on each interval, replace the graph by a rectangle. A lower approximation uses the infimum of the function on each interval as the rectangle height; an upper approximation uses the supremum. If lower and upper approximations can be made arbitrarily close, their common limiting value is the integral.

The definition applies equally well to functions that take negative values. In that setting, the integral is a signed area: portions below the x-axis contribute negatively.

7.2 Partitions and Darboux sums

Definition 7.1. Let $a < b$. A *partition* of $[a, b]$ is a finite sequence

$$P = (t_0, t_1, \ldots, t_n)$$

with

$$a = t_0 < t_1 < \cdots < t_n = b.$$

For $i = 1, \ldots, n$, set

$$\Delta t_i = t_i - t_{i-1}.$$

Definition 7.2. Let $f : [a, b] \to \mathbb{R}$ be bounded, and let $P = (t_0, t_1, \ldots, t_n)$ be a partition of $[a, b]$. For $i = 1, \ldots, n$, define

$$m_i = \inf \{ f(x) : x \in [t_{i-1}, t_i] \}, \qquad M_i = \sup \{ f(x) : x \in [t_{i-1}, t_i] \}.$$

The *lower Darboux sum* and *upper Darboux sum* of f with respect to P are

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta t_i, \qquad U(f, P) = \sum_{i=1}^{n} M_i \Delta t_i.$$

The numbers m_i and M_i exist because f is bounded and each interval $[t_{i-1}, t_i]$ is nonempty. The lower sum is the total area of rectangles lying below the graph, whereas the upper sum is the total area of rectangles lying above the graph.

Lemma 7.3. *For every bounded $f : [a, b] \to \mathbb{R}$ and every partition P of $[a, b]$,*

$$L(f, P) \leq U(f, P).$$

Proof. For each i, we have $m_i \leq M_i$, and $\Delta t_i > 0$. Hence

$$m_i \Delta t_i \leq M_i \Delta t_i.$$

Summing over $i = 1, \ldots, n$ gives the result. □

7.3 Refinements of partitions

Definition 7.4. Let P and Q be partitions of $[a, b]$. We say that Q is a *refinement* of P if every partition point of P is also a partition point of Q.

Adding partition points can only increase lower sums and decrease upper sums.

Lemma 7.5. *Let*

$$P = (t_0, \ldots, t_k, t_{k+1}, \ldots, t_n)$$

be a partition, and let $u \in (t_k, t_{k+1})$. Let Q be the partition obtained by adding u to P. Then

$$L(f, P) \leq L(f, Q) \qquad \text{and} \qquad U(f, Q) \leq U(f, P).$$

Proof. All summands except the one corresponding to $[t_k, t_{k+1}]$ are unchanged. Let

$$\begin{aligned} m &= \inf \{f(x) : x \in [t_k, t_{k+1}]\}, \\ m_1 &= \inf \{f(x) : x \in [t_k, u]\}, \\ m_2 &= \inf \{f(x) : x \in [u, t_{k+1}]\}. \end{aligned}$$

Since

$$[t_k, u] \subseteq [t_k, t_{k+1}], \qquad [u, t_{k+1}] \subseteq [t_k, t_{k+1}],$$

we have $m \leq m_1$ and $m \leq m_2$. Therefore

$$\begin{aligned} m(t_{k+1} - t_k) &= m(u - t_k) + m(t_{k+1} - u) \\ &\leq m_1(u - t_k) + m_2(t_{k+1} - u). \end{aligned}$$

This proves $L(f, P) \leq L(f, Q)$.

The proof for upper sums is analogous: if M, M_1, M_2 are the respective suprema, then $M_1 \leq M$ and $M_2 \leq M$, which gives

$$M_1(u - t_k) + M_2(t_{k+1} - u) \leq M(t_{k+1} - t_k).$$

□

Corollary 7.6. *If Q is a refinement of P, then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. Apply Lemma 7.5 repeatedly, adding one new partition point at a time, and use Lemma 7.3. □

Definition 7.7. If P and Q are partitions of $[a,b]$, their *common refinement*, denoted by $P \cup Q$, is the partition whose points are all points belonging to either P or Q.

Lemma 7.8. *For any two partitions P and Q of $[a,b]$,*

$$L(f,P) \le U(f,Q).$$

Proof. The common refinement $P \cup Q$ refines both P and Q. Thus, by Corollary 7.6,

$$L(f,P) \le L(f,P\cup Q) \le U(f,P\cup Q) \le U(f,Q).$$

□

7.4 Definition of the Darboux integral

Definition 7.9. Let $f : [a,b] \to \mathbb{R}$ be bounded. The *lower integral* and *upper integral* of f on $[a,b]$ are defined by

$$\underline{\int_a^b} f := \sup\{L(f,P) : P \text{ is a partition of } [a,b]\},$$

and

$$\overline{\int_a^b} f := \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}.$$

The two quantities exist. Indeed, the set of lower sums is nonempty and bounded above by every upper sum, by Lemma 7.8. Similarly, the set of upper sums is nonempty and bounded below by every lower sum.

Lemma 7.10. *For every bounded $f : [a,b] \to \mathbb{R}$,*

$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

Moreover, for every partition P,

$$L(f,P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P).$$

Proof. The first and last inequalities follow directly from the definitions of supremum and infimum. The middle inequality follows from Lemma 7.8: every lower sum is at most every upper sum, so the supremum of the lower sums is at most the infimum of the upper sums. □

Definition 7.11. A bounded function $f : [a,b] \to \mathbb{R}$ is *Darboux integrable* on $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

The common value is the *Darboux integral* of f from a to b, denoted by

$$\int_a^b f(x)\,dx \qquad \text{or simply} \qquad \int_a^b f.$$

For convenience, define

$$\int_a^a f = 0.$$

Later, when $b < a$, we will use the convention

$$\int_a^b f := -\int_b^a f.$$

7.5 The Darboux criterion for integrability

Theorem 7.12 (Darboux criterion)**.** *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if, for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof. Suppose first that f is integrable, with

$$I = \int_a^b f.$$

By the definition of supremum, there is a partition P_1 with

$$L(f, P_1) > I - \frac{\varepsilon}{2}.$$

By the definition of infimum, there is a partition P_2 with

$$U(f, P_2) < I + \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. By Corollary 7.6,

$$L(f, P) \geq L(f, P_1) > I - \frac{\varepsilon}{2},$$

and

$$U(f, P) \leq U(f, P_2) < I + \frac{\varepsilon}{2}.$$

Consequently,

$$U(f, P) - L(f, P) < \varepsilon.$$

Conversely, suppose that for every $\varepsilon > 0$, there is a partition P with

$$U(f, P) - L(f, P) < \varepsilon.$$

By Lemma 7.10,

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(f, P) - L(f, P) < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, the difference between the upper and lower integrals is 0. Thus they are equal, and f is integrable. □

Corollary 7.13. *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if, for every $n \in \mathbb{N}$, there exists a partition P_n such that*

$$U(f, P_n) - L(f, P_n) < \frac{1}{n}.$$

Proof. This is an immediate consequence of Theorem 7.12 and the fact that, for every $\varepsilon > 0$, there is $n \in \mathbb{N}$ with $1/n < \varepsilon$. □

7.6 ‡ The Riemann integral

The Darboux integral is defined through lower and upper sums. The more familiar *Riemann integral* approximates by sampling the function at one point of each subinterval. The two definitions are equivalent, as this section proves; it supplies the justification for the claim made in the chapter introduction.

Definition 7.14. Let $f : [a, b] \to \mathbb{R}$ be bounded. A *tagged partition* of $[a, b]$ is a partition

$$P = (t_0, t_1, \dots, t_n)$$

together with tags $\xi_i \in [t_{i-1}, t_i]$ for each i. The *Riemann sum* of f for the tagged partition is

$$R(f; P, \xi) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

and the *mesh* of P is

$$\|P\| = \max_{1 \le i \le n} (t_i - t_{i-1}).$$

Definition 7.15. The function f is *Riemann integrable* on $[a, b]$ with integral $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|R(f; P, \xi) - L| < \varepsilon$$

for every tagged partition with $\|P\| < \delta$.

Lemma 7.16. *For every tagged partition,*

$$L(f, P) \le R(f; P, \xi) \le U(f, P).$$

Proof. On each subinterval $[t_{i-1}, t_i]$, the number $f(\xi_i)$ lies between the infimum and supremum of f there (Definition 7.2). Summing the resulting inequalities

$$m_i(t_i - t_{i-1}) \le f(\xi_i)(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})$$

over i gives the claim. □

Lemma 7.17. *Let f be Darboux integrable on $[a, b]$. Choose $M > 0$ such that $|f| \le M$ on $[a, b]$; such an M exists because a Darboux integrable function is bounded, and if $f \equiv 0$ we take $M = 1$. Then*

$$\lim_{\|P\| \to 0} U(f, P) = \int_a^b f \qquad \text{and} \qquad \lim_{\|P\| \to 0} L(f, P) = \int_a^b f.$$

Proof. Let $\varepsilon > 0$. Since the upper integral of f equals $\int_a^b f$ (Definition 7.9), there is a partition Q with

$$U(f, Q) < \int_a^b f + \frac{\varepsilon}{2}.$$

Let k be the number of partition points of Q lying strictly inside (a, b), and set

$$\delta = \frac{\varepsilon}{4Mk}$$

if $k > 0$, and $\delta = 1$ if $k = 0$. Let P be any partition with $\|P\| < \delta$.

The common refinement $P \cup Q$ is obtained from P by adding at most k points. Adding one point changes the upper sum by at most $2M \|P\|$: the new point splits one subinterval of length at most $\|P\|$ into two parts, and since every supremum involved has absolute value at most M, the new contribution differs from the old one by at most $2M$ times the length of the split subinterval. Hence

$$U(f, P) \leq U(f, P \cup Q) + 2Mk \|P\| .$$

Since $P \cup Q$ refines Q, Corollary 7.6 gives

$$U(f, P \cup Q) \leq U(f, Q).$$

Combining the estimates,

$$U(f, P) \leq U(f, Q) + 2Mk \|P\| < \int_a^b f + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \int_a^b f + \varepsilon.$$

On the other hand, $U(f, P) \geq \int_a^b f$ always, because the upper integral is the infimum of the upper sums. Thus

$$\left| U(f, P) - \int_a^b f \right| < \varepsilon$$

whenever $\|P\| < \delta$, proving the first limit. The second follows by applying the first to $-f$. □

Theorem 7.18. *A bounded function $f : [a, b] \to \mathbb{R}$ is Darboux integrable if and only if it is Riemann integrable, and the two integrals coincide.*

Proof. Suppose first that f is Darboux integrable with integral I. Given $\varepsilon > 0$, Lemma 7.17 provides $\delta > 0$ such that

$$I - \varepsilon < L(f, P) \leq U(f, P) < I + \varepsilon$$

whenever $\|P\| < \delta$. By Lemma 7.16, every Riemann sum for such a partition lies between $L(f, P)$ and $U(f, P)$, hence within ε of I. Thus f is Riemann integrable with integral I.

Conversely, suppose that f is Riemann integrable with integral L. Given $\varepsilon > 0$, choose $\delta > 0$ such that every tagged partition with $\|P\| < \delta$ satisfies

$$|R(f; P, \xi) - L| < \frac{\varepsilon}{2}.$$

Fix such a partition P. Taking the supremum of the Riemann sums over all choices of tags gives

$$U(f, P) \leq L + \frac{\varepsilon}{2},$$

and taking the infimum gives

$$L(f, P) \geq L - \frac{\varepsilon}{2}.$$

Hence

$$U(f, P) - L(f, P) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the Darboux criterion (Theorem 7.12) shows that f is Darboux integrable. Its integral equals L, because the upper and lower integrals both lie between $L - \varepsilon$ and $L + \varepsilon$ for every $\varepsilon > 0$. □

This completes the proof of the equivalence asserted at the beginning of the chapter: the Darboux integral and the Riemann integral are two descriptions of the same limiting process.

7.7 First examples

Example 7.19 (Constant functions)**.** Let $f(x) = c$ on $[a, b]$. Then f is integrable and

$$\int_a^b c\,dx = c(b-a).$$

Proof. For every partition $P = (t_0, \dots, t_n)$, we have $m_i = M_i = c$. Hence

$$\begin{aligned} L(f,P) = U(f,P) &= \sum_{i=1}^{n} c(t_i - t_{i-1}) \\ &= c\sum_{i=1}^{n}(t_i - t_{i-1}) \\ &= c(b-a). \end{aligned}$$

Thus the lower and upper integrals are equal to $c(b-a)$. □

Example 7.20 (The identity function)**.** The function $f(x) = x$ is integrable on $[0, 1]$, and

$$\int_0^1 x\,dx = \frac{1}{2}.$$

Proof. For $n \in \mathbb{N}$, let

$$P_n = \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right).$$

Since $f(x) = x$ is increasing, on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ we have

$$m_i = \frac{i-1}{n}, \qquad M_i = \frac{i}{n}.$$

Therefore

$$\begin{aligned} L(f,P_n) &= \sum_{i=1}^{n} \frac{i-1}{n}\frac{1}{n} = \frac{1}{n^2}\sum_{j=0}^{n-1} j = \frac{n-1}{2n}, \\ U(f,P_n) &= \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{n+1}{2n}. \end{aligned}$$

Hence

$$U(f,P_n) - L(f,P_n) = \frac{1}{n} \to 0.$$

By Theorem 7.12, f is integrable. Since

$$L(f,P_n) = \frac{1}{2} - \frac{1}{2n}, \qquad U(f,P_n) = \frac{1}{2} + \frac{1}{2n},$$

Lemma 7.10 gives

$$\frac{1}{2} - \frac{1}{2n} \le \int_0^1 x\,dx \le \frac{1}{2} + \frac{1}{2n}$$

for every n. Letting $n \to \infty$ yields

$$\int_0^1 x\,dx = \frac{1}{2}.$$

□

Example 7.21 (Dirichlet's function is not integrable)**.** Let

$$d(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Then d is not Darboux integrable on any nondegenerate interval $[a, b]$.

Proof. Every subinterval of $[a, b]$ contains both rational and irrational numbers. Thus, for every partition P,

$$m_i = 0, \qquad M_i = 1$$

for every subinterval of the partition. Therefore

$$L(d, P) = 0, \qquad U(d, P) = b - a.$$

Since $a < b$, the lower and upper integrals are different. Hence d is not integrable. □

Example 7.22 (A discontinuous integrable function)**.** Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then f is integrable on $[0, 1]$, and

$$\int_0^1 f(x)\,dx = 0.$$

Proof. Every interval contains a point that is not of the form $1/n$, so every lower Darboux sum is 0. Hence the lower integral is 0.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $2/N < \varepsilon$. Pick a point u with

$$\frac{1}{N+1} < u < \frac{1}{N}.$$

Then the interval $[0, u]$, of length $u < 1/N$, contains the spikes $1/(N+1), 1/(N+2), \ldots$ but does not reach the spike $1/N$. The remaining spikes $1/N, 1/(N-1), \ldots, 1$ are finitely many points of $[u, 1]$. Cover each of them by a small interval, pairwise disjoint, with total length less than $1/N$: for the spike 1, take a half-open interval $J_1 = (u_1, 1]$, and for each spike $1/k$ with $2 \le k \le N$ take an open interval $J_k = (u_k, v_k)$ containing $1/k$, with all these intervals lying in $(u, 1]$ and disjoint.

Let P be the partition of $[0, 1]$ whose points are 0, 1, u, and the endpoints of all the J_k. Every subinterval of P is contained either in $[0, u]$, or in the closure $\overline{J_k}$ of some J_k, or in a gap between these intervals. The gaps contain no spike, so the only subintervals on which the supremum of f is 1 are $[0, u]$ and the $\overline{J_k}$. Since $u < 1/N$ and the J_k have total length less than $1/N$, the total length of these subintervals is less than $1/N + 1/N = 2/N$, and therefore

$$U(f, P) < \frac{2}{N} < \varepsilon.$$

Since $L(f, P) = 0$, Theorem 7.12 shows that f is integrable, and its integral is 0. □

7.8 A useful estimate

Lemma 7.23. *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable, and that*

$$m \le f(x) \le M \qquad \text{for every } x \in [a, b].$$

Then

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Proof. For every partition P, each lower rectangle height is at least m, and each upper rectangle height is at most M. Thus

$$m(b - a) \le L(f, P) \le \int_a^b f \le U(f, P) \le M(b - a).$$

□

7.9 Exercises

Exercise 7.24. Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove directly that the set of lower Darboux sums is bounded above and the set of upper Darboux sums is bounded below.

Exercise 7.25. Prove Corollary 7.6 by induction on the number of points added to a partition.

Exercise 7.26. Show that

$$\int_0^b x\,dx = \frac{b^2}{2} \qquad \text{for every } b > 0.$$

Use equally spaced partitions.

Exercise 7.27. Use equally spaced partitions to prove that

$$\int_0^b x^2\,dx = \frac{b^3}{3} \qquad \text{for every } b > 0.$$

Exercise 7.28. Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove that f is integrable if and only if, for every $\varepsilon > 0$, there are partitions P and Q such that

$$U(f, Q) - L(f, P) < \varepsilon.$$

7.10 Additivity over adjacent intervals

The integral over an interval may be decomposed at any intermediate point. This property is fundamental both for calculations and for the interpretation of the integral as signed area.

Theorem 7.29 (Additivity over adjacent intervals)**.** *Let $a < c < b$, and let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if it is integrable on both $[a, c]$ and $[c, b]$. In that case,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. Suppose first that f is integrable on $[a,b]$. Let $\varepsilon > 0$. By Theorem 7.12, there is a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Refine P, if necessary, by adding the point c. By Corollary 7.6, the difference between upper and lower sums does not increase. Thus we may assume that c is a point of P.

Write P_1 for the restriction of P to $[a,c]$, and P_2 for the restriction of P to $[c,b]$. Then

$$L(f,P) = L(f,P_1) + L(f,P_2), \qquad U(f,P) = U(f,P_1) + U(f,P_2).$$

Consequently,

$$\begin{aligned} 0 \le U(f,P_1) - L(f,P_1) \le U(f,P) - L(f,P) < \varepsilon, \\ 0 \le U(f,P_2) - L(f,P_2) \le U(f,P) - L(f,P) < \varepsilon. \end{aligned}$$

The Darboux criterion shows that f is integrable on both subintervals.

Conversely, suppose that f is integrable on $[a,c]$ and $[c,b]$. Let $\varepsilon > 0$. Choose partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ satisfying

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}, \qquad U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}.$$

The union $P = P_1 \cup P_2$ is a partition of $[a,b]$, and

$$\begin{aligned} U(f,P) - L(f,P) &= \big(U(f,P_1) - L(f,P_1)\big) + \big(U(f,P_2) - L(f,P_2)\big) \\ &< \varepsilon. \end{aligned}$$

Hence f is integrable on $[a,b]$.

It remains to prove the stated equality. For any partition P of $[a,b]$ that contains c, with induced partitions P_1, P_2, we have

$$L(f,P_1) \le \int_a^c f \le U(f,P_1), \qquad L(f,P_2) \le \int_c^b f \le U(f,P_2).$$

Adding gives

$$L(f,P) \le \int_a^c f + \int_c^b f \le U(f,P).$$

Taking the supremum of lower sums and the infimum of upper sums yields

$$\int_a^b f \le \int_a^c f + \int_c^b f \le \int_a^b f.$$

Therefore equality holds. □

Corollary 7.30. *Let $a,b,c \in \mathbb{R}$, and suppose that all relevant integrals exist. With the convention*

$$\int_b^a f := -\int_a^b f,$$

we have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

7.11 Linearity of the integral

Theorem 7.31 (Linearity)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be integrable, and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is integrable and*

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Proof. We first prove closure under addition. Let $P = (t_0, \ldots, t_n)$ be a partition of $[a, b]$. Write

$$m_i^f = \inf\{f(x) : x \in [t_{i-1}, t_i]\}, \qquad M_i^f = \sup\{f(x) : x \in [t_{i-1}, t_i]\},$$
$$m_i^g = \inf\{g(x) : x \in [t_{i-1}, t_i]\}, \qquad M_i^g = \sup\{g(x) : x \in [t_{i-1}, t_i]\}.$$

For every $x \in [t_{i-1}, t_i]$,

$$f(x) + g(x) \geq m_i^f + m_i^g, \qquad f(x) + g(x) \leq M_i^f + M_i^g.$$

Therefore

$$L(f + g, P) \geq L(f, P) + L(g, P), \qquad U(f + g, P) \leq U(f, P) + U(g, P).$$

Let $\varepsilon > 0$. Choose partitions P_1, P_2 satisfying

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2},$$
$$U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Refinement gives

$$U(f, P) - L(f, P) < \frac{\varepsilon}{2},$$
$$U(g, P) - L(g, P) < \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned} U(f + g, P) - L(f + g, P) &\leq U(f, P) + U(g, P) - L(f, P) - L(g, P) \\ &< \varepsilon. \end{aligned}$$

By the Darboux criterion, $f + g$ is integrable.

Let

$$I_f = \int_a^b f, \qquad I_g = \int_a^b g.$$

For a partition P chosen as above, the inequalities for the Darboux sums also give

$$U(f + g, P) \leq U(f, P) + U(g, P) \leq I_f + I_g + \varepsilon,$$
$$L(f + g, P) \geq L(f, P) + L(g, P) \geq I_f + I_g - \varepsilon.$$

Since

$$L(f + g, P) \leq \int_a^b (f + g) \leq U(f + g, P),$$

we obtain

$$\left| \int_a^b (f + g) - I_f - I_g \right| \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary,

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Next consider scalar multiplication. If $\alpha > 0$, then for every partition P,

$$L(\alpha f, P) = \alpha L(f, P), \qquad U(\alpha f, P) = \alpha U(f, P).$$

The Darboux criterion and the definitions of upper and lower integrals then imply

$$\int_a^b \alpha f = \alpha \int_a^b f.$$

If $\alpha < 0$, then

$$L(\alpha f, P) = \alpha U(f, P), \qquad U(\alpha f, P) = \alpha L(f, P),$$

and the same conclusion follows. The case $\alpha = 0$ is immediate.

The general formula follows by combining scalar multiplication and addition. □

7.12 Order properties and comparison

Theorem 7.32 (Comparison theorem)**.** *Let $f, g : [a, b] \to \mathbb{R}$ be integrable. If*

$$f(x) \le g(x) \qquad \text{for every } x \in [a, b],$$

then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof. For every partition P, the lower and upper heights satisfy

$$m_i^f \le m_i^g, \qquad M_i^f \le M_i^g.$$

In particular,

$$L(f, P) \le L(g, P), \qquad U(f, P) \le U(g, P).$$

Since

$$\int_a^b f \le U(f, P) \le U(g, P)$$

for every partition P, taking the infimum over P gives

$$\int_a^b f \le \int_a^b g.$$

□

Corollary 7.33. *If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \ge 0$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \ge 0.$$

Proof. Apply Theorem 7.32 to the zero function and f. □

Corollary 7.34. *If $f : [a, b] \to \mathbb{R}$ is integrable and*

$$m \le f(x) \le M \qquad \text{for all } x \in [a, b],$$

then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Proof. Apply Theorem 7.32 to the constant functions m and M, then use Example 7.19. □

7.13 Changing finitely many values

Lemma 7.35. *Fix $c \in [a,b]$. Define*

$$\mathbf{1}_{\{c\}}(x) = \begin{cases} 1, & x = c, \\ 0, & x \neq c. \end{cases}$$

Then $\mathbf{1}_{\{c\}}$ is integrable on $[a,b]$, and

$$\int_a^b \mathbf{1}_{\{c\}}(x)\,dx = 0.$$

Proof. Every lower Darboux sum is 0, because every nondegenerate subinterval contains a point distinct from c. Thus the lower integral is 0.

Let $\varepsilon > 0$. Choose a partition containing points $u < c < v$ such that $v - u < \varepsilon$, with the obvious one-sided modification if $c = a$ or $c = b$. The upper sum is at most $v - u < \varepsilon$, because only the subinterval containing c can have supremum 1. Thus the upper integral is also 0, and the function is integrable. □

Theorem 7.36 (Finite modification theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be integrable. Suppose $g : [a,b] \to \mathbb{R}$ differs from f at only finitely many points. Then g is integrable and*

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. Let $c_1, \dots, c_N$ be the points at which f and g differ. Then

$$g(x) = f(x) + \sum_{j=1}^{N} (g(c_j) - f(c_j))\mathbf{1}_{\{c_j\}}(x).$$

By Lemma 7.35, every singleton indicator is integrable with integral 0. Linearity, Theorem 7.31, now implies that g is integrable and

$$\int_a^b g = \int_a^b f + \sum_{j=1}^{N} (g(c_j) - f(c_j)) \int_a^b \mathbf{1}_{\{c_j\}} = \int_a^b f.$$

□

7.14 Monotone functions are integrable

Theorem 7.37 (Monotone functions are integrable)**.** *Every monotone function $f : [a,b] \to \mathbb{R}$ is Darboux integrable on $[a,b]$.*

Proof. It suffices to treat a nondecreasing f: a nonincreasing f becomes nondecreasing after applying $-f$, and Theorem 7.31 transfers integrability. A nondecreasing f is bounded on $[a,b]$, since $f(a) \le f(x) \le f(b)$ for every x, so the Darboux integral of f is at least defined.

For each $n \in \mathbb{N}$, let $P_n = (t_0, t_1, \dots, t_n)$ be the equally spaced partition of $[a,b]$, with

$$t_i = a + \frac{i}{n}(b-a), \qquad h = t_i - t_{i-1} = \frac{b-a}{n}.$$

Because f is nondecreasing, on each subinterval $[t_{i-1}, t_i]$ the infimum is attained at the left endpoint and the supremum at the right endpoint:

$$m_i = f(t_{i-1}), \qquad M_i = f(t_i).$$

Hence

$$\begin{aligned} U(f,P_n)-L(f,P_n) &= \sum_{i=1}^{n}\big(f(t_i)-f(t_{i-1})\big)h \\ &= h\sum_{i=1}^{n}\big(f(t_i)-f(t_{i-1})\big) \\ &= h\big(f(b)-f(a)\big) = \frac{b-a}{n}\big(f(b)-f(a)\big), \end{aligned}$$

where the third equality is a telescoping sum. Since $f(b)-f(a)\geq 0$, this tends to 0 as $n\to\infty$. By the Darboux criterion (Theorem 7.12), f is integrable. □

Example 7.38 (The floor function)**.** The floor function $f(x)=\lfloor x\rfloor$ is nondecreasing, so it is Darboux integrable on every closed interval $[a,b]$. For instance, split $[0,3]$ at 1 and 2. On $[0,1]$, $[1,2]$, $[2,3]$, the function $\lfloor x\rfloor$ differs from the constant 0, 1, 2 respectively only at the single point 1, 2, 3. Hence, by Theorem 7.36, additivity (Theorem 7.29), and Example 7.19,

$$\int_0^3 \lfloor x\rfloor\,dx = \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx = 0+1+2 = 3.$$

7.15 Absolute values and products

Lemma 7.39. *Let $A\subseteq\mathbb{R}$ be nonempty and bounded. Then*

$$\sup\{|x| : x\in A\} - \inf\{|x| : x\in A\} \leq \sup A - \inf A.$$

Proof. For every $x,y\in A$, the reverse triangle inequality gives

$$\big||x|-|y|\big| \leq |x-y|.$$

Since $\inf A\leq x,y\leq \sup A$, we have

$$|x-y|\leq \sup A - \inf A.$$

Taking the supremum over $x,y\in A$ yields the result. □

Theorem 7.40. *If $f:[a,b]\to\mathbb{R}$ is integrable, then $|f|$ is integrable.*

Proof. Let $P=(t_0,\dots,t_n)$ be a partition. On each subinterval $[t_{i-1},t_i]$, Lemma 7.39 gives

$$M_i^{|f|} - m_i^{|f|} \leq M_i^f - m_i^f.$$

Therefore

$$U(|f|,P)-L(|f|,P)\leq U(f,P)-L(f,P).$$

Given $\varepsilon>0$, choose a partition P for which

$$U(f,P)-L(f,P)<\varepsilon.$$

Then the same partition satisfies

$$U(|f|,P)-L(|f|,P)<\varepsilon.$$

The Darboux criterion proves integrability of $|f|$. □

Theorem 7.41. *If $f, g : [a, b] \to \mathbb{R}$ are integrable, then fg is integrable.*

Proof. We first assume that $f \geq 0$ and $g \geq 0$. Let M_f and M_g be upper bounds for f and g, respectively. On each interval of a partition P, we have

$$M_i^{fg} \leq M_i^f M_i^g, \qquad m_i^{fg} \geq m_i^f m_i^g.$$

Thus

$$\begin{aligned} M_i^{fg} - m_i^{fg} &\leq M_i^f M_i^g - m_i^f m_i^g \\ &= (M_i^f - m_i^f) M_i^g + m_i^f (M_i^g - m_i^g) \\ &\leq M_g (M_i^f - m_i^f) + M_f (M_i^g - m_i^g). \end{aligned}$$

Multiplying by Δt_i and summing gives

$$U(fg, P) - L(fg, P) \leq M_g \big(U(f, P) - L(f, P)\big) + M_f \big(U(g, P) - L(g, P)\big).$$

Choose a common refinement on which the two differences on the right are sufficiently small. The Darboux criterion proves that fg is integrable.

For general f and g, define their positive and negative parts by

$$f^+ = \max\{f, 0\} = \frac{|f| + f}{2}, \qquad f^- = \max\{-f, 0\} = \frac{|f| - f}{2}.$$

Both f^+ and f^- are nonnegative, and

$$f = f^+ - f^-.$$

Since

$$f^+ = \frac{|f| + f}{2}, \qquad f^- = \frac{|f| - f}{2},$$

Theorems 7.31 and 7.40 show that f^+, f^- are integrable; similarly for g^+, g^-. Moreover,

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+ g^+ + f^- g^- - f^+ g^- - f^- g^+.$$

Each of the four products on the right is a product of nonnegative integrable functions, and hence is integrable by the first part of the proof. Linearity now gives integrability of fg. □

7.16 Integral inequalities

Theorem 7.42 (Triangle inequality for integrals)**.** *If $f : [a, b] \to \mathbb{R}$ is integrable, then*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof. Since

$$-|f(x)| \leq f(x) \leq |f(x)|$$

for every $x \in [a, b]$, the comparison theorem gives

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

This is equivalent to the stated inequality. □

Theorem 7.43 (Cauchy–Schwarz inequality)**.** *If $f, g : [a, b] \to \mathbb{R}$ are integrable, then*

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right).$$

Proof. Set

$$A = \int_a^b g(x)^2\,dx, \qquad B = \int_a^b f(x)g(x)\,dx, \qquad C = \int_a^b f(x)^2\,dx.$$

For every $t \in \mathbb{R}$, the function $(f - tg)^2$ is nonnegative. Therefore, by Corollary 7.33 and linearity,

$$0 \le \int_a^b (f(x) - tg(x))^2\,dx = C - 2tB + t^2A.$$

If $A = 0$, then the inequality

$$0 \le C - 2tB + t^2A = C - 2tB$$

holds for every $t \in \mathbb{R}$. If $B \neq 0$, the function $t \mapsto C - 2tB$ is a nonconstant affine function, which is unbounded below on $\mathbb{R}$ and hence cannot be nonnegative for all real t. Therefore $B = 0$, and the desired conclusion $B^2 \le AC$ holds with equality. Assume $A > 0$. The quadratic polynomial

$$At^2 - 2Bt + C$$

is nonnegative for every real t. Its discriminant is therefore nonpositive:

$$(-2B)^2 - 4AC \le 0.$$

Thus $B^2 \le AC$, as required. □

7.17 Exercises

Exercise 7.44. Prove the additivity formula in Theorem 7.29 first for lower and upper Darboux sums, then deduce the integral formula.

Exercise 7.45. Let f, g be integrable on $[a, b]$. Prove directly from the Darboux criterion that $f - g$ is integrable and

$$\int_a^b (f - g) = \int_a^b f - \int_a^b g.$$

Exercise 7.46. Suppose that f is integrable and $f(x) \ge m$ for every $x \in [a, b]$. Prove that

$$\int_a^b f(x)\,dx \ge m(b - a).$$

Exercise 7.47. Prove that changing an integrable function at a single point does not change its Darboux integral, directly from the Darboux criterion.

Exercise 7.48. Prove the affine-function fact used in the zero case of the proof of Theorem 7.43: if $A = 0$ and

$$C - 2tB + t^2A \ge 0 \qquad \text{for every } t \in \mathbb{R},$$

then $B = 0$. (Hint: if $B \neq 0$, the expression is a nonconstant affine function of t, hence unbounded below.)

Exercise 7.49. Use Theorem 7.43 to prove that

$$\left|\int_0^1 f(x)\,dx\right| \le \left(\int_0^1 f(x)^2\,dx\right)^{1/2}$$

for every integrable $f : [0, 1] \to \mathbb{R}$.

7.18 Uniform continuity

The next goal is to prove that every continuous function on a closed interval is Darboux integrable. The relevant strengthening of continuity is uniform continuity.

Definition 7.50. Let $f : I \to \mathbb{R}$, where I is an interval. The function f is *uniformly continuous on* I if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in I$,

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Uniform continuity differs from ordinary continuity in the order of the quantifiers. In ordinary continuity, the number δ may depend on both ε and the point at which continuity is considered. Under uniform continuity, for each fixed ε, one value of δ works for every pair of points in the interval.

Theorem 7.51. *If $f : [a, b] \to \mathbb{R}$ is bounded and uniformly continuous, then f is Darboux integrable on $[a, b]$.*

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{b - a}$$

for all $x, y \in [a, b]$.

Choose a partition

$$P = (t_0, t_1, \ldots, t_n)$$

of $[a, b]$ such that every subinterval has length less than δ:

$$t_i - t_{i-1} < \delta \qquad (i = 1, \ldots, n).$$

For every $x, y \in [t_{i-1}, t_i]$, we then have

$$|x - y| < \delta,$$

and hence

$$|f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

It follows that

$$M_i - m_i \leq \frac{\varepsilon}{b - a}$$

for every subinterval of the partition. Therefore

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) \\ &\leq \frac{\varepsilon}{b - a} \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= \varepsilon. \end{aligned}$$

By Theorem 7.12, f is integrable. □

Corollary 7.52. *Every continuous function $f : [a, b] \to \mathbb{R}$ is Darboux integrable on $[a, b]$.*

Proof. By the Extreme Value Theorem, f is bounded on $[a, b]$. By the Heine–Cantor Theorem, Theorem 4.32, f is uniformly continuous on $[a, b]$. The conclusion follows from Theorem 7.51. □

Theorem 7.53 (Piecewise-continuous functions are integrable)**.** *If $f : [a, b] \to \mathbb{R}$ is bounded and is continuous at every point of $[a, b]$ except possibly at finitely many points, then f is Darboux integrable on $[a, b]$.*

Proof. Let $c_1, \dots, c_N \in [a, b]$ be the finitely many exceptional points, and let $M > 0$ satisfy $|f| \le M$ on $[a, b]$. Given $\varepsilon > 0$, surround the exceptional points by small pairwise disjoint intervals $J_1, \dots, J_N \subseteq [a, b]$ of total length less than $\varepsilon/(4M)$, each containing its c_j (for an interior c_j, take an open interval (u_j, v_j) with $u_j < c_j < v_j$; for an endpoint a or b, take a half-open interval).

The set $K = [a, b] \setminus \bigcup_{j=1}^{N} J_j$ is a finite union of closed intervals $K_1, \dots, K_r$, and f is continuous on each K_ℓ. By the Heine–Cantor theorem (Theorem 4.32), f is uniformly continuous on each K_ℓ; taking the minimum of the finitely many δ supplied by uniform continuity yields $\delta > 0$ such that, for every ℓ and all $x, y \in K_\ell$,

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2(b - a)}.$$

Choose a partition P of $[a, b]$ whose points include a, b, and the endpoints of every J_j, and which is fine enough that every subinterval lying inside a single K_ℓ has length less than δ. Each subinterval of P is then either the closure $\overline{J_j}$ of some J_j, or is contained in a single K_ℓ.

On a subinterval of the first kind, the oscillation satisfies $M_i - m_i \le 2M$, and the total length of all such subintervals is less than $\varepsilon/(4M)$ (the total length of the J_j); their combined contribution to $U(f, P) - L(f, P)$ is less than $2M \cdot \varepsilon/(4M) = \varepsilon/2$. On a subinterval of the second kind, its length is less than δ, so

$$M_i - m_i \le \frac{\varepsilon}{2(b - a)};$$

the total length of all such subintervals is at most $b - a$, so their combined contribution is at most $\varepsilon/(2(b - a)) \cdot (b - a) = \varepsilon/2$. Therefore

$$U(f, P) - L(f, P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By the Darboux criterion (Theorem 7.12), f is integrable. □

7.19 The integral as an accumulation function

Let I be an interval, and suppose that f is integrable on every closed subinterval of I. Fix $a \in I$, and define

$$F(x) = \int_a^x f(t)\, dt.$$

The variable t is a dummy variable of integration; the output $F(x)$ is a number depending on the endpoint x.

Theorem 7.54. *Let $f : I \to \mathbb{R}$ be integrable on every closed subinterval of I. Fix $a \in I$, and define*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F is continuous on I: two-sidedly at every interior point of I, and one-sidedly at every endpoint that belongs to I.

Proof. Fix $c \in I$. Choose a closed interval $J \subseteq I$ containing c on which the continuity of F at c is to be tested: if c is an interior point of I, take $J = [u, v] \subseteq I$ with $u < c < v$; if c is a left endpoint of I, take $J = [c, v] \subseteq I$ with $c < v$; if c is a right endpoint of I, take $J = [u, c] \subseteq I$ with $u < c$. Since f is integrable on J, it is bounded there, so there is $M > 0$ such that

$$|f(t)| \leq M \qquad \text{for every } t \in J.$$

Let $x \in J$. By additivity of the integral (Corollary 7.30),

$$F(x) - F(c) = \int_c^x f(t)\,dt.$$

If $x \geq c$, then $[c, x] \subseteq J$, and

$$|F(x) - F(c)| = \left| \int_c^x f(t)\,dt \right| \leq \int_c^x |f(t)|\,dt \leq M(x - c) = M\,|x - c|\,,$$

using Theorem 7.42. If $x \leq c$, the same argument on $[x, c]$ gives

$$|F(x) - F(c)| \leq M(c - x) = M\,|x - c|\,.$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon/M$. Then for every $x \in J$,

$$|x - c| < \delta \quad \Longrightarrow \quad |F(x) - F(c)| < \varepsilon.$$

For an interior point c, the point x may approach c from either side, so this is two-sided continuity; for an endpoint, x is restricted to one side, so this is the corresponding one-sided continuity. □

7.20 The Fundamental Theorem of Calculus, Part I

Theorem 7.55 (Fundamental Theorem of Calculus, Part I)**.** *Let $f : I \to \mathbb{R}$ be integrable on every closed subinterval of I. Fix $a \in I$, and define*

$$F(x) = \int_a^x f(t)\,dt.$$

Suppose that f is continuous at $c \in I$. If c is an interior point of I, then F is differentiable at c and

$$F'(c) = f(c).$$

If c is a left endpoint of I, the right derivative of F at c exists and equals $f(c)$; if c is a right endpoint of I, the left derivative of F at c exists and equals $f(c)$.

Proof. We first consider the right-hand derivative. Let $h > 0$ be sufficiently small that $[c, c + h] \subseteq I$. By additivity,

$$\frac{F(c + h) - F(c)}{h} = \frac{1}{h} \int_c^{c+h} f(t)\,dt.$$

Let

$$m_h = \inf\{f(t) : t \in [c, c + h]\}\,, \qquad M_h = \sup\{f(t) : t \in [c, c + h]\}\,.$$

The integral bounds give

$$m_h \leq \frac{F(c + h) - F(c)}{h} \leq M_h.$$

We claim that

$$\lim_{h\to 0^+} m_h = f(c) \qquad \text{and} \qquad \lim_{h\to 0^+} M_h = f(c).$$

Let $\varepsilon > 0$. Since f is continuous at c, there is $\delta > 0$ such that

$$|t - c| < \delta \quad \Longrightarrow \quad |f(t) - f(c)| < \varepsilon.$$

If $0 < h < \delta$, then every $t \in [c, c+h]$ satisfies

$$f(c) - \varepsilon < f(t) < f(c) + \varepsilon.$$

Hence

$$f(c) - \varepsilon \le m_h \le M_h \le f(c) + \varepsilon.$$

This proves the claim. The squeeze theorem now gives

$$\lim_{h\to 0^+} \frac{F(c+h) - F(c)}{h} = f(c).$$

For the left-hand derivative, let $h > 0$ be sufficiently small that $[c-h, c] \subseteq I$. Again by additivity,

$$\frac{F(c) - F(c-h)}{h} = \frac{1}{h}\int_{c-h}^{c} f(t)\,dt.$$

The same argument, with the infimum and supremum of f on $[c-h, c]$, gives

$$\lim_{h\to 0^+} \frac{F(c) - F(c-h)}{h} = f(c).$$

This is equivalent to the left-hand derivative of F at c. If c is an interior point of I, both one-sided derivatives exist and equal $f(c)$, so F is differentiable at c with $F'(c) = f(c)$. If c is a left endpoint of I, only the right-hand computation is available and it yields the right derivative equal to $f(c)$; if c is a right endpoint of I, only the left-hand computation is available and it yields the left derivative equal to $f(c)$. □

Corollary 7.56. *If $f : [a, b] \to \mathbb{R}$ is continuous, then the function*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f on $[a, b]$; that is,

$$F'(x) = f(x)$$

at every interior point, with the corresponding one-sided statement at the endpoints.

7.21 The Fundamental Theorem of Calculus, Part II

Theorem 7.57 (Fundamental Theorem of Calculus, Part II)**.** *Let $f : [a, b] \to \mathbb{R}$ be integrable. Suppose that there is a function $G : [a, b] \to \mathbb{R}$, continuous on $[a, b]$ and differentiable on (a, b), such that*

$$G'(x) = f(x) \qquad \text{for every } x \in (a, b).$$

Then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Proof. Let $P = (t_0, t_1, \dots, t_n)$ be any partition of $[a, b]$. By the Mean Value Theorem, for each $i = 1, \dots, n$, there exists $\xi_i \in (t_{i-1}, t_i)$ such that

$$G(t_i) - G(t_{i-1}) = G'(\xi_i)(t_i - t_{i-1}) = f(\xi_i)(t_i - t_{i-1}).$$

Since

$$m_i \le f(\xi_i) \le M_i,$$

we have

$$m_i(t_i - t_{i-1}) \le G(t_i) - G(t_{i-1}) \le M_i(t_i - t_{i-1}).$$

Summing over i gives

$$L(f, P) \le \sum_{i=1}^{n} (G(t_i) - G(t_{i-1})) \le U(f, P).$$

The sum telescopes:

$$\sum_{i=1}^{n} (G(t_i) - G(t_{i-1})) = G(b) - G(a).$$

Therefore

$$L(f, P) \le G(b) - G(a) \le U(f, P)$$

for every partition P. Taking the supremum of lower sums and the infimum of upper sums, and using integrability of f, yields

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

□

Corollary 7.58 (Evaluation by antiderivatives)**.** *If G is continuous on $[a, b]$, $G' = f$ on (a, b), and f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

In standard notation,

$$\int_a^b f(x)\,dx = G(x)\Big|_a^b.$$

Remark 7.59. The hypothesis that G be continuous on the closed interval $[a, b]$ is essential, not a technicality: differentiability of G only on (a, b) does not constrain its endpoint values. For example, take $f = 0$ on $[0, 1]$, $G(0) = 1$, and $G(x) = 0$ for $0 < x \le 1$. Then $G' = f$ on $(0, 1)$, but

$$\int_0^1 f(x)\,dx = 0 \ne G(1) - G(0) = -1.$$

Example 7.60. For every integer $n \ge 0$,

$$\int_a^b x^n\,dx = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Proof. The function

$$G(x) = \frac{x^{n+1}}{n+1}$$

satisfies $G'(x) = x^n$. Apply Theorem 7.57. □

Theorem 7.61 (Mean Value Theorem for integrals)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in (a, b)$ such that*

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Proof. Consider the accumulation function

$$F(x) = \int_a^x f(t)\,dt.$$

By Theorem 7.54, F is continuous on $[a, b]$, and by Theorem 7.55 it is differentiable on (a, b) with

$$F'(x) = f(x).$$

Applying the Mean Value Theorem (Theorem 6.9) to F on $[a, b]$ gives a point $c \in (a, b)$ such that

$$F(b) - F(a) = F'(c)(b - a) = f(c)(b - a).$$

Since $F(b) = \int_a^b f$ and $F(a) = 0$, the result follows. □

Remark 7.62. The quantity

$$\frac{1}{b - a} \int_a^b f$$

is the average value of f on $[a, b]$, and Theorem 7.61 says that a continuous function attains its average value at some interior point. The theorem is the integral analogue of the Mean Value Theorem.

7.22 Improper integrals

The Darboux integral is defined only for bounded functions on closed bounded intervals. Important extensions are obtained by limits.

Definition 7.63. Suppose that f is integrable on $[a, R]$ for every $R > a$. The *improper integral* of f on $[a, \infty)$ is defined by

$$\int_a^\infty f(x)\,dx := \lim_{R \to \infty} \int_a^R f(x)\,dx,$$

provided the limit exists as a finite real number.

Definition 7.64. Suppose that f is integrable on $[a + \varepsilon, b]$ for every ε with $0 < \varepsilon < b - a$. The *improper integral* of f on $(a, b]$ is defined by

$$\int_a^b f(x)\,dx := \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx,$$

provided the limit exists as a finite real number.

Analogous definitions apply at the right endpoint b (a limit of $\int_a^{b-\varepsilon} f$) and, more generally, at any interior singularity: if f is singular only at $c \in (a, b)$, one sets

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

requiring each improper integral on the right to converge separately. By additivity of the ordinary integral, the value of an improper integral is independent of any intermediate base point chosen to compute the limit.

Example 7.65. For $p > 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}.$$

For $p \le 1$, the improper integral diverges.

Proof. For $p \neq 1$,

$$\int_1^R \frac{1}{x^p}\,dx = \frac{1 - R^{1-p}}{p-1}.$$

If $p > 1$, then $R^{1-p} \to 0$, yielding $1/(p-1)$. If $p < 1$, the quantity tends to $+\infty$. For $p = 1$,

$$\int_1^R \frac{1}{x}\,dx = \log R \to \infty.$$

□

Theorem 7.66 (Comparison test for improper integrals)**.** *Suppose that*

$$0 \le g(x) \le f(x) \qquad \text{for every } x \in [a, \infty),$$

and that f and g are integrable on every compact subinterval of $[a, \infty)$. If

$$\int_a^\infty f(x)\,dx$$

converges, then

$$\int_a^\infty g(x)\,dx$$

converges and

$$0 \le \int_a^\infty g(x)\,dx \le \int_a^\infty f(x)\,dx.$$

Proof. For $R > a$, comparison on $[a, R]$ gives

$$0 \le \int_a^R g(x)\,dx \le \int_a^R f(x)\,dx.$$

The left-hand side is nondecreasing in R, while the right-hand side is bounded above by the convergent improper integral of f. Therefore $\int_a^R g$ has a finite limit as $R \to \infty$, by completeness. Taking limits gives the stated inequality. □

Definition 7.67 (Two-sided improper integrals)**.** Suppose f is integrable on every closed subinterval of $\mathbb{R}$. The improper integrals over the half-lines are

$$\int_{-\infty}^c f(x)\,dx := \lim_{R\to\infty} \int_{-R}^c f(x)\,dx, \qquad \int_c^\infty f(x)\,dx := \lim_{R\to\infty} \int_c^R f(x)\,dx,$$

whenever the respective limit exists as a finite real number. For any fixed $c \in \mathbb{R}$, the *two-sided* improper integral of f over $\mathbb{R}$ is then

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

provided that both improper integrals on the right converge independently. The value does not depend on the choice of c, by additivity.

Warning 7.68. Symmetric cancellation is not convergence. Although

$$\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty} 0 = 0,$$

the two-sided improper integral $\int_{-\infty}^{\infty} x\,dx$ diverges: each of the one-sided integrals

$$\int_{-\infty}^{0} x\,dx \qquad \text{and} \qquad \int_{0}^{\infty} x\,dx$$

diverges (to $-\infty$ and $+\infty$, respectively). Consequently Definition 7.67 does not apply; one may not take $R\to\infty$ symmetrically in the two directions.

Theorem 7.69 (Cauchy criterion for improper integrals)**.** *Suppose that f is integrable on every compact subinterval of $[a,\infty)$. Then $\int_a^\infty f$ converges if and only if, for every $\varepsilon>0$, there exists $R_0>a$ such that*

$$R_0 < r < s \quad\Longrightarrow\quad \left|\int_r^s f(x)\,dx\right| < \varepsilon.$$

Proof. Set $F(R)=\int_a^R f(x)\,dx$. The improper integral converges exactly when $F(R)$ has a finite limit as $R\to\infty$, which is equivalent to the Cauchy condition for F at infinity: for every $\varepsilon>0$ there is $R_0>a$ with

$$|F(s)-F(r)|<\varepsilon \qquad \text{whenever } R_0<r<s.$$

By additivity,

$$F(s)-F(r)=\int_r^s f(x)\,dx,$$

so this is exactly the stated condition. □

Theorem 7.70 (Absolute convergence implies convergence)**.** *Suppose that f is integrable on every compact subinterval of $[a,\infty)$. If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges, and*

$$\left|\int_a^\infty f(x)\,dx\right| \le \int_a^\infty |f(x)|\,dx.$$

Proof. For $r<s$, Theorem 7.42 gives

$$\left|\int_r^s f(x)\,dx\right| \le \int_r^s |f(x)|\,dx.$$

Since $\int_a^\infty |f|$ converges, the Cauchy criterion (Theorem 7.69) applied to $|f|$ shows that the right-hand side is smaller than any prescribed ε whenever r,s are large enough; hence so is $|\int_r^s f|$. Applying the Cauchy criterion to f shows that $\int_a^\infty f$ converges. Finally, taking $R\to\infty$ in

$$\left|\int_a^R f(x)\,dx\right| \le \int_a^R |f(x)|\,dx,$$

valid for every $R>a$, yields the stated inequality. □

7.23 Exercises

Exercise 7.71. Let $f : [a, b] \to \mathbb{R}$ be uniformly continuous. Give a direct ε–δ proof that the oscillation of f on every sufficiently short interval is small.

Exercise 7.72. Use Theorem 7.55 to show that

$$\frac{d}{dx} \int_0^x \cos(t)\, dt = \cos x.$$

Exercise 7.73. Let

$$F(x) = \int_1^{x^2} \frac{dt}{1 + t^2}.$$

Compute $F'(x)$.

Exercise 7.74. Use Theorem 7.57 to compute

$$\int_0^1 (3x^2 - 4x + 2)\, dx.$$

Exercise 7.75. Determine whether each improper integral converges:

$$\int_1^\infty \frac{dx}{x^{3/2}}, \qquad \int_1^\infty \frac{dx}{\sqrt{x}}, \qquad \int_0^1 \frac{dx}{\sqrt{x}}.$$

Exercise 7.76. Show that

$$\int_0^1 \frac{dx}{1 + x^2}$$

converges, and evaluate it after the inverse tangent function has been introduced.

Chapter 8

Inverse Functions

In the early discussion of functions, we saw how to form new functions from old ones by addition, multiplication, composition, and other operations. There is one further operation that will be essential in what follows: inversion. An inverse function reverses the action of a given function. The purpose of this chapter is to determine precisely when this operation is possible and to prove how inversion interacts with continuity and differentiation.

The results here will be used repeatedly. In particular, Chapter 9 defines the logarithm by an integral, and it is the inverse-function theory developed in this chapter that then produces the exponential function as the inverse of the logarithm. The same theory will be used to obtain the inverse trigonometric functions by restricting trigonometric functions to intervals on which they are one-to-one.

8.1 Injective, surjective, and bijective functions

Recall that the notation

$$f : A \to B$$

means that f is a function with domain A and codomain B. Thus every element of A occurs exactly once as the first coordinate of a pair belonging to f, while every second coordinate belongs to B. The actual set of values attained by f is its range:

$$\operatorname{ran}(f) := \{f(x) : x \in A\}.$$

The range is always a subset of the codomain, but it need not equal the codomain.

Definition 8.1. Let $f : A \to B$.

- The function f is *injective*, or *one-to-one*, if

  $$f(x) = f(y) \quad \Longrightarrow \quad x = y$$

  for all $x, y \in A$.
- The function f is *surjective*, or *onto*, if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.
- The function f is *bijective* if it is both injective and surjective.

The injectivity condition can equivalently be written as

$$x \neq y \quad \Longrightarrow \quad f(x) \neq f(y).$$

A function is surjective precisely when its range equals its codomain.

Example 8.2. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is bijective. It is injective because $x^3 = y^3$ implies $x = y$, and it is surjective because every $r \in \mathbb{R}$ has the form $r = (\sqrt[3]{r})^3$.

The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ is not injective, since $g(1) = g(-1)$. It is also not surjective onto $\mathbb{R}$, since it has no negative values. However, when regarded as a function

$$g : [0, \infty) \to [0, \infty), \qquad g(x) = x^2,$$

it is bijective.

Proposition 8.3. *A function $f : A \to B$ is injective if and only if each element of* $\operatorname{ran}(f)$ *has a unique preimage under f.*

Proof. Suppose first that f is injective, and let $y \in \operatorname{ran}(f)$. By definition of range, there is at least one $x \in A$ with $f(x) = y$. If also $f(x') = y$, then $f(x) = f(x')$, so injectivity gives $x = x'$. Thus the preimage is unique.

Conversely, suppose each element of the range has a unique preimage. If $f(x) = f(y)$, then the common value belongs to the range and has both x and y as preimages. Uniqueness gives $x = y$. Hence f is injective. □

8.2 Definition and elementary properties of inverses

If f is injective, then each value in its range comes from exactly one input. It is therefore possible to reverse the ordered pairs defining f.

Definition 8.4. Let $f : A \to B$ be injective, and let $R = \operatorname{ran}(f)$. The *inverse function* of f is the function

$$f^{-1} : R \to A$$

defined by

$$f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y.$$

Equivalently, the ordered pair (y, x) belongs to f^{-1} exactly when (x, y) belongs to f.

The notation f^{-1} denotes a function inverse, not the reciprocal $1/f$. These are entirely different objects: if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$, whereas $1/f(x) = x^{-3}$ whenever $x \neq 0$.

Theorem 8.5. *Let $f : A \to B$ be injective, with range R. Then $f^{-1} : R \to A$ is well defined and injective. Moreover,*

$$f^{-1}(f(x)) = x \qquad (x \in A),$$

and

$$f(f^{-1}(y)) = y \qquad (y \in R).$$

Finally,

$$(f^{-1})^{-1} = f.$$

Proof. For each $y \in R$, there is at least one $x \in A$ with $f(x) = y$, because y belongs to the range. By Proposition 8.3, this x is unique. Thus the rule defining f^{-1} is well defined.

Let $y_1, y_2 \in R$ and suppose that $f^{-1}(y_1) = f^{-1}(y_2)$. Apply f to both sides. By the defining relation for the inverse,

$$y_1 = f(f^{-1}(y_1)) = f(f^{-1}(y_2)) = y_2.$$

Therefore f^{-1} is injective.

For $x \in A$, the equality $f^{-1}(f(x)) = x$ follows directly from the fact that x is the unique point mapped by f to $f(x)$. Likewise, if $y \in R$, then $f(f^{-1}(y)) = y$ by the definition of f^{-1}.

The final assertion follows because forming the inverse reverses every ordered pair. Reversing them a second time returns the original ordered pairs of f. □

Corollary 8.6. *If $f : A \to B$ is bijective, then its inverse is a bijection*

$$f^{-1} : B \to A.$$

Proof. If f is surjective, then $\operatorname{ran}(f) = B$, so the domain of f^{-1} is B. Theorem 8.5 shows that f^{-1} is injective. It is surjective onto A because, for every $x \in A$,

$$f^{-1}(f(x)) = x.$$

□

Proposition 8.7. *Let $f : A \to B$ and $g : B \to A$ be functions. If*

$$g \circ f = \mathrm{id}_A \qquad \text{and} \qquad f \circ g = \mathrm{id}_B,$$

then f and g are bijections and $g = f^{-1}$.

Proof. To prove that f is injective, suppose $f(x) = f(y)$. Applying g and using $g \circ f = \mathrm{id}_A$ gives

$$x = g(f(x)) = g(f(y)) = y.$$

To prove surjectivity, let $b \in B$. Then $b = f(g(b))$, so b lies in the range of f. Thus f is bijective.

Since $g(f(x)) = x$ for every $x \in A$ and $f(g(b)) = b$ for every $b \in B$, the defining identities in Theorem 8.5 show that $g = f^{-1}$. □

8.3 Monotonicity and invertibility

Strict monotonicity gives an immediate and important source of injective functions.

Proposition 8.8. *Let I be an interval. If $f : I \to \mathbb{R}$ is strictly increasing or strictly decreasing, then f is injective. Moreover, f^{-1} has the same type of strict monotonicity on its range.*

Proof. Suppose first that f is strictly increasing. If $x \neq y$, then either $x < y$ or $y < x$. In the first case $f(x) < f(y)$, and in the second case $f(y) < f(x)$. Thus $f(x) \neq f(y)$, proving injectivity.

Let $u, v \in \operatorname{ran}(f)$ with $u < v$. Write $u = f(x)$ and $v = f(y)$. If $x \geq y$, then strict increase would give $f(x) \geq f(y)$, contrary to $u < v$. Hence $x < y$, and therefore

$$f^{-1}(u) = x < y = f^{-1}(v).$$

Thus f^{-1} is strictly increasing.

The decreasing case is analogous: $x < y$ implies $f(x) > f(y)$, and the inequalities reverse in the proof for the inverse. □

The converse is false without additional hypotheses. There are injective functions on intervals that are neither increasing nor decreasing. Continuity restores the converse.

Lemma 8.9. *Let $f : I \to \mathbb{R}$ be injective, where I is an interval. If f is neither strictly increasing nor strictly decreasing, then there are points $x < y < z$ in I such that either*

$$f(y) > \max\{f(x), f(z)\}$$

or

$$f(y) < \min\{f(x), f(z)\}.$$

Proof. Because f is not strictly increasing, there are $p < q$ in I with $f(p) \geq f(q)$. Since f is injective and $p \neq q$, equality is impossible, so $f(p) > f(q)$. Because f is not strictly decreasing, there are $r < s$ in I with $f(r) \leq f(s)$, and injectivity gives $f(r) < f(s)$.

Thus one pair of points witnesses a decrease, $f(p) > f(q)$ with $p < q$, and another witnesses an increase, $f(r) < f(s)$ with $r < s$. List the distinct points among p, q, r, s in increasing order as

$$t_1 < t_2 < \cdots < t_m,$$

and write $v_i = f(t_i)$. Injectivity makes $v_1, \ldots, v_m$ pairwise distinct. The decrease forces $v_i > v_j$ for some $i < j$, so the sequence $v_1, \ldots, v_m$ is not strictly increasing; the increase forces $v_i < v_j$ for some $i < j$, so it is not strictly decreasing. Because the v_i are pairwise distinct, a finite sequence fails to be strictly increasing exactly when some successive difference $v_{i+1} - v_i$ is negative, and it fails to be strictly decreasing exactly when some such difference is positive. Hence the differences $d_i = v_{i+1} - v_i$ are all nonzero and take both signs.

Consequently the sign of d_i changes at least once as i runs from 1 to $m - 1$. Let $k \geq 2$ be the first index at which d_k and d_{k-1} have opposite signs. Then either

$$v_{k-1} < v_k > v_{k+1} \qquad \text{or} \qquad v_{k-1} > v_k < v_{k+1}.$$

Setting $x = t_{k-1}$, $y = t_k$, and $z = t_{k+1}$ gives

$$f(y) > \max\{f(x), f(z)\} \qquad \text{or} \qquad f(y) < \min\{f(x), f(z)\}.$$

□

Theorem 8.10 (Continuous injective functions on intervals are monotone)**.** *Let I be an interval, and let $f : I \to \mathbb{R}$ be continuous and injective. Then f is either strictly increasing or strictly decreasing on I.*

Proof. Suppose, for contradiction, that f is neither strictly increasing nor strictly decreasing. By Lemma 8.9, there are $x < y < z$ in I for which either

$$f(y) > \max\{f(x), f(z)\}$$

or

$$f(y) < \min\{f(x), f(z)\}.$$

We treat the first case; the second is analogous after replacing f by $-f$.

Set

$$m = \max\{f(x), f(z)\}.$$

Then $m < f(y)$. Choose a number w such that

$$m < w < f(y).$$

The function f is continuous on $[x, y]$. Since w lies between $f(x)$ and $f(y)$, the Intermediate Value Theorem gives $u \in (x, y)$ with $f(u) = w$. Similarly, w lies between $f(z)$ and $f(y)$, so continuity on $[y, z]$ gives $v \in (y, z)$ with $f(v) = w$.

The points u and v are distinct, but $f(u) = f(v)$, contradicting injectivity. Therefore f must be strictly increasing or strictly decreasing. □

Remark 8.11. Continuity and the fact that the domain is an interval are both important in Theorem 8.10. The proof uses the Intermediate Value Theorem on subintervals. This is one of the points at which the completeness of the real numbers enters the theory of inverse functions.

Corollary 8.12. *Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. If f is increasing, then*

$$\operatorname{ran}(f) = [f(a), f(b)],$$

and if f is decreasing, then

$$\operatorname{ran}(f) = [f(b), f(a)].$$

Proof. Assume that f is increasing. Then $f(a) \le f(x) \le f(b)$ for every $x \in [a, b]$, so the range is contained in $[f(a), f(b)]$. Conversely, if $y \in [f(a), f(b)]$, the Intermediate Value Theorem supplies $x \in [a, b]$ with $f(x) = y$. Hence y belongs to the range. The decreasing case is identical with the endpoint values reversed. □

8.4 Continuity of inverse functions

An inverse need not be continuous for an arbitrary injective function. For a continuous one-to-one function defined on an interval, however, the inverse is automatically continuous.

Lemma 8.13. *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous. Then the image*

$$f(I) = \operatorname{ran}(f)$$

is an interval.

Proof. Let $u < v$ be points of $\operatorname{ran}(f)$, and let w satisfy $u < w < v$. Write $u = f(p)$ and $v = f(q)$ with $p, q \in I$. Since I is an interval, the closed interval between p and q lies in I, and f is continuous there. By the general Intermediate Value Theorem (Corollary 4.23), there is a point t between p and q with $f(t) = w$. Hence $w \in \operatorname{ran}(f)$. Thus every point lying between two points of $\operatorname{ran}(f)$ again belongs to $\operatorname{ran}(f)$, so $\operatorname{ran}(f)$ is an interval. □

Theorem 8.14 (Continuity of the inverse)**.** *Let I be an interval, and let $f : I \to \mathbb{R}$ be continuous and injective. Then*

$$f^{-1} : \operatorname{ran}(f) \to I$$

is continuous.

Proof. By Theorem 8.10, f is strictly increasing or strictly decreasing. Replacing f by $-f$ if necessary, it is enough to consider the increasing case. By Lemma 8.13, $\operatorname{ran}(f)$ is an interval, so the domain of f^{-1} is an interval. If I consists of a single point, then so does $\operatorname{ran}(f)$, and continuity is immediate; assume I is nondegenerate.

Fix $b \in \operatorname{ran}(f)$, and put

$$a = f^{-1}(b).$$

We show that $f^{-1}(y) \to a$ as $y \to b$ through points of $\operatorname{ran}(f)$.

Let $\varepsilon > 0$. We first replace ε by a possibly smaller positive number ε' whose neighboring points lie in I.

If a is an interior point of I, choose $\varepsilon' > 0$ with $\varepsilon' \le \varepsilon$ and

$$a - \varepsilon' \in I, \qquad a + \varepsilon' \in I.$$

Strict increase gives

$$f(a - \varepsilon') < f(a) = b < f(a + \varepsilon'),$$

so that

$$\delta = \min\{b - f(a - \varepsilon'),\ f(a + \varepsilon') - b\} > 0.$$

If $y \in \operatorname{ran}(f)$ and $|y - b| < \delta$, then

$$f(a - \varepsilon') < y < f(a + \varepsilon').$$

Applying the strictly increasing function f^{-1} gives

$$a - \varepsilon' < f^{-1}(y) < a + \varepsilon',$$

so $|f^{-1}(y) - a| < \varepsilon' \leq \varepsilon$.

If a is an included endpoint of I, only the one-sided estimate is needed. Suppose, for example, that a is the left endpoint of I, so that no point of I lies to the left of a. Choose $\varepsilon' > 0$ with $\varepsilon' \leq \varepsilon$ and $a + \varepsilon' \in I$, and set

$$\delta = f(a + \varepsilon') - b > 0.$$

Since f is increasing, $b = f(a)$ is the minimum value of f on I, so every $y \in \operatorname{ran}(f)$ satisfies $y \geq b$. If also $|y - b| < \delta$, then

$$b \leq y < b + \delta = f(a + \varepsilon'),$$

and applying f^{-1} gives

$$a \leq f^{-1}(y) < a + \varepsilon'.$$

Hence $|f^{-1}(y) - a| < \varepsilon' \leq \varepsilon$. The case in which a is the right endpoint of I is analogous.

Thus f^{-1} is continuous at every point of its domain. □

Corollary 8.15. *A continuous bijection from an interval I onto an interval J has a continuous inverse. Thus it gives a one-to-one continuous correspondence between the two intervals.*

Example 8.16. The function

$$f : [0, \infty) \to [0, \infty), \qquad f(x) = x^2,$$

is continuous and strictly increasing. Its inverse is

$$f^{-1}(y) = \sqrt{y}.$$

Theorem 8.14 proves continuity of the square-root function without requiring a separate limit calculation.

8.5 Differentiability of inverse functions

The derivative of an inverse function is obtained by comparing two corresponding difference quotients. The continuity theorem above is essential: it ensures that points close to b under the inverse correspond to points close to $a = f^{-1}(b)$ under the original function.

Theorem 8.17 (Derivative of an inverse function)**.** *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous and injective. Fix $a \in I$, set*

$$b = f(a),$$

and suppose that f is differentiable at a.

1. *If $f'(a) \neq 0$, then f^{-1} is differentiable at b, and*

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

2. *If $f'(a) = 0$, then f^{-1} is not differentiable at b as a finite-valued derivative.*

Proof. By Theorem 8.14, f^{-1} is continuous at b. Let $h \neq 0$ be such that $b + h \in \operatorname{ran}(f)$, and define

$$k = f^{-1}(b + h) - a.$$

Then

$$f(a + k) = b + h,$$

so

$$h = f(a + k) - f(a).$$

Because $f^{-1}(b+h) \to f^{-1}(b) = a$ as $h \to 0$, we have $k \to 0$ as $h \to 0$. Moreover, $k \neq 0$ when $h \neq 0$, because f^{-1} is injective. Therefore

$$\frac{f^{-1}(b + h) - f^{-1}(b)}{h} = \frac{k}{f(a + k) - f(a)} = \left(\frac{f(a + k) - f(a)}{k}\right)^{-1}.$$

If $f'(a) \neq 0$, the expression in parentheses tends to the nonzero number $f'(a)$ as $k \to 0$. By the reciprocal limit law,

$$\lim_{h \to 0} \frac{f^{-1}(b + h) - f^{-1}(b)}{h} = \frac{1}{f'(a)}.$$

This proves the first assertion.

Now suppose $f'(a) = 0$. If f^{-1} had a finite derivative at b, the chain rule applied to

$$f \circ f^{-1} = \mathrm{id}$$

at b would give

$$1 = (f \circ f^{-1})'(b) = f'(f^{-1}(b))(f^{-1})'(b) = f'(a)(f^{-1})'(b) = 0,$$

a contradiction. Hence f^{-1} is not differentiable at b with a finite derivative. □

Remark 8.18. The conclusion in the second part of Theorem 8.17 concerns ordinary real-valued differentiability. Geometrically, a zero derivative of f at a corresponds, after reflection across the line $y = x$, to a vertical tangent for the inverse at $(f(a), a)$.

Corollary 8.19. *For $x > 0$,*

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Proof. Restrict $f(t) = t^2$ to $[0, \infty)$. This function is continuous and strictly increasing, and $f'(t) = 2t$. Its inverse is $f^{-1}(x) = \sqrt{x}$. For $x > 0$, take $a = \sqrt{x} > 0$ in Theorem 8.17. Then

$$(\sqrt{x})' = \frac{1}{2a} = \frac{1}{2\sqrt{x}}.$$

□

Corollary 8.20. *Let $n \in \mathbb{N}$. On $(0, \infty)$, the function $x \mapsto x^{1/n}$ is differentiable and*

$$\frac{d}{dx} x^{1/n} = \frac{1}{n x^{(n-1)/n}}.$$

Proof. Apply Theorem 8.17 to $f(t) = t^n$ restricted to $[0, \infty)$. It is continuous and strictly increasing, and its inverse is $x^{1/n}$. At $a = x^{1/n} > 0$,

$$f'(a) = n a^{n-1} = n x^{(n-1)/n},$$

which yields the stated formula. □

8.6 A mean-value formulation

The Mean Value Theorem gives another useful description of the derivative formula. It also shows directly that a nonzero derivative near a point controls the local behavior of the inverse.

Proposition 8.21. *Let $f : I \to \mathbb{R}$ be continuous and injective on an interval I, and differentiable on the interior of I. Let $x_1, x_2 \in I$ be distinct. Then there is a point c strictly between x_1 and x_2 such that*

$$\frac{f^{-1}(f(x_2)) - f^{-1}(f(x_1))}{f(x_2) - f(x_1)} = \frac{1}{f'(c)}.$$

Proof. Since $f^{-1}(f(x_i)) = x_i$, the left-hand side is

$$\frac{x_2 - x_1}{f(x_2) - f(x_1)}.$$

By the Mean Value Theorem, there is a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Because f is injective, $f(x_2) \neq f(x_1)$, so $f'(c) \neq 0$ in this equality. Rearranging gives the claim. □

Corollary 8.22. *Let $f : I \to \mathbb{R}$ be continuous and injective on an interval I, differentiable on the interior of I, and suppose that*

$$|f'(x)| \geq m > 0$$

for every interior point $x \in I$. Then, for all $u, v \in \operatorname{ran}(f)$,

$$|f^{-1}(u) - f^{-1}(v)| \leq \frac{1}{m} |u - v|.$$

Proof. Write $u = f(x_1)$ and $v = f(x_2)$. If $u = v$, the result is immediate. Otherwise, Proposition 8.21 gives a point c between x_1 and x_2 for which

$$\frac{x_2 - x_1}{f(x_2) - f(x_1)} = \frac{1}{f'(c)}.$$

Taking absolute values and using $|f'(c)| \geq m$ yields

$$|f^{-1}(u) - f^{-1}(v)| = |x_1 - x_2| \leq \frac{1}{m} |u - v|.$$

□

8.7 Examples and applications

Example 8.23 (A cubic inverse)**.** The function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^3 + x,$$

is strictly increasing, because for $x < y$,

$$y^3 + y - (x^3 + x) = (y - x)(x^2 + xy + y^2 + 1) > 0.$$

It is therefore injective, and its continuity together with its behavior as $x \to \pm\infty$ shows that its range is all of $\mathbb{R}$. Hence it has an inverse on $\mathbb{R}$.

Although there is no especially simple formula for f^{-1}, the derivative of the inverse is explicit. If $y = f(x)$, then

$$(f^{-1})'(y) = \frac{1}{3x^2 + 1} = \frac{1}{3(f^{-1}(y))^2 + 1}.$$

This illustrates an important point: an inverse function can be useful even when it cannot be written in a simple elementary form.

Example 8.24 (A restriction is necessary)**.** The function $x \mapsto x^2$ on $\mathbb{R}$ cannot be inverted because 1 and -1 have the same image. Restricting the domain to $[0, \infty)$ gives the inverse square-root function, while restricting to $(-\infty, 0]$ gives the inverse $x \mapsto -\sqrt{x}$. Thus a function may have different inverses after different restrictions of its domain.

Example 8.25 (Failure at a zero derivative)**.** Let $f(x) = x^3$. This is a continuous bijection from $\mathbb{R}$ to $\mathbb{R}$, with inverse $f^{-1}(y) = y^{1/3}$. Since $f'(0) = 0$, Theorem 8.17 says that f^{-1} is not differentiable at 0 in the ordinary sense. Indeed,

$$\frac{y^{1/3} - 0}{y - 0} = y^{-2/3},$$

which is unbounded as $y \to 0$.

Corollary 8.26 (Rational power rule)**.** *Let $r = m/n \in \mathbb{Q}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. For $x > 0$,*

$$\frac{d}{dx} x^r = r x^{r-1}.$$

Proof. Write $x^{m/n} = (x^{1/n})^m$. By Theorem 5.18, Corollary 5.17, and Corollary 8.20,

$$\frac{d}{dx} x^{m/n} = m(x^{1/n})^{m-1} \cdot \frac{d}{dx} x^{1/n} = m(x^{1/n})^{m-1} \cdot \frac{1}{n} x^{(1-n)/n} = \frac{m}{n} x^{m/n-1} = r x^{r-1}.$$

□

Example 8.27 (Tangents to a circle)**.** Consider the circle

$$x^2 + y^2 = 25$$

and the point $(3, 4)$ on it. Near $x = 3$, the upper branch of the circle is the explicit function

$$y = \varphi(x) = (25 - x^2)^{1/2},$$

which is differentiable on $(-5, 5)$: by Theorem 5.18 and the rational power rule (Corollary 8.26),

$$\varphi'(x) = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) = -\frac{x}{(25 - x^2)^{1/2}}.$$

At $x = 3$,

$$\varphi'(3) = -\frac{3}{(25-9)^{1/2}} = -\frac{3}{4}.$$

Thus the tangent line to the circle at $(3, 4)$ has slope $-3/4$.

Remark 8.28. In practice one does not solve for the upper branch before differentiating. Treating y as a differentiable function of x and differentiating the equation $x^2 + y^2 = 25$ implicitly gives

$$2x + 2yy' = 0, \qquad \text{so} \qquad y' = -\frac{x}{y}.$$

At $(3, 4)$ this gives $y' = -3/4$, in agreement with the direct computation. The manipulation is legitimate because Theorem 8.17 guarantees that near $(3, 4)$ the equation defines y as a differentiable function of x: writing $x^2 = 25 - y^2$, the map $y \mapsto 25 - y^2$ has derivative $-2y \neq 0$ at $y = 4$, so it has a differentiable inverse there, and composing that inverse with $x \mapsto x^2$ near $x = 3$ recovers the differentiable upper branch.

8.8 Exercises

Exercise 8.29. For each function below, determine whether it is injective, surjective, and bijective, with the stated domain and codomain:

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x - 3,$$

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2,$$

$$h : [0, \infty) \to [0, \infty), \quad h(x) = x^2.$$

Exercise 8.30. Prove directly from the definitions that a function $f : A \to B$ is bijective if and only if each $b \in B$ has exactly one preimage in A.

Exercise 8.31. Let $f : A \to B$ be injective. Prove that the graph of f^{-1} is obtained from the graph of f by reflection across the line $y = x$.

Exercise 8.32. Give an example of an injective function $f : [0, 1] \to \mathbb{R}$ that is not monotone. Explain why this does not contradict Theorem 8.10.

Exercise 8.33. Prove the decreasing case of Proposition 8.8.

Exercise 8.34. Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly increasing. Prove directly that

$$\operatorname{ran}(f) = [f(a), f(b)].$$

State and prove the analogous result for a strictly decreasing function.

Exercise 8.35. Let

$$f : (-\infty, 0] \to (-\infty, 0], \qquad f(x) = x^3.$$

Find f^{-1} and determine every point at which f^{-1} is differentiable.

Exercise 8.36. Use Theorem 8.17 to derive the formula

$$\frac{d}{dx} x^{1/n} = \frac{1}{n x^{(n-1)/n}}$$

for $x > 0$, beginning with the function $t \mapsto t^n$ on $[0, \infty)$.

Exercise 8.37. Let $f(x) = x + x^5$. Prove that $f : \mathbb{R} \to \mathbb{R}$ is bijective. Express $(f^{-1})'(y)$ in terms of $f^{-1}(y)$.

Exercise 8.38. Let $f : I \to \mathbb{R}$ be continuous and injective. Prove that f^{-1} is continuous at an endpoint of $\mathrm{ran}(f)$ using the appropriate one-sided definition of continuity.

Exercise 8.39. Suppose $f : I \to \mathbb{R}$ is continuous and injective, and differentiable on the interior of I. Prove that $f'(x)$ cannot change sign on the interior of I without vanishing somewhere.

Exercise 8.40. Let $f : [1, \infty) \to [2, \infty)$ be given by $f(x) = x + 1/x$. Determine whether f is invertible. If it is, find a formula for its inverse; if not, find a restriction of the domain on which it becomes invertible.

Remark 8.41. The codomain $[2, \infty)$ is chosen deliberately: f attains its minimum value 2 at $x = 1$ and is strictly increasing thereafter (its derivative $1 - 1/x^2$ is positive on $(1, \infty)$), so its range is $[2, \infty)$, and f is a bijection onto that range. Had the codomain been declared to be $[1, \infty)$ instead, the same rule would still define an injective function, but one that is not surjective, since no $x \geq 1$ satisfies $x + 1/x = 3/2$. Such a function is still invertible on its range $[2, \infty)$; only the declared codomain differs. This is precisely the range-versus-codomain distinction recorded in Section 8.1.

Exercise 8.42. Use the Mean Value Theorem to prove Corollary 8.22.

Exercise 8.43. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Verify directly from the definition of derivative that $f^{-1}(x) = x^{1/3}$ is not differentiable at 0.

Exercise 8.44. Explain precisely why the chain-rule argument in the second part of Theorem 8.17 cannot be used to prove the first part before differentiability of f^{-1} has been established.

Chapter 9

Logarithmic, Exponential, and Trigonometric Functions

This chapter develops the logarithmic, exponential, trigonometric, inverse trigonometric, and hyperbolic functions from the preceding theory of integration and inverse functions. It concludes with a definition of the length of the graph of a function and a proof of the arc-length formula.

The emphasis is on the logical dependencies. The logarithm is defined by an integral; the exponential function is its inverse. The inverse-function theorem then supplies the derivative of the exponential. The trigonometric functions are treated as differentiable functions satisfying the usual identities, and their inverses are defined only after suitable restrictions of domain. Hyperbolic functions are expressed in terms of the exponential function. Finally, the Darboux integral and the Mean Value Theorem identify the length of a sufficiently smooth graph.

9.1 The natural logarithm

Definition 9.1. For $x > 0$, define the *natural logarithm* of x by

$$\log x := \int_1^x \frac{dt}{t}.$$

The integral is well defined for every $x > 0$, since $t \mapsto 1/t$ is continuous on every closed interval contained in $(0, \infty)$. The convention for reversed limits gives $\log x < 0$ when $0 < x < 1$.

Theorem 9.2. *The logarithm is continuous and differentiable on* $(0, \infty)$*, and*

$$(\log x)' = \frac{1}{x}.$$

It is strictly increasing and strictly concave on $(0, \infty)$*.*

Proof. The Fundamental Theorem of Calculus, Part I, applied to the continuous function $t \mapsto 1/t$, gives

$$(\log x)' = \frac{1}{x}.$$

Since $1/x > 0$ on $(0, \infty)$, the Mean Value Theorem shows that log is strictly increasing. Differentiating once more gives

$$(\log x)'' = -\frac{1}{x^2} < 0,$$

so the second-derivative test for concavity shows that log is strictly concave. □

Theorem 9.3 (Logarithm laws)**.** *For all $a, b > 0$,*

$$\log(ab) = \log a + \log b, \qquad \log\left(\frac{a}{b}\right) = \log a - \log b.$$

Proof. Fix $b > 0$, and define

$$F(a) = \log(ab) - \log a.$$

By the chain rule and Theorem 9.2,

$$F'(a) = \frac{b}{ab} - \frac{1}{a} = 0.$$

Thus F is constant on $(0, \infty)$. Evaluating at $a = 1$ gives

$$F(a) = F(1) = \log b,$$

which proves the first identity. For the second,

$$\log a = \log\left(\frac{a}{b}b\right) = \log\left(\frac{a}{b}\right) + \log b.$$

Rearranging yields the result. □

By induction on the product law,

$$\log(x_1 \cdots x_n) = \log x_1 + \cdots + \log x_n,$$

and, taking all x_i equal, $\log(y^n) = n \log y$; equivalently

$$\log(y^{1/n}) = \frac{1}{n} \log y.$$

Corollary 9.4 (Arithmetic–geometric mean inequality)**.** *For $x_1, \ldots, x_n > 0$,*

$$(x_1 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n},$$

with equality if and only if $x_1 = \cdots = x_n$.

Proof. By Theorem 9.2, log is strictly concave on $(0, \infty)$, so $f = -\log$ is strictly convex there. Let

$$A = \frac{x_1 + \cdots + x_n}{n}, \qquad G = (x_1 \cdots x_n)^{1/n}$$

be the arithmetic and geometric means. Jensen's inequality (Theorem 6.42) applied to f with equal weights $t_1 = \cdots = t_n = 1/n$ gives

$$-\log A = f(A) \leq \frac{1}{n} \sum_{i=1}^{n} f(x_i) = -\frac{1}{n} \sum_{i=1}^{n} \log x_i.$$

By the logarithm laws just proved (Theorem 9.3),

$$\frac{1}{n} \sum_{i=1}^{n} \log x_i = \frac{1}{n} \log(x_1 \cdots x_n) = \log G,$$

so $-\log A \leq -\log G$, that is, $\log A \geq \log G$. Since log is strictly increasing (Theorem 9.2), it follows that $A \geq G$, which is the stated inequality.

If equality holds, then $A = G$, so $\log A = \log G$ and hence equality holds in Jensen's inequality. Because f is strictly convex and all the weights are positive, the strict form of Theorem 6.42 forces $x_1 = \cdots = x_n$. Conversely, if $x_1 = \cdots = x_n$, then plainly $A = G$. Thus equality holds if and only if $x_1 = \cdots = x_n$. □

This completes the program announced in Chapter 6 (see the remark following Example 6.43), where the strict concavity of log was used to predict exactly this inequality.

Proposition 9.5. *The range of* log *is all of* $\mathbb{R}$. *Equivalently,*

$$\lim_{x\to 0^+} \log x = -\infty, \qquad \lim_{x\to\infty} \log x = \infty.$$

Proof. For every integer $n \geq 1$,

$$\log(2^n) = n \log 2.$$

Since $\log 2 > 0$, this tends to ∞ as $n \to \infty$. Thus log is unbounded above. By the quotient law,

$$\log(2^{-n}) = -n \log 2,$$

so it is unbounded below as well.

Because log is continuous and strictly increasing on $(0, \infty)$, its range is an interval. An interval unbounded above and below is all of $\mathbb{R}$. □

9.2 The exponential function and the number e

By Theorem 9.2 and Proposition 9.5, the logarithm is a continuous strictly increasing bijection

$$\log : (0, \infty) \to \mathbb{R}.$$

It therefore has a continuous strictly increasing inverse.

Definition 9.6. The *exponential function* is the inverse of the natural logarithm:

$$\exp : \mathbb{R} \to (0, \infty), \qquad \exp = (\log)^{-1}.$$

The number

$$e := \exp(1)$$

is called the *base of the natural logarithms*. We also write

$$e^x := \exp x.$$

Thus

$$\log(e^x) = x \qquad (x \in \mathbb{R}),$$

and

$$e^{\log y} = y \qquad (y > 0).$$

Theorem 9.7. *The exponential function is differentiable on* $\mathbb{R}$ *and satisfies*

$$(e^x)' = e^x.$$

Moreover,

$$e^{x+y} = e^x e^y \qquad (x, y \in \mathbb{R}).$$

Proof. Since $\log'(u) = 1/u \neq 0$ for every $u > 0$, the derivative-of-an-inverse theorem gives

$$(\exp x)' = \frac{1}{\log'(\exp x)} = \exp x = e^x.$$

For the addition law, use the first logarithm law:

$$\log(e^x e^y) = \log(e^x) + \log(e^y) = x + y.$$

Applying the inverse function exp to both sides gives

$$e^x e^y = e^{x+y}.$$

□

Corollary 9.8. *For all real x, y,*

$$e^0 = 1, \qquad e^{-x} = \frac{1}{e^x}, \qquad e^{x-y} = \frac{e^x}{e^y}.$$

The exponential function is strictly increasing and strictly convex.

Proof. The identities follow from Theorem 9.7 and the defining relation $e^0 = \exp(0) = 1$. Since

$$(e^x)' = e^x > 0$$

and

$$(e^x)'' = e^x > 0,$$

the Mean Value Theorem and the second-derivative criterion imply strict increase and strict convexity. □

Definition 9.9. For $a > 0$ and $x \in \mathbb{R}$, define

$$a^x := e^{x \log a}.$$

Proposition 9.10. *If $a > 0$, then the function $x \mapsto a^x$ is differentiable on $\mathbb{R}$, with*

$$\frac{d}{dx} a^x = (\log a) a^x.$$

Proof. By definition, $a^x = e^{x \log a}$. The chain rule gives

$$\frac{d}{dx} a^x = e^{x \log a} \log a = (\log a) a^x.$$

□

Proposition 9.11 (Power rule for real exponents)**.** *For every real α, the function $x \mapsto x^\alpha$ is differentiable on $(0, \infty)$, with*

$$\frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}.$$

Proof. By definition,

$$x^\alpha = e^{\alpha \log x}.$$

The chain rule and Theorem 9.2 give

$$\frac{d}{dx} x^\alpha = e^{\alpha \log x} \cdot \frac{\alpha}{x} = \alpha x^\alpha x^{-1} = \alpha x^{\alpha - 1}.$$

□

Remark 9.12. For rational $\alpha = m/n$ this agrees with the rational power rule of Corollary 8.26, so the two definitions of x^α are compatible. The exponent laws

$$x^{\alpha+\beta} = x^\alpha x^\beta, \qquad (x^\alpha)^\beta = x^{\alpha\beta}$$

follow from the definition $x^\alpha = e^{\alpha \log x}$ and the logarithm laws (Theorem 9.3); the first is immediate, and for the second,

$$(x^\alpha)^\beta = e^{\beta \log(x^\alpha)} = e^{\beta\alpha \log x} = x^{\alpha\beta}.$$

The endpoint behavior of x^α at 0 and ∞ used for improper integrals and the binomial series follows from the corresponding behavior of the exponential and logarithm.

9.3 Trigonometric functions

The trigonometric functions are constructed here from an integral, in the same spirit as the logarithm: cosine is defined first as the inverse of an integral, and sine is built from it. No geometric facts are presupposed. (The derivative formulas of Chapter 5, used there provisionally, are justified below.)

Definition 9.13. For $-1 \le x \le 1$, define

$$A(x) = x\sqrt{1-x^2} + 2\int_x^1 \sqrt{1-t^2}\,dt.$$

For $0 \le x \le 1$, the first summand is the area of the triangle with vertices $(0,0)$, $(x,0)$, and $(x,\sqrt{1-x^2})$, and the integral is the area under the circle between x and 1; thus $A(x)$ is the area of the circular sector cut out by the point $(x,\sqrt{1-x^2})$ on the unit circle. Only the analytic properties below are needed, and these do not depend on this geometric interpretation.

Definition 9.14. The number π is defined by

$$\pi := 2\int_{-1}^1 \sqrt{1-t^2}\,dt.$$

The integrand is continuous and positive on $(-1,1)$, so $\pi > 0$; it is twice the area of the unit semicircle, that is, the area of the unit disk.

Proposition 9.15. *The function A is continuous and strictly decreasing on $[-1,1]$, with*

$$A(1) = 0, \qquad A(-1) = \pi, \qquad A(0) = \frac{\pi}{2},$$

and

$$A'(x) = -\frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1).$$

Proof. The function A is continuous on $[-1,1]$ because the first summand is continuous and the integral has a continuous integrand (Theorem 7.54). Its values at $0, \pm 1$ are immediate from Definition 9.13 and Definition 9.14. For $-1 < x < 1$, the product rule and the Fundamental Theorem of Calculus, Part I (Theorem 7.55), give

$$A'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} - 2\sqrt{1-x^2} = -\frac{1}{\sqrt{1-x^2}} < 0.$$

Since A is continuous and $A' < 0$ on $(-1,1)$, A is strictly decreasing on $[-1,1]$ (Theorem 6.18). □

Definition 9.16. Since A is continuous and strictly decreasing, it is a bijection

$$A : [-1,1] \to [0,\pi].$$

The *cosine* function is its inverse:

$$\cos := A^{-1} : [0,\pi] \to [-1,1].$$

For $0 \le x \le \pi$, define

$$\sin x := \sqrt{1-\cos^2 x} \ge 0.$$

Thus

$$\cos 0 = 1, \qquad \cos\frac{\pi}{2} = 0, \qquad \cos\pi = -1,$$

and

$$\sin 0 = 0, \qquad \sin\frac{\pi}{2} = 1, \qquad \sin\pi = 0.$$

By definition,

$$\sin^2 x + \cos^2 x = 1 \qquad (0 \le x \le \pi).$$

Definition 9.17. Extend sin and cos to all of $\mathbb{R}$ by first defining, for $0 \le x \le \pi$,

$$\cos(x+\pi) = -\cos x, \qquad \sin(x+\pi) = -\sin x,$$

and then, having defined the functions on $[0, 2\pi]$, extending by periodicity:

$$\cos(x+2\pi) = \cos x, \qquad \sin(x+2\pi) = \sin x \qquad (x \in \mathbb{R}).$$

Every real number differs from a unique point of $[0, 2\pi)$ by an integer multiple of 2π, so the extension is well defined. At the seams the two descriptions agree, so sin and cos are continuous on all of $\mathbb{R}$; for example $\sin\pi = -\sin 0 = 0$ and $\sin 2\pi = -\sin\pi = 0$. The Pythagorean identity

$$\sin^2 x + \cos^2 x = 1$$

continues to hold on all of $\mathbb{R}$, because each extension step preserves it.

Theorem 9.18 (Derivatives of sine and cosine)**.** *The functions* sin *and* cos *are differentiable on* $\mathbb{R}$*, and*

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x.$$

Proof. For $0 < x < \pi$, we have $\sin x > 0$. The inverse-function theorem (Theorem 8.17) applied to A gives, at $x = \cos y \in (-1, 1)$,

$$(\cos y)' = \frac{1}{A'(\cos y)} = -\sqrt{1 - \cos^2 y} = -\sin y.$$

Then, from $\sin x = \sqrt{1 - \cos^2 x}$,

$$(\sin x)' = -\frac{\cos x\,(\cos x)'}{\sqrt{1 - \cos^2 x}} = -\frac{\cos x\,(-\sin x)}{\sin x} = \cos x.$$

This proves both formulas on $(0, \pi)$, and the extension rules of Definition 9.17 carry them to the remaining points: for $0 < x < \pi$,

$$(\sin(x+\pi))' = (-\sin x)' = -\cos x = \cos(x+\pi),$$

and similarly for cosine. At the multiples of π the left and right derivatives of sin both equal cos, and those of cos both equal $-\sin$; by the one-sided-derivative criterion, the derivatives exist there as well, and periodicity transfers the formulas to all of $\mathbb{R}$. □

Theorem 9.19 (Addition formulas)**.** *For all* $x, y \in \mathbb{R}$*,*

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Proof. Fix $y \in \mathbb{R}$, and define

$$F(x) = \sin(x+y) - \sin x \cos y - \cos x \sin y,$$

$$G(x) = \cos(x+y) - \cos x \cos y + \sin x \sin y.$$

Using Theorem 9.18,

$$F'(x) = \cos(x+y) - \cos x \cos y + \sin x \sin y = G(x),$$

and

$$G'(x) = -\sin(x+y) + \sin x \cos y + \cos x \sin y = -F(x).$$

Therefore

$$(F^2 + G^2)' = 2FF' + 2GG' = 2FG - 2GF = 0,$$

so $F^2 + G^2$ is constant (Corollary 6.10). At $x = 0$,

$$F(0) = \sin y - \sin y = 0, \qquad G(0) = \cos y - \cos y = 0,$$

so the constant is 0. Hence $F = G = 0$ identically, which is exactly the two addition formulas. □

Corollary 9.20. *For all $x \in \mathbb{R}$,*

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x,$$

$$\sin(x + \pi/2) = \cos x, \qquad \cos(x + \pi/2) = -\sin x.$$

In particular,

$$\lim_{h \to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

Proof. The parity formulas follow from

$$\sin(-x) = \sin(0 - x) = \sin 0 \cos x - \cos 0 \sin x = -\sin x,$$

$$\cos(-x) = \cos(0 - x) = \cos 0 \cos x + \sin 0 \sin x = \cos x.$$

The shift formulas follow from

$$\sin(x + \pi/2) = \sin x \cos(\pi/2) + \cos x \sin(\pi/2) = \cos x,$$

and similarly for cosine. The limits are the definitions of the derivatives at 0:

$$\lim_{h \to 0} \frac{\sin h}{h} = \sin'(0) = \cos 0 = 1,$$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \cos'(0) = -\sin 0 = 0.$$

□

Definition 9.21. The remaining trigonometric functions are defined, where their denominators are nonzero, by

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x},$$

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}.$$

Theorem 9.22. *At every point in the relevant domain,*

$$(\tan x)' = \sec^2 x, \qquad (\cot x)' = -\csc^2 x,$$

$$(\sec x)' = \sec x \tan x, \qquad (\csc x)' = -\csc x \cot x.$$

Proof. The quotient and reciprocal rules give

$$(\tan x)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x,$$

and

$$(\sec x)' = -\frac{-\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

The remaining two formulas are proved similarly. □

9.4 Inverse trigonometric functions

The trigonometric functions are periodic and hence are not one-to-one on all of $\mathbb{R}$. To define inverse functions, we restrict their domains to intervals on which they are continuous and strictly monotone.

Definition 9.23. The inverse sine, inverse cosine, and inverse tangent functions are defined by

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

as the inverse of the restriction of sin to $[-\pi/2, \pi/2]$;

$$\arccos : [-1, 1] \to [0, \pi],$$

as the inverse of the restriction of cos to $[0, \pi]$; and

$$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

as the inverse of the restriction of tan to $(-\pi/2, \pi/2)$.

Proposition 9.24. *The restrictions in Definition 9.23 are continuous bijections onto the stated ranges. Consequently,* arcsin*,* arccos*, and* arctan *are continuous on their domains.*

Proof. On $[-\pi/2, \pi/2]$, one has $\cos x \geq 0$, with strict positivity in the interior. The Mean Value Theorem therefore implies that sine is strictly increasing there. Its endpoint values are -1 and 1, so the Intermediate Value Theorem gives the stated range.

On $[0, \pi]$, one has $-\sin x \leq 0$, with strict negativity in the interior, so cosine is strictly decreasing. Its endpoint values are 1 and -1.

On $(-\pi/2, \pi/2)$, the derivative $\sec^2 x$ is strictly positive, so tangent is strictly increasing. Its limits at the two endpoints are $-\infty$ and ∞, respectively; hence its range is all of $\mathbb{R}$. The continuity of the inverse functions follows from Theorem 8.14. □

Theorem 9.25 (Derivatives of inverse trigonometric functions)**.** *For $-1 < x < 1$,*

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}, \qquad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1 - x^2}}.$$

For every $x \in \mathbb{R}$,

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}.$$

Proof. Let $y = \arcsin x$. Then $y \in (-\pi/2, \pi/2)$, $\sin y = x$, and $\cos y > 0$. By the derivative-of-an-inverse theorem,

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}.$$

The Pythagorean identity gives

$$\cos^2(\arcsin x) = 1 - \sin^2(\arcsin x) = 1 - x^2.$$

Because the cosine is positive on the principal range of arcsine,

$$\cos(\arcsin x) = \sqrt{1 - x^2}.$$

This proves the first formula.

For $y = \arccos x$, one has $y \in (0, \pi)$ and $\sin y > 0$. Thus

$$(\arccos x)' = \frac{1}{-\sin(\arccos x)} = -\frac{1}{\sqrt{1 - x^2}}.$$

Finally, if $y = \arctan x$, then $\tan y = x$ and $y \in (-\pi/2, \pi/2)$. Hence

$$(\arctan x)' = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1 + \tan^2(\arctan x)} = \frac{1}{1 + x^2}.$$

□

Definition 9.26. The inverse secant is defined by restricting secant to

$$D = \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

On the first component $(0, \pi/2)$ the derivative $\sec' \theta = \sec\theta \tan\theta$ is strictly positive, and on the second $(\pi/2, \pi)$ it is also strictly positive ($\sec\theta < 0$ and $\tan\theta < 0$ there), so secant is continuous and strictly increasing on each component separately. Its images are

$$\sec([0, \pi/2)) = [1, \infty), \qquad \sec((\pi/2, \pi]) = (-\infty, -1],$$

which are disjoint, so the restriction is injective and has an inverse

$$\operatorname{arcsec} : (-\infty, -1] \cup [1, \infty) \to \left[0, \frac{\pi}{2}\right) \cup \left[\frac{\pi}{2}, \pi\right].$$

Proposition 9.27. *For $|x| > 1$,*

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Proof. The inverse-function theorem applies separately on the two open components, where $\sec' \neq 0$. For $y = \operatorname{arcsec} x$,

$$(\operatorname{arcsec} x)' = \frac{1}{\sec y \tan y}.$$

Since $\sec y = x$ and

$$\tan^2 y = \sec^2 y - 1 = x^2 - 1,$$

we have $|\tan y| = \sqrt{x^2 - 1}$, and $\tan y$ has the sign of x on the chosen range: $\tan y > 0$ for $y \in (0, \pi/2)$, where $x > 0$, and $\tan y < 0$ for $y \in (\pi/2, \pi)$, where $x < 0$. Therefore

$$\sec y \tan y = |x|\sqrt{x^2 - 1},$$

which proves the formula. □

9.5 Hyperbolic functions

The hyperbolic functions are naturally expressed using the exponential function. Their names arise from their relationship with the hyperbola $u^2 - v^2 = 1$, just as sine and cosine are related to the unit circle.

Definition 9.28. For $x \in \mathbb{R}$, define

$$\cosh x := \frac{e^x + e^{-x}}{2}, \qquad \sinh x := \frac{e^x - e^{-x}}{2}.$$

Where defined, set

$$\tanh x := \frac{\sinh x}{\cosh x}, \qquad \coth x := \frac{\cosh x}{\sinh x},$$
$$\operatorname{sech} x := \frac{1}{\cosh x}, \qquad \operatorname{csch} x := \frac{1}{\sinh x}.$$

Theorem 9.29 (Basic hyperbolic identities)**.** *For every $x \in \mathbb{R}$,*

$$\cosh^2 x - \sinh^2 x = 1.$$

Moreover,

$$\sinh(-x) = -\sinh x, \qquad \cosh(-x) = \cosh x,$$

and

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

Proof. Using the exponential definitions,

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = 1.$$

The parity and addition identities follow by direct expansion and the exponential addition law. □

Theorem 9.30 (Derivatives of hyperbolic functions)**.** *At all points in the corresponding domains,*

$$(\sinh x)' = \cosh x, \qquad (\cosh x)' = \sinh x,$$
$$(\tanh x)' = \operatorname{sech}^2 x, \qquad (\coth x)' = -\operatorname{csch}^2 x,$$
$$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x, \qquad (\operatorname{csch} x)' = -\operatorname{csch} x \coth x.$$

Proof. Differentiate the exponential definitions to obtain the first two formulas. For example,

$$(\tanh x)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x.$$

The remaining formulas follow from the quotient and reciprocal rules. □

Proposition 9.31. *The function $\sinh : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing bijection. The restriction*

$$\cosh : [0, \infty) \to [1, \infty)$$

is a continuous strictly increasing bijection. The function

$$\tanh : \mathbb{R} \to (-1, 1)$$

is a continuous strictly increasing bijection.

Proof. Since $(\sinh x)' = \cosh x > 0$, sinh is strictly increasing. The exponential formulas show that $\sinh x \to \infty$ as $x \to \infty$ and $\sinh x \to -\infty$ as $x \to -\infty$, so its range is $\mathbb{R}$.

On $[0, \infty)$, $(\cosh x)' = \sinh x \geq 0$, with strict positivity for $x > 0$, and $\cosh 0 = 1$. Also $\cosh x \to \infty$ as $x \to \infty$, proving the second statement.

Finally,

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

This is strictly increasing because its derivative is $\operatorname{sech}^2 x > 0$, and its limits at $-\infty$ and ∞ are -1 and 1, respectively. □

9.6 Inverse hyperbolic functions

Definition 9.32. The inverse hyperbolic functions are the inverses supplied by Proposition 9.31:

$$\operatorname{arsinh} : \mathbb{R} \to \mathbb{R}, \qquad \operatorname{arcosh} : [1, \infty) \to [0, \infty),$$
$$\operatorname{artanh} : (-1, 1) \to \mathbb{R}.$$

They are also commonly denoted by

$$\sinh^{-1} x, \qquad \cosh^{-1} x, \qquad \tanh^{-1} x,$$

respectively. These superscripts denote inverse functions, not reciprocal powers.

Theorem 9.33 (Derivatives of inverse hyperbolic functions)**.** *For every $x \in \mathbb{R}$,*

$$\frac{d}{dx} \operatorname{arsinh} x = \frac{1}{\sqrt{1 + x^2}}.$$

For $x > 1$,

$$\frac{d}{dx} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2 - 1}}.$$

For $|x| < 1$,

$$\frac{d}{dx} \operatorname{artanh} x = \frac{1}{1 - x^2}.$$

Proof. Let $y = \operatorname{arsinh} x$. Then $\sinh y = x$. By the inverse derivative theorem,

$$(\operatorname{arsinh} x)' = \frac{1}{\cosh y}.$$

Since $\cosh y > 0$ and $\cosh^2 y = 1 + \sinh^2 y = 1 + x^2$,

$$\cosh y = \sqrt{1 + x^2}.$$

This proves the first formula.

If $y = \operatorname{arcosh} x$, then $y \geq 0$, $\cosh y = x$, and $\sinh y \geq 0$. Thus

$$(\operatorname{arcosh} x)' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}.$$

Finally, if $y = \operatorname{artanh} x$, then

$$(\operatorname{artanh} x)' = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}.$$

□

Theorem 9.34 (Logarithmic formulas)**.** *For $x \in \mathbb{R}$,*

$$\operatorname{arsinh} x = \log\left(x + \sqrt{1+x^2}\right).$$

For $x \geq 1$,

$$\operatorname{arcosh} x = \log\left(x + \sqrt{x^2-1}\right).$$

For $|x| < 1$,

$$\operatorname{artanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Proof. Let $y = \operatorname{arsinh} x$. Then

$$x = \frac{e^y - e^{-y}}{2}.$$

Multiplying by $2e^y$ gives

$$e^{2y} - 2xe^y - 1 = 0.$$

The positive solution for e^y is

$$e^y = x + \sqrt{1+x^2},$$

so

$$y = \log\left(x + \sqrt{1+x^2}\right).$$

For $y = \operatorname{arcosh} x$, the equation

$$x = \frac{e^y + e^{-y}}{2}$$

gives

$$e^{2y} - 2xe^y + 1 = 0.$$

Since $y \geq 0$, the appropriate root is

$$e^y = x + \sqrt{x^2-1}.$$

Finally, if $y = \operatorname{artanh} x$, then

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}.$$

Solving for e^{2y} yields

$$e^{2y} = \frac{1+x}{1-x},$$

and taking logarithms proves the formula. □

9.7 Length of a graph

We now give a definition of length for the graph of a bounded function. The definition extends the familiar fact that the distance between two points in the plane is the length of the straight segment joining them.

Definition 9.35. Let $f : [a,b] \to \mathbb{R}$ be bounded, and let

$$P = (t_0, t_1, \ldots, t_n)$$

be a partition of $[a,b]$. The *polygonal length of the graph of f with respect to P* is

$$\ell(f,P) := \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + \big(f(t_i) - f(t_{i-1})\big)^2}.$$

The graph of f is called *rectifiable* on $[a,b]$ if the set

$$\{\ell(f,P) : P \text{ is a partition of } [a,b]\}$$

is bounded above. In that case, its *length* is

$$\mathcal{L}(f;[a,b]) := \sup\{\ell(f,P) : P \text{ is a partition of } [a,b]\}.$$

Lemma 9.36 (Refinement increases polygonal length)**.** *If Q is a refinement of a partition P, then*

$$\ell(f,P) \le \ell(f,Q).$$

Proof. It suffices to consider the insertion of one point $u \in (t_{k-1}, t_k)$. All segments other than the kth are unchanged. The old segment joins

$$A = (t_{k-1}, f(t_{k-1})) \quad \text{to} \quad B = (t_k, f(t_k)),$$

whereas the refined polygonal path uses the two segments from A to

$$C = (u, f(u))$$

and from C to B. The triangle inequality in the Euclidean plane gives

$$|A - B| \le |A - C| + |C - B|.$$

Thus the new polygonal length is at least the old one. Repeating the argument proves the result for every refinement. □

Theorem 9.37 (Arc-length formula)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable. Then its graph is rectifiable and*

$$\mathcal{L}(f;[a,b]) = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx.$$

Proof. Set

$$g(x) = \sqrt{1 + \big(f'(x)\big)^2}.$$

Since f' is continuous, g is continuous and hence Darboux integrable on $[a,b]$.

Let $P = (t_0, \dots, t_n)$ be a partition. By the Mean Value Theorem, for each i there is $c_i \in (t_{i-1}, t_i)$ such that

$$f(t_i) - f(t_{i-1}) = f'(c_i)(t_i - t_{i-1}).$$

Therefore

$$\ell(f,P) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sqrt{1 + \big(f'(c_i)\big)^2}.$$

For each i, the value $g(c_i)$ lies between the infimum and supremum of g on $[t_{i-1}, t_i]$. Hence

$$L(g,P) \le \ell(f,P) \le U(g,P).$$

The upper bound is now transferred to the integral. Fix the partition P, and let Q be any other partition. The common refinement $P \cup Q$ refines P, so by Lemma 9.36,

$$\ell(f, P) \le \ell(f, P \cup Q) \le U(g, P \cup Q) \le U(g, Q),$$

where the last step is Corollary 7.6. Taking the infimum over all partitions Q gives

$$\ell(f, P) \le \int_a^b g(x)\,dx.$$

Since this holds for every partition P, the set of polygonal lengths is bounded above; the graph is rectifiable and

$$\mathcal{L}(f; [a, b]) \le \int_a^b g(x)\,dx.$$

Conversely, the left-hand inequality gives

$$L(g, P) \le \ell(f, P) \le \mathcal{L}(f; [a, b])$$

for every partition P. Taking the supremum over all lower Darboux sums yields

$$\int_a^b g(x)\,dx \le \mathcal{L}(f; [a, b]).$$

Combining the two inequalities proves the formula. □

Example 9.38 (Length of a line segment)**.** Let $f(x) = mx + d$ on $[a, b]$. Then $f'(x) = m$, so Theorem 9.37 gives

$$\mathcal{L}(f; [a, b]) = \int_a^b \sqrt{1 + m^2}\,dx = (b - a)\sqrt{1 + m^2}.$$

This agrees with the Euclidean distance between $(a, ma + d)$ and $(b, mb + d)$.

Example 9.39 (The catenary)**.** For $f(x) = \cosh x$ on $[0, a]$, we have $f'(x) = \sinh x$. By the hyperbolic identity,

$$\sqrt{1 + \sinh^2 x} = \cosh x.$$

Thus

$$\mathcal{L}(\cosh; [0, a]) = \int_0^a \cosh x\,dx = \sinh a.$$

Example 9.40 (A semicircle)**.** The upper unit semicircle is the graph

$$f(x) = \sqrt{1 - x^2}, \qquad -1 \le x \le 1.$$

For $0 < r < 1$, apply the arc-length formula on $[-r, r]$, where f is continuously differentiable:

$$f'(x) = -\frac{x}{\sqrt{1 - x^2}}, \qquad \sqrt{1 + (f'(x))^2} = \frac{1}{\sqrt{1 - x^2}},$$

so

$$\mathcal{L}(f; [-r, r]) = \int_{-r}^{r} \frac{dx}{\sqrt{1 - x^2}} = 2 \arcsin r.$$

The arc-length theorem itself does not apply on $[-1, 1]$: the derivative is unbounded at the endpoints. The general definition of length, however, does apply, since f is bounded on $[-1, 1]$, and the truncated lengths determine it. For any partition P of $[-1, 1]$, choose

$r < 1$ so that all interior points of P lie in $[-r, r]$; adding the points $\pm r$ to P only increases polygonal length, and the resulting partition of $[-r, r]$ has length at most $\mathcal{L}(f; [-r, r])$. Hence

$$\mathcal{L}(f; [-1, 1]) \leq \sup_{0<r<1} \mathcal{L}(f; [-r, r]).$$

Conversely, every partition of $[-r, r]$ is a partition of $[-1, 1]$, so

$$\mathcal{L}(f; [-r, r]) \leq \mathcal{L}(f; [-1, 1]).$$

The truncated lengths $\mathcal{L}(f; [-r, r]) = 2 \arcsin r$ increase with r and are bounded above by π, so their limit as $r \to 1^-$ exists. Combining the two inequalities,

$$\mathcal{L}(f; [-1, 1]) = \lim_{r \to 1^-} \mathcal{L}(f; [-r, r]) = 2 \arcsin 1 = \pi.$$

Thus the upper unit semicircle has length π, and the unit circle has circumference 2π, in agreement with Definition 9.14.

9.8 Exercises

Exercise 9.41. Prove directly from Definition 9.1 and additivity of the integral that

$$\log(ab) = \log a + \log b$$

for $a, b > 0$.

Exercise 9.42. Use the definition of logarithm to prove that $\log x < 0$ for $0 < x < 1$, $\log 1 = 0$, and $\log x > 0$ for $x > 1$.

Exercise 9.43. Prove that $e > 2$ and $e < 3$ using integral estimates for $\log 2$ and $\log 3$.

Exercise 9.44. Let $a > 0$. Prove that $x \mapsto a^x$ is increasing, constant, or decreasing according as $a > 1$, $a = 1$, or $0 < a < 1$.

Exercise 9.45. Use the derivative formulas to prove the double-angle identities

$$\sin(2x) = 2 \sin x \cos x, \qquad \cos(2x) = \cos^2 x - \sin^2 x.$$

Exercise 9.46. Verify that the indicated restrictions of sine, cosine, and tangent in Definition 9.23 are bijective onto their stated ranges.

Exercise 9.47. Use Theorem 9.25 to show that

$$\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$$

on every interval contained in $(-1, 1)$.

Exercise 9.48. Prove, for $x > 0$,

$$\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$

Exercise 9.49. Prove all the addition formulas in Theorem 9.29 directly from the exponential definitions.

Exercise 9.50. Show that

$$\operatorname{arsinh} x = \log\left(x + \sqrt{1+x^2}\right)$$

by differentiating the right-hand side and checking its value at $x = 0$.

Exercise 9.51. Use the inverse-hyperbolic derivative formulas to find antiderivatives of

$$\frac{1}{\sqrt{1+x^2}}, \qquad \frac{1}{\sqrt{x^2-1}} \quad (x > 1), \qquad \frac{1}{1-x^2} \quad (|x| < 1).$$

Exercise 9.52. Prove Lemma 9.36 carefully from the Euclidean triangle inequality.

Exercise 9.53. Let $f(x) = x^2/2$ on $[0, 1]$. Use Theorem 9.37 to express the length of its graph as an integral. Do not assume that the integral has an elementary antiderivative.

Exercise 9.54. Find the length of the graph of

$$f(x) = \frac{x^3}{3} + \frac{1}{4x}$$

on $[1, 2]$.

Exercise 9.55. Explain exactly where the continuity of f' is used in the proof of Theorem 9.37. Can it be weakened if one assumes directly that $\sqrt{1 + (f')^2}$ is Darboux integrable?

Chapter 10

Integration Techniques

The Fundamental Theorem of Calculus reduces the calculation of a definite integral to the problem of finding a primitive. This chapter develops several systematic methods for finding primitives. Unlike differentiation, antidifferentiation is not governed by a single universal algorithm. The purpose of the methods below is to recognize particular structures in an integrand and to transform them into forms whose primitives are known.

Every identity involving an indefinite integral is understood on an interval on which the functions involved are defined. This qualification is essential: for example, $1/x$ has different primitive constants on the two components of its domain.

10.1 Primitives and notation

Definition 10.1. Let I be an interval and let $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is a *primitive* or *antiderivative* of f on I if

$$F'(x) = f(x) \qquad \text{for every } x \in I.$$

The notation

$$\int f(x)\,dx = F(x)$$

is shorthand for the assertion that F is a primitive of f on the interval under discussion. It is not an equality between a number on the left and a function on the right.

Theorem 10.2 (Uniqueness of primitives up to a constant)**.** *Let I be an interval. If F and G are primitives of the same function $f : I \to \mathbb{R}$, then there is a constant C such that*

$$F(x) = G(x) + C \qquad \text{for every } x \in I.$$

Proof. The function $H = F - G$ is differentiable on I, and

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0.$$

By the consequence of the Mean Value Theorem stating that a differentiable function with zero derivative on an interval is constant, $H(x) = C$ on I. Thus $F(x) = G(x) + C$. □

Remark 10.3. The interval hypothesis in Theorem 10.2 cannot be omitted. On $\mathbb{R} \setminus \{0\}$, both

$$F(x) = \log|x|$$

and the function

$$G(x) = \begin{cases} \log x, & x > 0, \\ \log(-x) + 1, & x < 0 \end{cases}$$

have derivative $1/x$, but $F - G$ is not a single constant on the disconnected domain $\mathbb{R} \setminus \{0\}$.

10.2 Integration by substitution

The substitution rule is the integral form of the chain rule. The proof for definite integrals follows directly from the Fundamental Theorem of Calculus and does not depend on heuristic manipulation of differentials.

Theorem 10.4 (Substitution rule)**.** *Let I and J be intervals, let $f : J \to \mathbb{R}$ be continuous, and let F be a primitive of f on J. Suppose that $g : I \to J$ is differentiable. Then*

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Equivalently,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \qquad u = g(x).$$

If, moreover, $a, b \in I$ and g is continuously differentiable on the closed interval with endpoints a and b, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. By the chain rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

at every point of I at which g is differentiable. Thus $F \circ g$ is a primitive of $f(g)g'$ on I, proving the indefinite statement.

For the definite statement, the additional hypothesis makes $f(g)g'$ continuous, hence Darboux integrable by Corollary 7.52, on the closed interval with endpoints a and b. Since F is differentiable it is continuous, and g is continuous, so $F \circ g$ is continuous there; by the chain rule it is a primitive of $f(g)g'$ on the interior. The Fundamental Theorem of Calculus, Part II (Theorem 7.57), applied to $f(g)g'$ and its primitive $F \circ g$, gives

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)).$$

The same theorem, now applied to f and its primitive F on the interval with endpoints $g(a)$ and $g(b)$, gives

$$F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du.$$

□

Remark 10.5. The requirement that g be continuously differentiable on the interval of integration is stronger than necessary for the definite formula, but it is the cleanest hypothesis for a first course: it guarantees that $f(g)g'$ is continuous, hence integrable. A more general sufficient condition is that $f(g)g'$ be Darboux integrable on the interval of integration and that the primitive $F \circ g$ be continuous at the endpoints, which is exactly the endpoint continuity required by the Fundamental Theorem of Calculus, Part II (Theorem 7.57). No monotonicity of g is required in either formulation: the definite identity holds when g is decreasing just as when it is increasing.

Remark 10.6. The notation $du = g'(x)\,dx$ is useful shorthand, but the theorem—not formal cancellation of symbols—is the justification for substitution. For a definite integral, one must either change the limits to $g(a)$ and $g(b)$ or return to the original variable before evaluating.

Example 10.7. Since $\frac{d}{dx}(x^2) = 2x$, Theorem 10.4 gives

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Likewise,

$$\int_0^1 \frac{3x^2}{1+x^3}\,dx = \int_1^2 \frac{du}{u} = \log 2.$$

Corollary 10.8. *If g is differentiable and nonzero on an interval I, then*

$$\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + C.$$

Proof. On an interval where g has constant sign, $\log|g(x)|$ is differentiable and

$$\frac{d}{dx}\log|g(x)| = \frac{g'(x)}{g(x)}.$$

□

10.3 Integration by parts

Integration by parts reverses the product rule.

Theorem 10.9 (Integration by parts)**.** *Let u and v be differentiable functions on an interval I, with u' and v' continuous on I. Then*

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

In differential notation,

$$\int u\,dv = uv - \int v\,du.$$

If $a, b \in I$, then

$$\int_a^b u(x)v'(x)\,dx = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\,dx.$$

Proof. The product rule gives

$$(uv)' = u'v + uv'.$$

Rearranging gives

$$uv' = (uv)' - u'v.$$

Taking primitives proves the indefinite formula. Integrating the same equality from a to b and applying the Fundamental Theorem of Calculus proves the definite formula. □

Example 10.10. To calculate

$$\int xe^x\,dx,$$

take $u = x$ and $v' = e^x$. Thus $u' = 1$ and $v = e^x$, so

$$\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x-1) + C.$$

Example 10.11. Even when an integrand does not visibly appear as a product, one may write it as one. For $x > 0$,

$$\int \log x \, dx = \int (\log x) \cdot 1 \, dx.$$

Taking $u = \log x$ and $v' = 1$ gives

$$\int \log x \, dx = x \log x - \int 1 \, dx = x \log x - x + C.$$

Example 10.12 (A repeated application)**.** Let

$$I = \int e^x \cos x \, dx.$$

Integrating by parts with $u = \cos x$ and $v' = e^x$ gives

$$I = e^x \cos x + \int e^x \sin x \, dx.$$

Applying integration by parts to the remaining integral gives

$$\int e^x \sin x \, dx = e^x \sin x - I.$$

Thus

$$2I = e^x(\sin x + \cos x),$$

and hence

$$\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C.$$

Proposition 10.13 (Reduction formula for powers of sine)**.** *For each integer $n \geq 2$,*

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

Proof. Write

$$\int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx.$$

Take $u = \sin^{n-1} x$ and $dv = \sin x \, dx$. Then

$$du = (n-1) \sin^{n-2} x \cos x \, dx, \qquad v = -\cos x.$$

Integration by parts gives

$$\begin{aligned} \int \sin^n x \, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx. \end{aligned}$$

Move the final term to the left and divide by n to obtain the formula. □

10.4 Trigonometric integrals

The identities

$$\sin^2 x + \cos^2 x = 1, \qquad 1 + \tan^2 x = \sec^2 x, \qquad \sec^2 x - 1 = \tan^2 x$$

allow many trigonometric integrals to be reduced to substitution or integration by parts.

Proposition 10.14 (Odd powers of sine or cosine)**.** *Let m, n be nonnegative integers.*

- *If m is odd, then an integral of the form*

$$\int \sin^m x \cos^n x \, dx$$

can be reduced by separating one factor of $\sin x$, replacing the remaining even power of sine using $\sin^2 x = 1 - \cos^2 x$, and using $u = \cos x$.

- *If n is odd, the analogous reduction uses one factor of $\cos x$, the identity $\cos^2 x = 1 - \sin^2 x$, and $u = \sin x$.*

Proof. Suppose that $m = 2r + 1$. Then

$$\sin^m x \cos^n x = (1 - \cos^2 x)^r \cos^n x \sin x.$$

Under $u = \cos x$, $du = -\sin x \, dx$, so the integral becomes a polynomial integral in u. The second assertion is proved symmetrically. □

Example 10.15. Using Proposition 10.14,

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\ &= -\int (u^2 - u^4) \, du \\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.\end{aligned}$$

Proposition 10.16 (Even powers)**.** *When both powers are even, the half-angle identities*

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

reduce the powers.

Example 10.17.

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \frac{1}{4} \int \sin^2(2x) \, dx \\ &= \frac{1}{8} \int (1 - \cos 4x) \, dx \\ &= \frac{x}{8} - \frac{\sin 4x}{32} + C.\end{aligned}$$

Proposition 10.18 (Powers of tangent and secant)**.** *Let m, n be nonnegative integers, and consider*

$$\int \tan^m x \sec^n x \, dx.$$

- *If n is even with $n \geq 2$, reserve one factor of $\sec^2 x$ and express the remaining power of secant in terms of $\tan x$ using $\sec^2 x = 1 + \tan^2 x$. The substitution $u = \tan x$, $du = \sec^2 x\, dx$, then reduces the integral to that of a polynomial in u.*
- *If m is odd with $m \geq 1$, and $n \geq 1$, reserve one factor of $\sec x \tan x$ and express the remaining power of tangent in terms of $\sec x$ using $\tan^2 x = \sec^2 x - 1$. The substitution $u = \sec x$, $du = \sec x \tan x\, dx$, then reduces the integral to that of a polynomial in u.*

Reserving $\sec^2 x$ requires a secant power at least 2; reserving $\sec x \tan x$ requires an available factor of $\sec x$ (that is, $n \geq 1$). The pure cases are handled separately. When $n = 0$, the integral is a pure power of tangent: repeatedly writing $\tan^2 x = \sec^2 x - 1$ lowers the exponent until only $\int dx = x + C$ or $\int \tan x\, dx = -\log|\cos x| + C$ remains. When $m = 0$, the integral is a pure power of secant, reduced by the integration-by-parts recursion (the case $n = 3$ is computed in the exercises).

Example 10.19.

$$\int \tan^3 x \sec^2 x\, dx = \int u^3\, du = \frac{\tan^4 x}{4} + C,$$

where $u = \tan x$.

10.4.1 The Weierstrass substitution

For $x \in (-\pi, \pi)$, set

$$t = \tan\left(\frac{x}{2}\right).$$

As x runs through $(-\pi, \pi)$, t runs through all of $\mathbb{R}$. This substitution expresses $\sin x$, $\cos x$, and dx rationally in t. From

$$1 + \tan^2\left(\frac{x}{2}\right) = \sec^2\left(\frac{x}{2}\right)$$

we obtain

$$\cos^2\left(\frac{x}{2}\right) = \frac{1}{1+t^2}.$$

Therefore

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2},$$

and

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\tan\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{2t}{1+t^2}.$$

Finally, differentiating $t = \tan(x/2)$ gives

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1+t^2}{2}, \qquad \text{so} \qquad dx = \frac{2\,dt}{1+t^2}.$$

Consequently, for any rational function R of two variables, the substitution converts

$$\int R(\sin x, \cos x)\, dx$$

into the integral of a rational function of t alone, to which the methods of Section 10.6 apply.

Example 10.20. Evaluate

$$\int \frac{dx}{1+\sin x}.$$

With $t = \tan(x/2)$,

$$1+\sin x = 1+\frac{2t}{1+t^2} = \frac{1+t^2+2t}{1+t^2} = \frac{(1+t)^2}{1+t^2},$$

and $dx = 2\,dt/(1+t^2)$. Hence

$$\int \frac{dx}{1+\sin x} = \int \frac{1+t^2}{(1+t)^2}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{(1+t)^2}\,dt = -\frac{2}{1+t}+C = -\frac{2}{1+\tan(x/2)}+C.$$

Differentiation confirms the result: if $F(x) = -2/(1+\tan(x/2))$, then

$$F'(x) = \frac{1+\tan^2(x/2)}{(1+\tan(x/2))^2} = \frac{1+t^2}{(1+t)^2} = \frac{1}{1+\sin x}.$$

Example 10.21. Evaluate

$$\int \frac{dx}{2+\cos x}.$$

With $t = \tan(x/2)$,

$$2+\cos x = 2+\frac{1-t^2}{1+t^2} = \frac{3+t^2}{1+t^2},$$

so

$$\int \frac{dx}{2+\cos x} = \int \frac{1+t^2}{3+t^2}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{t^2+3}\,dt = \frac{2}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)+C = \frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right)+C.$$

10.5 Trigonometric substitution

Trigonometric substitution is a method for removing square roots of quadratic expressions. The domain restrictions below are part of the method: they determine the sign of the square root after a trigonometric identity is used.

Theorem 10.22 (Substitution for a^2-x^2)**.** *Let $a>0$. On an interval contained in $(-a,a)$, set*

$$x = a\sin\theta, \qquad \theta = \arcsin(x/a).$$

Then

$$\sqrt{a^2-x^2} = a\cos\theta, \qquad dx = a\cos\theta\,d\theta.$$

Proof. Because $\theta \in (-\pi/2, \pi/2)$, one has $\cos\theta > 0$. Therefore

$$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)} = a\sqrt{\cos^2\theta} = a\cos\theta.$$

The formula for dx follows by differentiation. □

Theorem 10.23 (Substitution for a^2+x^2)**.** *Let $a>0$. Set*

$$x = a\tan\theta, \qquad \theta = \arctan(x/a).$$

Then

$$\sqrt{a^2+x^2} = a\sec\theta, \qquad dx = a\sec^2\theta\,d\theta.$$

Proof. The principal range of arctan is $(-\pi/2, \pi/2)$, where $\sec\theta > 0$. Hence

$$\sqrt{a^2 + x^2} = a\sqrt{1 + \tan^2\theta} = a\sec\theta.$$

The differential relation follows by differentiation. □

Theorem 10.24 (Substitution for $x^2 - a^2$)**.** *Let $a > 0$. On either interval (a, ∞) or $(-\infty, -a)$, set*

$$x = a\sec\theta$$

with a branch of arcsec *appropriate to that interval. Then*

$$\sqrt{x^2 - a^2} = a|\tan\theta|, \qquad dx = a\sec\theta\tan\theta\, d\theta.$$

On the branch $\theta \in [0, \pi/2)$ corresponding to $x \geq a$, this simplifies to

$$\sqrt{x^2 - a^2} = a\tan\theta.$$

Proof. The identity $\sec^2\theta - 1 = \tan^2\theta$ gives

$$\sqrt{x^2 - a^2} = a\sqrt{\sec^2\theta - 1} = a|\tan\theta|.$$

The sign simplification on the positive branch follows because tangent is nonnegative there. □

Example 10.25. For $a > 0$,

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx &= a^2 \int \cos^2\theta\, d\theta \\ &= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C.\end{aligned}$$

Example 10.26. For $a > 0$,

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + a^2}} &= \int \sec\theta\, d\theta \\ &= \log|\sec\theta + \tan\theta| + C \\ &= \log\left|x + \sqrt{x^2 + a^2}\right| + C.\end{aligned}$$

10.6 Rational functions and partial fractions

A rational function is a quotient $P(x)/Q(x)$ of polynomials, with Q not identically zero. The partial-fraction method reduces the integration of rational functions to a finite collection of elementary primitives.

Theorem 10.27 (Polynomial division)**.** *Let P and Q be polynomials with $Q \neq 0$. Then there are unique polynomials A and B such that*

$$P = AQ + B,$$

where either $B = 0$ or

$$\deg B < \deg Q.$$

Remark 10.28. The quotient P/Q should be divided first whenever $\deg P \geq \deg Q$. After division, only the proper rational part B/Q requires partial fractions.

Theorem 10.29 (Partial-fraction form)**.** *Let P/Q be a proper rational function, and factor Q over $\mathbb{R}$ into powers of distinct linear factors and distinct irreducible quadratic factors. Then its partial-fraction decomposition has the form*

$$\sum_{i} \left(\frac{A_{i,1}}{x - r_i} + \frac{A_{i,2}}{(x - r_i)^2} + \cdots + \frac{A_{i,m_i}}{(x - r_i)^{m_i}} \right)$$

together with terms

$$\sum_{j} \left(\frac{B_{j,1}x + C_{j,1}}{q_j(x)} + \frac{B_{j,2}x + C_{j,2}}{q_j(x)^2} + \cdots + \frac{B_{j,n_j}x + C_{j,n_j}}{q_j(x)^{n_j}} \right),$$

where each q_j is an irreducible quadratic polynomial.

Remark 10.30. The existence and uniqueness of the decomposition are algebraic facts. In applications, the unknown coefficients are found by multiplying through by the denominator and comparing polynomial coefficients, or by substituting suitable values of x.

Example 10.31 (Distinct linear factors)**.** Decompose

$$\frac{1}{x^2 - x - 2} = \frac{1}{(x - 2)(x + 1)} = \frac{A}{x - 2} + \frac{B}{x + 1}.$$

Multiplying by $(x - 2)(x + 1)$ gives

$$1 = A(x + 1) + B(x - 2).$$

Setting $x = 2$ gives $A = 1/3$, and setting $x = -1$ gives $B = -1/3$. Thus

$$\int \frac{dx}{x^2 - x - 2} = \frac{1}{3}\log|x - 2| - \frac{1}{3}\log|x + 1| + C.$$

Example 10.32 (Repeated linear factor)**.** Suppose

$$\frac{x + 1}{x(x - 1)^2} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}.$$

Multiplication by $x(x - 1)^2$ and comparison of coefficients yields

$$A = 1, \qquad B = -1, \qquad C = 2.$$

Therefore

$$\int \frac{x + 1}{x(x - 1)^2}\,dx = \log|x| - \log|x - 1| - \frac{2}{x - 1} + C.$$

Example 10.33 (An irreducible quadratic)**.** Completing the square gives

$$\int \frac{dx}{x^2 + 6x + 13} = \int \frac{dx}{(x + 3)^2 + 2^2} = \frac{1}{2}\arctan\left(\frac{x + 3}{2}\right) + C.$$

Proposition 10.34. *Let $q(x) = x^2 + bx + c$ have no real roots. Every integral of the form*

$$\int \frac{Ax + B}{q(x)}\,dx$$

reduces to a logarithmic term and an inverse-tangent term after writing the numerator as a multiple of $q'(x) = 2x + b$ plus a constant and completing the square in the denominator.

Proof. There are constants α, β such that

$$Ax + B = \alpha(2x + b) + \beta.$$

Thus the integral is the sum of

$$\alpha \int \frac{q'(x)}{q(x)}\,dx$$

and

$$\beta \int \frac{dx}{q(x)}.$$

The first is logarithmic by Corollary 10.8; completing the square converts the second to a constant multiple of the derivative form of arctan. □

Proposition 10.35 (Repeated irreducible quadratic factors)**.** *For $a > 0$ and each integer $k \geq 1$, set*

$$I_k = \int \frac{dx}{(x^2 + a^2)^k}.$$

Then

$$I_1 = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C,$$

and for every $k \geq 1$,

$$I_{k+1} = \frac{1}{2ka^2}\left(\frac{x}{(x^2 + a^2)^k} + (2k - 1)I_k\right).$$

Proof. The formula for I_1 is the standard arctangent primitive. For the recursion, integrate I_k by parts with

$$u = (x^2 + a^2)^{-k}, \qquad dv = dx,$$

so that $v = x$ and

$$du = -2kx(x^2 + a^2)^{-k-1}\,dx.$$

This gives

$$I_k = \frac{x}{(x^2 + a^2)^k} + 2k \int \frac{x^2}{(x^2 + a^2)^{k+1}}\,dx.$$

Writing $x^2 = (x^2 + a^2) - a^2$ in the numerator,

$$\int \frac{x^2}{(x^2 + a^2)^{k+1}}\,dx = \int \frac{dx}{(x^2 + a^2)^k} - a^2 \int \frac{dx}{(x^2 + a^2)^{k+1}} = I_k - a^2 I_{k+1}.$$

Substituting back gives

$$I_k = \frac{x}{(x^2 + a^2)^k} + 2k(I_k - a^2 I_{k+1}),$$

which rearranges to the stated formula. □

Example 10.36. Evaluate

$$\int \frac{dx}{(x^2 + 1)^2}.$$

This is I_2 with $a = 1$. Applying Proposition 10.35 with $k = 1$,

$$I_2 = \frac{1}{2}\left(\frac{x}{x^2 + 1} + I_1\right),$$

and $I_1 = \arctan x$. Therefore

$$\int \frac{dx}{(x^2 + 1)^2} = \frac{x}{2(x^2 + 1)} + \frac{1}{2}\arctan x + C.$$

Remark 10.37. Two algebraic facts are assumed in this section without proof. First, every real polynomial factors over $\mathbb{R}$ as a product of linear factors and irreducible quadratic factors; second, the partial-fraction decomposition of Theorem 10.29 exists and is unique. Given these facts, the integration of a rational function is complete: polynomial division produces a polynomial plus a proper fraction; the proper fraction decomposes into the displayed elementary terms; the linear terms are logarithmic or reciprocal powers; and each quadratic term $(Bx + C)/q(x)^k$ is handled by completing the square in q and writing $Bx + C = \alpha q'(x) + \beta$. The α-term is integrated by the substitution $u = q(x)$, and the β-term becomes a constant multiple of I_k in Proposition 10.35.

10.7 Improper integrals revisited

The Darboux integral developed earlier applies to bounded functions on closed bounded intervals. An improper integral is defined by a limit when either the interval is unbounded or the integrand becomes unbounded at an endpoint or interior point.

Definition 10.38. Suppose that f is integrable on $[a, R]$ for every $R > a$. The improper integral

$$\int_a^{\infty} f(x)\,dx$$

converges if the finite limit

$$\lim_{R \to \infty} \int_a^{R} f(x)\,dx$$

exists. Its value is defined to be that limit.

Definition 10.39. Suppose that f is integrable on $[a + \varepsilon, b]$ for every $\varepsilon > 0$. The improper integral

$$\int_a^{b} f(x)\,dx$$

converges at a if

$$\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx$$

exists as a finite real number.

Theorem 10.40 (p-integral criterion)**.** *For $p \in \mathbb{R}$,*

$$\int_1^{\infty} \frac{dx}{x^p}$$

converges if and only if $p > 1$. When it converges,

$$\int_1^{\infty} \frac{dx}{x^p} = \frac{1}{p-1}.$$

Also,

$$\int_0^{1} \frac{dx}{x^p}$$

converges if and only if $p < 1$, in which case its value is $1/(1-p)$.

Proof. For $p \neq 1$,

$$\int_1^R x^{-p}\,dx = \frac{R^{1-p}-1}{1-p}.$$

As $R \to \infty$, this has a finite limit precisely when $1-p<0$, or $p>1$, and the limit is $1/(p-1)$. For $p=1$,

$$\int_1^R \frac{dx}{x} = \log R \to \infty.$$

For the integral near zero, use

$$\int_\varepsilon^1 x^{-p}\,dx = \frac{1-\varepsilon^{1-p}}{1-p}$$

and let $\varepsilon \to 0^+$. □

Theorem 10.41 (Comparison test)**.** *Let $0 \le g(x) \le f(x)$ for all $x \ge a$, and suppose both functions are integrable on each compact subinterval of $[a,\infty)$.*

- *If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.*
- *If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.*

Proof. For $R > a$, comparison on the finite interval $[a,R]$ gives

$$0 \le \int_a^R g(x)\,dx \le \int_a^R f(x)\,dx.$$

If the improper integral of f converges, the increasing function of R given by $\int_a^R g$ is bounded above. By completeness, it has a finite limit, proving the first statement. The second is the contrapositive of the first. □

10.8 Strategy and limitations

The methods in this chapter are not a complete algorithm for finding primitives. They are rigorous transformations that apply when an integrand has an appropriate form.

A practical order of inquiry is often the following:

1. Look for a composite expression together with a multiple of its derivative; use substitution.
2. Look for a product in which differentiating one factor simplifies it; use integration by parts.
3. Use trigonometric identities to simplify powers or products of trigonometric functions.
4. Use a trigonometric substitution for radicals involving quadratic expressions.
5. For rational functions, divide first if necessary and then use partial fractions.

There are elementary functions that have no elementary primitive. Thus the failure of these techniques to produce an answer does not imply a failure of ingenuity; sometimes no elementary expression exists.

10.9 Numerical integration

The methods of this chapter produce exact primitives only for integrands of particular forms. Many elementary integrands—for example e^{-x^2}—have no elementary primitive at all, and even when a primitive exists it may be much harder to find than the value of the definite integral one actually needs. The Fundamental Theorem of Calculus is exact, but it is not always computationally usable. This section records two methods that approximate a definite integral directly from finitely many values of the integrand.

Definition 10.42 (Composite trapezoid rule)**.** Let $f : [a,b] \to \mathbb{R}$ be integrable, and let $n \in \mathbb{N}$. Subdivide $[a,b]$ into n equal subintervals with

$$h = \frac{b-a}{n}, \qquad x_k = a + kh \qquad (0 \le k \le n).$$

The *composite trapezoid rule* approximates $\int_a^b f(x)\,dx$ by

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Definition 10.43 (Simpson's rule)**.** With the notation of Definition 10.42 and with n even, *Simpson's rule* approximates $\int_a^b f(x)\,dx$ by

$$S_n = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Example 10.44. Approximate

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

With $n = 2$ we have $h = 1/2$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$, and $f(0) = 0$, $f(1/2) = 1/4$, $f(1) = 1$. Therefore

$$T_2 = \frac{1/2}{2}\left[0 + 2\cdot\frac{1}{4} + 1\right] = \frac{1}{4}\cdot\frac{3}{2} = \frac{3}{8},$$

while

$$S_2 = \frac{1/2}{3}\left[0 + 4\cdot\frac{1}{4} + 1\right] = \frac{1}{6}\cdot 2 = \frac{1}{3}.$$

Simpson's rule gives the exact value here, and in fact it is exact for every quadratic (and even every cubic) polynomial. The trapezoid rule with two subintervals slightly overshoots: $T_2 = 3/8 = 0.375$ against the true value $1/3 \approx 0.333$.

Remark 10.45 (‡ Classical error bounds). Under additional smoothness the errors obey the classical estimates

$$\left|\int_a^b f(x)\,dx - T_n\right| \le \frac{(b-a)^3}{12n^2}\sup_{[a,b]}|f''|,$$

and

$$\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{(b-a)^5}{180n^4}\sup_{[a,b]}|f''''|,$$

where the suprema are taken over $[a,b]$ and the indicated derivatives are assumed continuous. Their proofs use Taylor's theorem with remainder, which is developed in Chapter 11; they are not proved here.

10.10 Exercises

Exercise 10.46. Use Theorem 10.4 to prove that

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

when g is decreasing as well as when g is increasing.

Exercise 10.47. Evaluate:

$$\int x(1+x^2)^5\,dx, \qquad \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx, \qquad \int_0^1 xe^{x^2}\,dx.$$

Exercise 10.48. Use integration by parts to compute:

$$\int x\sin x\,dx, \qquad \int x^2\log x\,dx, \qquad \int e^x\sin x\,dx.$$

Exercise 10.49. Prove the corresponding reduction formula for

$$\int \cos^n x\,dx$$

for integers $n \geq 2$.

Exercise 10.50. Evaluate:

$$\int \sin^5 x\cos^4 x\,dx, \qquad \int \sin^2 x\,dx, \qquad \int \tan^4 x\sec^2 x\,dx.$$

Exercise 10.51. Use integration by parts to derive

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\log|\sec x + \tan x| + C.$$

Exercise 10.52. Evaluate, with correct domain restrictions,

$$\int \frac{dx}{\sqrt{9-x^2}}, \qquad \int \sqrt{x^2+4}\,dx, \qquad \int \frac{dx}{x^2\sqrt{x^2-1}}.$$

Exercise 10.53. Find the partial-fraction decomposition of

$$\frac{3x+5}{(x-1)(x+2)},$$

and then integrate the function.

Exercise 10.54. Evaluate

$$\int \frac{x^2+1}{x(x+1)^2}\,dx.$$

Exercise 10.55. Use polynomial division and then partial fractions to evaluate

$$\int \frac{x^3+1}{x^2-x}\,dx.$$

Exercise 10.56. Show directly that

$$\int \frac{2x+5}{x^2+4x+13}\,dx$$

is a sum of a logarithmic expression and an inverse-tangent expression.

Exercise 10.57. Determine whether each improper integral converges. Evaluate those that do:

$$\int_1^\infty \frac{dx}{x^3}, \qquad \int_1^\infty \frac{dx}{\sqrt{x}}, \qquad \int_0^1 \frac{dx}{\sqrt{x}}.$$

Exercise 10.58. Use the comparison test to determine whether

$$\int_1^\infty \frac{dx}{x^2 + \sin^2 x}$$

converges.

Exercise 10.59. Evaluate the improper integral

$$\int_0^1 \log x \, dx.$$

Exercise 10.60. Give an example of a continuous elementary function for which the methods of this chapter do not produce an elementary primitive. Explain why the chapter does not claim that every continuous function has an elementary primitive.

Part IV

Sequences and Series

Chapter 11

Taylor Polynomials

Polynomials are among the most tractable functions in calculus: they can be differentiated, integrated, and evaluated without difficulty. Taylor polynomials provide a systematic way to approximate a sufficiently differentiable function near a chosen point by a polynomial whose derivatives agree with those of the original function to a prescribed order.

The central question is not merely how to write down the polynomial, but how to measure the error. This chapter develops Taylor polynomials, proves their uniqueness, establishes Taylor's theorem with both exact integral and Lagrange forms of the remainder, and derives rigorous approximation estimates.

11.1 Taylor polynomials and contact of order

Definition 11.1. Let f have derivatives through order n at a. The *Taylor polynomial of degree n for f centered at a* is

$$T_{n,a}f(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

When the center a is clear from the context, we write $T_n f(x)$.

Thus

$$T_{n,a}f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The degree of $T_{n,a}f$ may be less than n if some of the final coefficients vanish.

Definition 11.2. Let f and g be functions defined at and near a, and let $n \geq 0$. We say that f and g *agree to order n at a* if

$$f(a) = g(a),$$

and, when $n \geq 1$, additionally

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0,$$

equivalently $f(x) - g(x) = o(|x-a|^n)$ as $x \to a$ (see Definition 11.17).

The phrase "agree to order n" therefore imposes two conditions: equality of the values at a, and, for $n \geq 1$, a difference that is negligible relative to $|x-a|^n$ as x approaches a. The deleted-neighbourhood limit alone says nothing about $f(a) - g(a)$. If f and g are differentiable at a, agreement to order 1 is equivalent to equality of their values and first derivatives at a; for arbitrary functions the asymptotic condition should not be read as asserting the existence of a common linear approximation.

Theorem 11.3 (Derivative matching)**.** *Suppose that f has derivatives through order n at a. Then*

$$\big(T_{n,a}f\big)^{(k)}(a) = f^{(k)}(a) \qquad \text{for } k = 0, 1, \dots, n.$$

Proof. Write

$$T_{n,a}f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

After differentiating k times, every term with $j < k$ has disappeared, and every term with $j > k$ still contains a positive power of $x - a$. The $j = k$ term becomes exactly $f^{(k)}(a)$. Therefore, evaluating at $x = a$ leaves only that term:

$$\big(T_{n,a}f\big)^{(k)}(a) = f^{(k)}(a).$$

□

Theorem 11.4 (Uniqueness of the Taylor polynomial)**.** *Let f have derivatives through order n at a. The Taylor polynomial $T_{n,a}f$ is the unique polynomial of degree at most n whose derivatives through order n agree with those of f at a.*

Proof. Theorem 11.3 proves existence. For uniqueness, suppose that P and Q are polynomials of degree at most n and that

$$P^{(k)}(a) = Q^{(k)}(a) \qquad (k = 0, 1, \dots, n).$$

Set $R = P - Q$. Then

$$R^{(k)}(a) = 0 \qquad (k = 0, 1, \dots, n).$$

Write R in powers of $x - a$:

$$R(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n.$$

The equality $R(a) = 0$ gives $c_0 = 0$. The equality $R'(a) = 0$ gives $c_1 = 0$, and continuing in this manner gives $c_k = 0$ for every k. Hence $R = 0$, so $P = Q$. □

Example 11.5. For $f(x) = e^x$, every derivative equals e^x. Centered at 0, the degree-four Taylor polynomial is

$$T_{4,0}e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

Example 11.6. For $f(x) = \sin x$, the derivatives repeat with period four:

$$\sin x, \quad \cos x, \quad -\sin x, \quad -\cos x.$$

At 0, these values are $0, 1, 0, -1$. Hence

$$T_{6,0}\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

11.2 Maclaurin polynomials

Definition 11.7. A Taylor polynomial centered at 0 is called a *Maclaurin polynomial*. Thus

$$T_n f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

Maclaurin polynomials are especially useful because the values of many familiar functions and their derivatives at 0 are simple.

Example 11.8. The degree-six Maclaurin polynomial for cosine is

$$T_6 \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Example 11.9. Let $f(x) = \log(1+x)$, defined near 0. For $k \geq 1$,

$$f^{(k)}(x) = (-1)^{k-1} \frac{(k-1)!}{(1+x)^k}.$$

Therefore

$$f^{(k)}(0) = (-1)^{k-1}(k-1)!.$$

The degree-four Maclaurin polynomial is

$$T_4 \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

11.3 The exact integral remainder

The Taylor polynomial is determined entirely by derivative data at a single point. Taylor's theorem identifies the precise difference between the function and its Taylor polynomial by an integral involving the next derivative.

Theorem 11.10 (Taylor's theorem with exact integral remainder)**.** *Let $n \geq 0$. Suppose that $f^{(n+1)}$ is continuous on an interval containing a and x. Then*

$$f(x) = T_{n,a} f(x) + R_{n,a} f(x),$$

where

$$R_{n,a} f(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$

Equivalently,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$

Proof. The case $n = 0$ is the Fundamental Theorem of Calculus:

$$f(x) = f(a) + \int_a^x f'(t) \, dt.$$

Assume $n \geq 1$, and set

$$I_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$

Integrate by parts, taking

$$u = (x-t)^n, \qquad dv = f^{(n+1)}(t) \, dt.$$

Then

$$du = -n(x-t)^{n-1}\,dt, \qquad v = f^{(n)}(t),$$

so

$$\begin{aligned} I_n(x) &= \frac{1}{n!}\left(\Big[f^{(n)}(t)(x-t)^n\Big]_a^x + n\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt\right) \\ &= -\frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt. \end{aligned}$$

Applying the same calculation repeatedly yields

$$\begin{aligned} I_n(x) &= f(x) - f(a) - f'(a)(x-a) - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(x) - T_{n,a}f(x). \end{aligned}$$

Rearranging proves the theorem. □

Remark 11.11. The integral remainder is an exact identity, not merely an estimate. The approximation enters only when the remainder is shown to be small.

Corollary 11.12 (Integral remainder bound)**.** *Suppose that*

$$|f^{(n+1)}(t)| \le M$$

for every t between a and x. Then

$$|R_{n,a}f(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Proof. Suppose first that $x \ge a$. By Theorem 11.10,

$$\begin{aligned} |R_{n,a}f(x)| &\le \frac{1}{n!}\int_a^x |f^{(n+1)}(t)|(x-t)^n\,dt \\ &\le \frac{M}{n!}\int_a^x (x-t)^n\,dt \\ &= \frac{M}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

The case $x < a$ follows by reversing the integral limits, giving the absolute-value form. □

11.4 The Lagrange remainder

The integral form immediately implies the more familiar Lagrange form by the weighted mean value theorem for integrals, which we prove first.

Theorem 11.13 (Weighted mean value theorem for integrals)**.** *Let f and g be continuous on $[a,b]$, with $g \ge 0$ on $[a,b]$. Then there is a point $c \in [a,b]$ such that*

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Proof. Since f is continuous on the compact interval $[a, b]$, the Extreme Value Theorem gives points where f attains its minimum m and maximum M, so that

$$m \le f(x) \le M \qquad \text{for all } x \in [a, b].$$

Because $g \ge 0$,

$$m\, g(x) \le f(x)g(x) \le M\, g(x),$$

and integrating over $[a, b]$ gives

$$m \int_a^b g(x)\, dx \le \int_a^b f(x)g(x)\, dx \le M \int_a^b g(x)\, dx.$$

If $\int_a^b g(x)\, dx = 0$, then both outer terms vanish, so $\int_a^b f(x)g(x)\, dx = 0$ and the identity holds for any $c \in [a, b]$. Otherwise $\int_a^b g(x)\, dx > 0$, and

$$m \le \frac{\int_a^b f(x)g(x)\, dx}{\int_a^b g(x)\, dx} \le M.$$

Since f is continuous and attains every value between m and M on $[a, b]$ (by the Intermediate Value Theorem), there is $c \in [a, b]$ with

$$f(c) = \frac{\int_a^b f(x)g(x)\, dx}{\int_a^b g(x)\, dx},$$

which is the desired identity. □

Theorem 11.14 (Taylor's theorem with Lagrange remainder)**.** *Let $n \ge 0$. Suppose that $f^{(n+1)}$ is continuous on an interval containing a and x. Then there is a point c between a and x such that*

$$f(x) = T_{n,a}f(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}.$$

Proof. If $x = a$, then $T_{n,a}f(a) = f(a)$ and the remainder term vanishes, so the formula holds with $c = a$. Assume henceforth that $x \ne a$, and first that $x > a$. The exact integral remainder gives

$$R_{n,a}f(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n\, dt.$$

The weight $(x - t)^n$ is nonnegative and continuous on $[a, x]$. By Theorem 11.13, there is a point $c \in [a, x]$ such that

$$\int_a^x f^{(n+1)}(t)(x - t)^n\, dt = f^{(n+1)}(c) \int_a^x (x - t)^n\, dt.$$

Since

$$\int_a^x (x - t)^n\, dt = \frac{(x - a)^{n+1}}{n + 1},$$

the desired formula follows. The case $x < a$ is analogous, with the orientation of the integral taken into account. □

Corollary 11.15 (Lagrange error estimate)**.** *Under the hypotheses of Theorem 11.14, if*

$$|f^{(n+1)}(t)| \le M$$

between a and x, then

$$|f(x) - T_{n,a}f(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}.$$

Proof. Take absolute values in the Lagrange remainder formula and use the bound on $f^{(n+1)}$. □

Remark 11.16. The point c in the Lagrange form generally cannot be found explicitly. Its value lies in certifying the existence of an error bound, rather than in providing a numerical formula for the error itself.

11.5 Agreement to order and Taylor approximation

Definition 11.17. Let f and g be functions defined on a deleted neighbourhood of a, with $g(x) \neq 0$ for every $x \neq a$ in that neighbourhood. We write

$$f(x) = o(g(x)) \qquad \text{as } x \to a$$

to mean that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

Theorem 11.18. *Suppose that $f^{(n+1)}$ is continuous near a. Then f and $T_{n,a}f$ agree to order n at a: the equality of values $T_{n,a}f(a) = f(a)$ holds by construction, and, for $n \geq 1$,*

$$\lim_{x \to a} \frac{f(x) - T_{n,a}f(x)}{(x-a)^n} = 0.$$

Proof. Choose a closed interval $[a-r, a+r]$ on which $f^{(n+1)}$ is continuous. By the Extreme Value Theorem, there is $M > 0$ with

$$|f^{(n+1)}(t)| \leq M$$

on that interval. Corollary 11.12 gives

$$|f(x) - T_{n,a}f(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for $|x-a| < r$. Dividing by $|x-a|^n$ yields

$$\left|\frac{f(x) - T_{n,a}f(x)}{(x-a)^n}\right| \leq \frac{M}{(n+1)!}|x-a|,$$

which tends to 0 as $x \to a$. □

Corollary 11.19. *Let P be a polynomial of degree at most n. Suppose $f^{(n+1)}$ is continuous near a and that f and P agree to order n at a. Then*

$$P = T_{n,a}f.$$

Proof. By Theorem 11.18, f and $T_{n,a}f$ agree to order n at a. Hence the polynomial

$$Q = P - T_{n,a}f$$

agrees to order n with the zero function at a: indeed $Q(a) = P(a) - T_{n,a}f(a) = f(a) - f(a) = 0$, and

$$\frac{Q(x)}{(x-a)^n} = \frac{P(x) - f(x)}{(x-a)^n} + \frac{f(x) - T_{n,a}f(x)}{(x-a)^n} \to 0.$$

Write

$$Q(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n.$$

If some coefficient were nonzero, let j be the smallest index with $c_j \neq 0$. Then

$$\frac{Q(x)}{(x-a)^j} = c_j + c_{j+1}(x-a) + \cdots + c_n(x-a)^{n-j} \to c_j \neq 0,$$

contradicting agreement to order n, which in particular implies agreement to order j. Therefore all coefficients vanish and $Q = 0$. □

Remark 11.20. By Theorem 11.18 and Corollary 11.19, the Taylor polynomial is the unique polynomial of degree at most n for which

$$f(x) = T_{n,a}f(x) + o\big((x-a)^n\big) \qquad \text{as } x \to a.$$

11.5.1 The Peano form of Taylor's theorem

Theorem 11.18 required continuity of $f^{(n+1)}$ near a. The asymptotic form $f(x) = T_{n,a}f(x) + o((x-a)^n)$ in fact holds under the weaker hypothesis that f is merely n times differentiable at a.

Theorem 11.21 (Taylor's theorem, Peano form)**.** *Let $n \geq 1$, and suppose that f is n times differentiable at a: $f^{(n-1)}$ is defined in a neighbourhood of a and $f^{(n)}(a)$ exists. Then*

$$f(x) = T_{n,a}f(x) + o\big((x-a)^n\big) \qquad \text{as } x \to a.$$

Proof. Set

$$R(x) = f(x) - T_{n,a}f(x).$$

By Theorem 11.3,

$$R(a) = R'(a) = \cdots = R^{(n)}(a) = 0.$$

For $n = 1$, the assertion is exactly the definition of $f'(a)$:

$$\frac{R(x)}{x-a} = \frac{f(x)-f(a)}{x-a} - f'(a) \to f'(a) - f'(a) = 0.$$

Assume $n \geq 2$. Since $f^{(n-1)}$ is defined near a and $T_{n,a}f$ is a polynomial, $R^{(k)}$ is differentiable near a for $0 \leq k \leq n-2$, and $R^{(k)}(a) = 0$ for $0 \leq k \leq n-1$. Thus both numerator and denominator of

$$\frac{R(x)}{(x-a)^n}$$

tend to 0 as $x \to a$, and L'Hôpital's rule (Theorem 6.26) applies. Applying it $n-1$ times gives

$$\lim_{x\to a} \frac{R(x)}{(x-a)^n} = \lim_{x\to a} \frac{R^{(n-1)}(x)}{n!\,(x-a)}.$$

Now

$$R^{(n-1)}(x) = f^{(n-1)}(x) - f^{(n-1)}(a) - f^{(n)}(a)(x-a),$$

because $(T_{n,a}f)^{(n-1)}(x) = f^{(n-1)}(a) + f^{(n)}(a)(x-a)$. Hence

$$\frac{R^{(n-1)}(x)}{n!\,(x-a)} = \frac{1}{n!}\left(\frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} - f^{(n)}(a)\right) \to \frac{1}{n!}\big(f^{(n)}(a) - f^{(n)}(a)\big) = 0,$$

using the definition of $f^{(n)}(a)$. Therefore the original quotient tends to 0, which is the Peano form. □

Remark 11.22 (The hierarchy of remainder forms). Derivative matching (Theorem 11.3) determines the Taylor polynomial; n-fold differentiability at a alone gives the Peano form (Theorem 11.21); and stronger hypotheses give the exact integral remainder (Theorem 11.10) and the Lagrange remainder (Theorem 11.14). Derivative bounds then convert these remainders into numerical estimates (Corollaries 11.12 and 11.15).

11.6 Standard Taylor polynomials

The following polynomial approximations are obtained directly from derivative patterns. Their convergence as degrees tend to infinity will be considered later, in the chapter on power series.

Proposition 11.23. *The Taylor polynomials centered at* 0 *have the following forms:*

$$\begin{aligned}
T_n e^x &= \sum_{k=0}^{n} \frac{x^k}{k!}, \\
T_{2n+1} \sin x &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \\
T_{2n} \cos x &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}, \\
T_n \log(1+x) &= \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k}, \\
T_{2n+1} \arctan x &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}.
\end{aligned}$$

The final two formulas are valid as Taylor polynomials at 0 *wherever the corresponding functions are defined near* 0.

Proof. For e^x, every derivative at 0 is 1. For sine and cosine, the derivative cycles of length four give the displayed alternating coefficients.

For $\log(1+x)$, the calculation in the preceding section gives

$$\frac{f^{(k)}(0)}{k!} = \frac{(-1)^{k-1}(k-1)!}{k!} = (-1)^{k-1}\frac{1}{k}.$$

For $\arctan x$, the displayed formula follows by differentiating $1/(1+x^2)$ repeatedly at 0, or by using the integral representation

$$\arctan x = \int_0^x \frac{dt}{1+t^2}$$

and the finite geometric identity developed below. □

Example 11.24 (Taylor polynomial for a square root)**.** Let

$$f(x) = \sqrt{1+x}.$$

Then

$$f(0) = 1, \qquad f'(0) = \frac{1}{2}, \qquad f''(0) = -\frac{1}{4}, \qquad f'''(0) = \frac{3}{8}.$$

Therefore

$$T_3 f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$$

11.7 Examples of rigorous approximation

Example 11.25 (Approximating $e^{0.1}$)**.** The degree-three Maclaurin polynomial gives

$$T_3 e^{0.1} = 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} = 1.105166\ldots.$$

For $0 \le t \le 0.1$, one has $e^t \le e^{0.1}$. Hence Corollary 11.15 gives

$$\left|e^{0.1} - T_3 e^{0.1}\right| \le \frac{e^{0.1}}{4!}(0.1)^4.$$

Thus the displayed polynomial value has a certified error smaller than 5×10^{-6}.

Example 11.26 (An alternating estimate for cosine)**.** For $x \in \mathbb{R}$,

$$T_4 \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Since every fifth derivative of cosine is either sin or $-\sin$, whose absolute values are at most 1, the Lagrange remainder gives

$$\left|\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\right| \le \frac{|x|^5}{5!}.$$

In particular, this gives a rigorous estimate for $\cos(0.2)$.

Example 11.27 (An exact sign of the exponential remainder)**.** For $x \ge 0$, Theorem 11.10 gives

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{1}{n!}\int_0^x e^t (x-t)^n \, dt.$$

The integrand is nonnegative. Therefore

$$\sum_{k=0}^{n} \frac{x^k}{k!} \le e^x \qquad (x \ge 0).$$

Thus each Maclaurin polynomial for e^x underestimates e^x on the nonnegative real axis.

Theorem 11.28. *The number e is irrational.*

Proof. First we record the crude bound $e < 4$. By Theorem 11.14 applied to e^x at $a = 0$, evaluated at $x = 1$, and using the Taylor polynomial from Proposition 11.23,

$$e = 1 + 1 + \frac{1}{2} + \frac{e^{\xi_2}}{6}$$

for some $\xi_2 \in (0, 1)$. Since $e^{\xi_2} < e$,

$$e < \frac{5}{2} + \frac{e}{6} < 3 + \frac{e}{6},$$

and therefore

$$e < \frac{18}{5} < 4.$$

More generally, the same theorem gives, for every integer $N \ge 0$,

$$e = \sum_{k=0}^{N} \frac{1}{k!} + \frac{e^{\xi_N}}{(N+1)!}$$

for some $\xi_N \in (0,1)$.

Suppose, for contradiction, that $e = p/q$ with $p, q \in \mathbb{N}$. Choose an integer N with $N \geq q$ and $N+1 > 4q$ (for example, $N = 4q$). Multiplying the displayed identity by $qN!$ gives

$$qN!e = qN! \sum_{k=0}^{N} \frac{1}{k!} + \frac{qN!e^{\xi_N}}{(N+1)!}.$$

The left-hand side is the integer

$$qN!e = qN! \cdot \frac{p}{q} = pN!.$$

In the first term on the right, each quotient $N!/k!$ is an integer because $k! \mid N!$ for every $k \leq N$; hence that term is an integer. For the second term,

$$0 < \frac{qN!e^{\xi_N}}{(N+1)!} = \frac{qe^{\xi_N}}{N+1} < \frac{4q}{N+1} < 1,$$

because $e^{\xi_N} < e < 4$ and $N+1 > 4q$. Thus an integer equals an integer plus a number strictly between 0 and 1, which is impossible. Hence e is irrational. □

Remark 11.29. The irrationality of π is substantially harder to prove.

11.8 Taylor polynomials from elementary identities

Taylor polynomials may sometimes be found more efficiently from a finite algebraic identity than from repeated differentiation.

Proposition 11.30 (Finite geometric identity)**.** *For every real number $u \neq -1$ and every integer $n \geq 0$,*

$$\frac{1}{1+u} = 1 - u + u^2 - \cdots + (-1)^n u^n + (-1)^{n+1} \frac{u^{n+1}}{1+u}.$$

Proof. The finite geometric-sum identity gives

$$(1+u) \sum_{k=0}^{n} (-u)^k = 1 - (-u)^{n+1}.$$

Dividing by $1+u$ yields

$$\sum_{k=0}^{n} (-u)^k = \frac{1-(-u)^{n+1}}{1+u}.$$

Rearranging gives the stated identity. □

Theorem 11.31 (Taylor polynomial for $\arctan x$)**.** *For every integer $n \geq 0$,*

$$\arctan x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\, dt.$$

Proof. Apply Proposition 11.30 with $u = t^2$:

$$\frac{1}{1+t^2} = \sum_{k=0}^{n} (-1)^k t^{2k} + (-1)^{n+1} \frac{t^{2n+2}}{1+t^2}.$$

Integrate from 0 to x. Since

$$\arctan x = \int_0^x \frac{dt}{1+t^2},$$

the result follows. □

Corollary 11.32. *If $|x| \le 1$, then*

$$\left|\arctan x - \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}\right| \le \frac{|x|^{2n+3}}{2n+3}.$$

Proof. For $|t| \le 1$,

$$\left|\frac{t^{2n+2}}{1+t^2}\right| \le |t|^{2n+2}.$$

Take absolute values in the remainder integral from Theorem 11.31 and integrate the bound. □

11.9 Operations on Taylor polynomials

If polynomial approximations are known for two functions, they can be combined by algebraic operations, with terms of degree above the desired order discarded. The justification comes from the order of the remainders. We first record the notation for remainders of a controlled size.

Definition 11.33 (Big-O notation)**.** Let r be defined on a deleted neighbourhood of a, and let $m \ge 0$. We write

$$r(x) = O(|x-a|^m) \qquad \text{as } x \to a$$

if there exist constants $C, \delta > 0$ such that

$$|r(x)| \le C|x-a|^m \qquad \text{whenever } 0 < |x-a| < \delta.$$

Remark 11.34 (Rules for big-O). The following rules follow directly from the definition and are used in the examples below.

- If $r_1 = O(|x-a|^{m_1})$ and $r_2 = O(|x-a|^{m_2})$ with $m_1 \le m_2$, then $r_1 + r_2 = O(|x-a|^{m_2})$: a finite sum of O-terms is dominated by the largest order.
- If $r_1 = O(|x-a|^{m_1})$ and $r_2 = O(|x-a|^{m_2})$, then $r_1 r_2 = O(|x-a|^{m_1+m_2})$.
- If $r = O(|x-a|^m)$, then $(x-a)^k r = O(|x-a|^{m+k})$.

Proposition 11.35. *Let f and g be defined near a. Suppose that*

$$f(x) = P(x) + O((x-a)^{n+1}), \qquad g(x) = Q(x) + O((x-a)^{n+1}),$$

where P and Q are polynomials. Then

$$f(x)g(x) = P(x)Q(x) + O((x-a)^{n+1}).$$

Consequently, if f and g have continuous $(n+1)$st derivatives near a, then the degree-n Taylor polynomial of fg is obtained by multiplying the Taylor polynomials $T_{n,a}f$ and $T_{n,a}g$ and discarding terms of degree greater than n. The product estimate itself is purely asymptotic and requires no differentiability of f and g.

Proof. Write

$$f = P + r, \qquad g = Q + s,$$

where $r = O((x-a)^{n+1})$ and $s = O((x-a)^{n+1})$. Then

$$fg = PQ + Ps + Qr + rs.$$

Because polynomials are bounded near a, both Ps and Qr are $O((x-a)^{n+1})$, and rs is of even higher order. Thus

$$fg - PQ = O((x-a)^{n+1}).$$

□

Example 11.36. To obtain a degree-four approximation for $e^x \cos x$, multiply

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O(x^5)$$

and

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6).$$

Keeping terms through degree four gives

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + O(x^5).$$

Example 11.37. Since

$$\sin u = u - \frac{u^3}{6} + O(u^5),$$

substituting $u = x^2$ gives

$$\sin(x^2) = x^2 - \frac{x^6}{6} + O(x^{10}).$$

Thus the degree-six Taylor polynomial of $\sin(x^2)$ at 0 is

$$x^2 - \frac{x^6}{6}.$$

11.10 Exercises

Exercise 11.38. Find the Taylor polynomial of degree 3 centered at $a = 1$ for $f(x) = \log x$.

Exercise 11.39. Find the Taylor polynomial of degree 4 centered at 0 for each function:

$$e^{2x}, \qquad \cos(3x), \qquad \frac{1}{1+x}.$$

Exercise 11.40. Prove Theorem 11.4 by showing directly that a polynomial of degree at most n whose value and derivatives of orders $1, \ldots, n$ all vanish at a must be the zero polynomial.

Exercise 11.41. Use Taylor's theorem with exact integral remainder to prove that

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{1}{n!} \int_0^x e^t (x-t)^n \, dt.$$

Exercise 11.42. Use the preceding exercise to prove that, for $x \geq 0$,

$$0 \leq e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \leq \frac{e^x x^{n+1}}{(n+1)!}.$$

Exercise 11.43. Find the degree-five Maclaurin polynomial for $\sin x$ and give a rigorous upper bound for the error in using it to approximate $\sin(0.3)$.

Exercise 11.44. Find the degree-four Taylor polynomial for $f(x) = \sqrt{x}$ centered at $a = 4$.

Exercise 11.45. Use the Lagrange remainder to determine an integer n for which the degree-n Maclaurin polynomial of e^x approximates $e^{0.1}$ with error less than 10^{-8}.

Exercise 11.46. Use Theorem 11.31 to approximate $\pi/4 = \arctan 1$ by a finite alternating sum, and give a rigorous error bound.

Exercise 11.47. Use the degree-three Taylor polynomial centered at 0 for $\sqrt{1+x}$ to approximate $\sqrt{0.98}$. Bound the error using Taylor's theorem.

Exercise 11.48. Show that the degree-six Taylor polynomial of $\log(1+x^2)$ at 0 is

$$x^2 - \frac{x^4}{2} + \frac{x^6}{3}.$$

Exercise 11.49. Use Taylor polynomials to evaluate

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3}.$$

Exercise 11.50. Let f have a continuous $(n+1)$st derivative near a. Prove that

$$f(x) - T_{n,a}f(x) = o((x-a)^n) \qquad \text{as } x \to a.$$

Exercise 11.51. Prove the weighted mean value theorem for integrals used in the proof of Theorem 11.14: if h is continuous on $[a,b]$ and w is continuous and nonnegative on $[a,b]$, then there is $c \in [a,b]$ such that

$$\int_a^b h(t)w(t)\,dt = h(c)\int_a^b w(t)\,dt.$$

Exercise 11.52. Explain why the infinite series suggested by the finite formulas in Proposition 11.23 are not yet asserted in this chapter to equal the corresponding functions. State what additional limiting statement would be needed.

Chapter 12

Sequences

A sequence is a function whose inputs are the positive integers. Sequences provide the basic language for approximation and limiting processes: the successive approximations in a numerical method form a sequence, the partial sums of a series form a sequence, and limits of functions can often be tested along sequences of points.

The central issue is convergence. The most important structural result in this chapter, the Monotone Convergence Theorem, is a direct consequence of the completeness axiom for the real numbers. This makes sequences one of the clearest settings in which the least-upper-bound property exerts its force.

12.1 Definitions and examples

Definition 12.1. A *sequence of real numbers* is a function

$$a : \mathbb{N} \to \mathbb{R}.$$

Its value at n is denoted by a_n, and the sequence itself is written

$$(a_n)_{n=1}^{\infty}, \qquad (a_n), \qquad \text{or} \qquad a_1, a_2, a_3, \ldots.$$

A sequence may be given by an explicit formula, such as

$$a_n = \frac{1}{n}, \qquad b_n = (-1)^n, \qquad c_n = \frac{n^2+1}{2n^2-3},$$

or recursively, by specifying an initial value and a rule producing each new term from preceding ones.

Example 12.2. The recursive specification

$$a_1 = 2, \qquad a_{n+1} = \frac{a_n + 3}{2}$$

defines the sequence

$$2,\ \frac{5}{2},\ \frac{11}{4},\ \frac{23}{8}, \ldots.$$

Each term is the average of the preceding term and 3.

Definition 12.3. Let (a_n) be a sequence. A *subsequence* of (a_n) is a sequence of the form

$$(a_{n_k})_{k=1}^{\infty},$$

where

$$n_1 < n_2 < n_3 < \cdots$$

are positive integers.

For example, (a_{2k}) is the even-indexed subsequence, and (a_{2k-1}) is the odd-indexed subsequence.

12.2 Limits of sequences

Definition 12.4. A sequence (a_n) *converges* to $L \in \mathbb{R}$, written

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L,$$

if, for every $\varepsilon > 0$, there is an integer N such that

$$n \geq N \quad \Longrightarrow \quad |a_n - L| < \varepsilon.$$

If no such L exists, the sequence *diverges*.

The integer N may depend on ε, but it must work simultaneously for every $n \geq N$. The definition disregards any finite initial part of a sequence.

Example 12.5. The sequence $a_n = 1/n$ converges to 0.

Indeed, let $\varepsilon > 0$. By the Archimedean property, choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Then, for every $n \geq N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus $1/n \to 0$.

Theorem 12.6 (Uniqueness of limits)**.** *A convergent sequence has exactly one limit.*

Proof. Suppose that $a_n \to L$ and $a_n \to M$. Assume, for contradiction, that $L \neq M$. Set

$$\varepsilon = \frac{|L - M|}{3} > 0.$$

There are integers N_1, N_2 such that

$$n \geq N_1 \Longrightarrow |a_n - L| < \varepsilon,$$

and

$$n \geq N_2 \Longrightarrow |a_n - M| < \varepsilon.$$

For $n \geq \max\{N_1, N_2\}$, the triangle inequality gives

$$|L - M| \leq |L - a_n| + |a_n - M| < 2\varepsilon = \frac{2}{3}|L - M|,$$

a contradiction. Hence $L = M$. □

Theorem 12.7 (Algebraic limit laws)**.** *Let $a_n \to L$ and $b_n \to M$, and let $c \in \mathbb{R}$. Then*

$$a_n + b_n \to L + M, \qquad a_n - b_n \to L - M, \qquad ca_n \to cL,$$

and

$$a_n b_n \to LM.$$

If $M \neq 0$, then $b_n \neq 0$ for all sufficiently large n, and

$$\frac{a_n}{b_n} \to \frac{L}{M}.$$

Proof. We prove the sum and product laws; the other statements follow by similar arguments.

Let $\varepsilon > 0$. Choose N_1, N_2 such that

$$n \geq N_1 \Longrightarrow |a_n - L| < \frac{\varepsilon}{2},$$

and

$$n \geq N_2 \Longrightarrow |b_n - M| < \frac{\varepsilon}{2}.$$

Then, for $n \geq \max\{N_1, N_2\}$,

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \varepsilon.$$

Thus $a_n + b_n \to L + M$.

For the product law, first note that (a_n) is bounded; this will also follow from Theorem 12.12, but it can be seen here directly. There is N_0 such that $|a_n - L| < 1$ for $n \geq N_0$, whence $|a_n| \leq |L| + 1$ for such n. Enlarging the bound to include the finitely many preceding terms gives a number K with

$$|a_n| \leq K \qquad \text{for all } n.$$

Now

$$\begin{aligned} |a_n b_n - LM| &= |a_n(b_n - M) + M(a_n - L)| \\ &\leq K|b_n - M| + |M||a_n - L|. \end{aligned}$$

If $M = 0$, choose N so that $|b_n| < \varepsilon/K$ for $n \geq N$, which proves the result. If $M \neq 0$, choose N_1, N_2 so that

$$|b_n - M| < \frac{\varepsilon}{2K}, \qquad |a_n - L| < \frac{\varepsilon}{2|M|}$$

for all sufficiently large n. The displayed estimate then gives $|a_n b_n - LM| < \varepsilon$.

For the quotient law, choose N_0 so that

$$|b_n - M| < \frac{|M|}{2}$$

for $n \geq N_0$. Then $|b_n| \geq |M|/2$, so $b_n \neq 0$. Moreover,

$$\begin{aligned} \left|\frac{a_n}{b_n} - \frac{L}{M}\right| &= \left|\frac{M(a_n - L) - L(b_n - M)}{M b_n}\right| \\ &\leq \frac{2}{|M|}\,|a_n - L| + \frac{2|L|}{|M|^2}\,|b_n - M|. \end{aligned}$$

The right-hand side tends to 0, proving the quotient law. □

Corollary 12.8. *If P and Q are polynomials and $Q(L) \neq 0$, then*

$$a_n \to L \quad \Longrightarrow \quad P(a_n) \to P(L), \qquad \frac{P(a_n)}{Q(a_n)} \to \frac{P(L)}{Q(L)}.$$

Example 12.9.

$$\lim_{n\to\infty} \frac{3n^2 - 2n + 5}{n^2 + 7} = \lim_{n\to\infty} \frac{3 - 2/n + 5/n^2}{1 + 7/n^2} = 3.$$

12.3 Order and boundedness

Definition 12.10. A sequence (a_n) is *bounded above* if there is $M \in \mathbb{R}$ such that

$$a_n \le M \qquad \text{for all } n.$$

It is *bounded below* if there is $m \in \mathbb{R}$ such that

$$m \le a_n \qquad \text{for all } n.$$

It is *bounded* if there is $K > 0$ such that

$$|a_n| \le K \qquad \text{for all } n.$$

Definition 12.11. A sequence (a_n) is *increasing* if

$$a_n \le a_{n+1} \qquad \text{for every } n,$$

and *strictly increasing* if all these inequalities are strict. It is *decreasing* if

$$a_{n+1} \le a_n \qquad \text{for every } n.$$

A sequence is *monotone* if it is increasing or decreasing.

Theorem 12.12 (A convergent sequence is bounded)**.** *Every convergent sequence is bounded.*

Proof. Suppose $a_n \to L$. Taking $\varepsilon = 1$ in Definition 12.4, there is N such that

$$n \ge N \Longrightarrow |a_n - L| < 1.$$

For such n,

$$|a_n| \le |a_n - L| + |L| < 1 + |L|.$$

The finite set

$$\{|a_1|, \dots, |a_{N-1}|, 1 + |L|\}$$

has a maximum, say K. Then $|a_n| \le K$ for every n. □

The converse is false. The sequence $((-1)^n)$ is bounded but does not converge.

Theorem 12.13 (Order preservation under limits)**.** *Suppose that $a_n \to L$, $b_n \to M$, and that*

$$a_n \le b_n$$

for all sufficiently large n. Then

$$L \le M.$$

Proof. Suppose instead that $L > M$. Set

$$\varepsilon = \frac{L - M}{3} > 0.$$

For all sufficiently large n,

$$|a_n - L| < \varepsilon, \qquad |b_n - M| < \varepsilon.$$

Hence

$$a_n > L - \varepsilon = \frac{2L + M}{3} > \frac{L + 2M}{3} = M + \varepsilon > b_n,$$

contradicting $a_n \le b_n$. Therefore $L \le M$. □

Corollary 12.14. *If $a_n \to L$ and $a_n \ge 0$ for all sufficiently large n, then $L \ge 0$.*

Proof. Apply Theorem 12.13 to the zero sequence and (a_n). □

12.4 The squeeze theorem

Theorem 12.15 (Squeeze theorem)**.** *Suppose that*

$$a_n \le b_n \le c_n$$

for all sufficiently large n. If

$$a_n \to L \qquad \text{and} \qquad c_n \to L,$$

then

$$b_n \to L.$$

Proof. Let $\varepsilon > 0$. There are integers N_1, N_2 such that

$$n \ge N_1 \Longrightarrow |a_n - L| < \varepsilon,$$

and

$$n \ge N_2 \Longrightarrow |c_n - L| < \varepsilon.$$

For all sufficiently large n, the hypotheses and these inequalities give

$$L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon.$$

Thus $|b_n - L| < \varepsilon$, proving $b_n \to L$. □

Example 12.16. Since $|\sin n| \le 1$,

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

The outer sequences tend to 0, so

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

12.5 The Monotone Convergence Theorem

The following theorem is a fundamental consequence of completeness. It is false in the rational numbers: an increasing bounded sequence of rationals may approach an irrational number and hence fail to have a rational limit.

Theorem 12.17 (Monotone Convergence Theorem)**.** *Let (a_n) be an increasing sequence that is bounded above. Then (a_n) converges and*

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

Likewise, if (a_n) is decreasing and bounded below, then (a_n) converges and

$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$

Proof. Suppose that (a_n) is increasing and bounded above. Set

$$S = \{a_n : n \in \mathbb{N}\}.$$

The set S is nonempty and bounded above. By completeness, it has a least upper bound; write

$$L = \sup S.$$

Let $\varepsilon > 0$. Since $L - \varepsilon$ is not an upper bound for S, there is an integer N such that

$$a_N > L - \varepsilon.$$

Because the sequence is increasing, for every $n \geq N$,

$$L - \varepsilon < a_N \leq a_n \leq L.$$

Therefore

$$0 \leq L - a_n < \varepsilon,$$

and hence

$$|a_n - L| < \varepsilon.$$

This proves $a_n \to L$.

Now suppose that (a_n) is decreasing and bounded below. The sequence $(-a_n)$ is increasing and bounded above. By the first part, $-a_n \to L$ for some real number L. Therefore $a_n \to -L$. Since

$$\sup\{-a_n : n \in \mathbb{N}\} = -\inf\{a_n : n \in \mathbb{N}\},$$

the stated infimum formula follows. □

Example 12.18. The sequence

$$a_n = 1 - \frac{1}{n}$$

is increasing, since

$$a_{n+1} - a_n = \frac{1}{n(n+1)} > 0.$$

It is bounded above by 1. Therefore it converges by Theorem 12.17; indeed, its limit is 1.

Corollary 12.19. *Let (a_n) be a convergent sequence satisfying*

$$a_{n+1} = g(a_n)$$

for all n, where g is continuous at $L = \lim a_n$. Then

$$L = g(L).$$

Proof. Because (a_n) converges to L, its shifted sequence (a_{n+1}) also converges to L. By continuity of g,

$$a_{n+1} = g(a_n) \to g(L).$$

Uniqueness of limits gives $L = g(L)$. □

Remark 12.20. Solving $L = g(L)$ identifies possible limits of a recursive sequence; it does not itself prove convergence. A separate argument, often based on monotonicity and boundedness, is necessary.

Example 12.21 (A recursive square-root sequence)**.** Let

$$a_1 = 1, \qquad a_{n+1} = \sqrt{2 + a_n}.$$

Then $a_n \to 2$.

First we prove by induction that

$$1 \le a_n \le 2 \qquad \text{for every } n.$$

This holds for $n = 1$. If $1 \le a_n \le 2$, then

$$\sqrt{3} \le a_{n+1} = \sqrt{2 + a_n} \le 2,$$

so the induction is complete. For $1 \le a_n \le 2$,

$$a_{n+1} \ge a_n$$

is equivalent, after squaring nonnegative quantities, to

$$2 + a_n \ge a_n^2,$$

or

$$(2 - a_n)(a_n + 1) \ge 0,$$

which is true. Thus (a_n) is increasing and bounded above by 2. The Monotone Convergence Theorem gives convergence.

Let $L = \lim a_n$. Corollary 12.19 gives

$$L = \sqrt{2 + L}.$$

Thus

$$L^2 - L - 2 = 0,$$

so $L = 2$ or $L = -1$. Since $a_n \ge 1$ for all n, Corollary 12.14 rules out -1. Hence $L = 2$.

Remark 12.22. The preceding example established convergence by monotonicity and boundedness. A second, widely applicable method replaces monotonicity with a Lipschitz estimate on the update map and deduces convergence from the Cauchy criterion; this is the contraction mapping theorem, stated and proved in Section 12.8 as Theorem 12.38.

Example 12.23 (The Euler–Mascheroni constant)**.** Let

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

be the nth harmonic number, and set

$$a_n = H_n - \log n.$$

Then (a_n) converges.

We use the logarithm inequalities, valid for $u > 0$,

$$\frac{u}{1+u} < \log(1+u) < u,$$

obtained by integrating $1/(1+u) < 1/(1+t) < 1$ over $[0, u]$; these are the same inequalities used in the proof of Theorem 12.46. First,

$$a_{n+1} - a_n = \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) < \frac{1}{n+1} - \frac{1}{n+1} = 0,$$

because the inequality $\log(1+u) > u/(1+u)$ with $u = 1/n$ gives $\log(1+1/n) > 1/(n+1)$. Thus (a_n) is decreasing.

Second,

$$\log(n+1) = \sum_{k=1}^{n} \log\left(1 + \frac{1}{k}\right) < \sum_{k=1}^{n} \frac{1}{k} = H_n,$$

using $\log(1+u) < u$ with $u = 1/k$. Hence

$$a_n = H_n - \log n > \log(n+1) - \log n > 0.$$

Thus (a_n) is bounded below by 0. By Theorem 12.17, (a_n) converges. Its limit is the *Euler–Mascheroni constant*

$$\gamma := \lim_{n\to\infty} (H_n - \log n),$$

and it lies in $[0,1]$: indeed $0 \le \gamma \le a_1 = H_1 - \log 1 = 1$.

Remark 12.24. The Euler–Mascheroni constant satisfies $\gamma \approx 0.57721\ldots$. It is unknown whether γ is rational.

12.5.1 ‡ Stolz–Cesaro

Theorem 12.25 (Stolz–Cesaro)**.** *Let (b_n) be a strictly increasing sequence with $b_n \to \infty$. If*

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} \to L$$

for some finite L, then

$$\frac{a_n}{b_n} \to L.$$

Proof. Let $\varepsilon > 0$. By hypothesis there is an integer N such that

$$k > N \qquad \Longrightarrow \qquad L - \varepsilon < \frac{a_k - a_{k-1}}{b_k - b_{k-1}} < L + \varepsilon.$$

Because (b_n) is strictly increasing, $b_k - b_{k-1} > 0$. Multiplying by this positive quantity and summing over $k = N+1, \ldots, n$ (for $n > N$) telescopes to give

$$(L-\varepsilon)(b_n - b_N) < a_n - a_N < (L+\varepsilon)(b_n - b_N).$$

Dividing by $b_n > 0$ (which holds for all sufficiently large n, since $b_n \to \infty$) yields

$$(L-\varepsilon)\left(1 - \frac{b_N}{b_n}\right) < \frac{a_n}{b_n} - \frac{a_N}{b_n} < (L+\varepsilon)\left(1 - \frac{b_N}{b_n}\right).$$

Adding a_N/b_n throughout and letting $n \to \infty$ (using $b_N/b_n \to 0$ and $a_N/b_n \to 0$, both because $b_n \to \infty$) gives

$$L - \varepsilon \le \liminf_{n\to\infty} \frac{a_n}{b_n} \le \limsup_{n\to\infty} \frac{a_n}{b_n} \le L + \varepsilon,$$

where the limit inferior and limit superior are those of Definition 12.27. Since $\varepsilon > 0$ is arbitrary, $\liminf a_n/b_n = \limsup a_n/b_n = L$; by Proposition 12.28 this equality characterizes convergence, so $a_n/b_n \to L$. □

Example 12.26. Let

$$a_n = H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \qquad b_n = \log n.$$

Then (b_n) is strictly increasing and $b_n \to \infty$. For $n \geq 2$,

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{1/n}{\log n - \log(n-1)} = \frac{1}{n \log\left(1 + \frac{1}{n-1}\right)}.$$

From the logarithm inequalities

$$\frac{u}{1+u} \leq \log(1+u) \leq u$$

used in the proof of Theorem 12.46, applied with $u = 1/(n-1)$, we get

$$\frac{1}{n} \leq \log\left(1 + \frac{1}{n-1}\right) \leq \frac{1}{n-1}.$$

Multiplying by n and taking reciprocals of the positive quantities gives

$$\frac{n-1}{n} \leq \frac{1}{n \log\left(1 + \frac{1}{n-1}\right)} \leq 1.$$

By the squeeze theorem (Theorem 12.15), this ratio tends to 1. Theorem 12.25 therefore gives

$$\frac{H_n}{\log n} \to 1.$$

12.6 Limit superior and limit inferior

For a sequence that need not converge, it is useful to record the extreme values approached by its tails.

Definition 12.27. Let (a_n) be a bounded sequence. Its *limit superior* (or *upper limit*) is

$$\limsup_{n\to\infty} a_n := \lim_{N\to\infty} \sup_{n \geq N} a_n,$$

and its *limit inferior* (or *lower limit*) is

$$\liminf_{n\to\infty} a_n := \lim_{N\to\infty} \inf_{n \geq N} a_n.$$

For each N the tail set $\{a_n : n \geq N\}$ is bounded, so its supremum and infimum exist. As N increases the tail shrinks, so the sequence $\left(\sup_{n\geq N} a_n\right)_N$ is decreasing and $\left(\inf_{n\geq N} a_n\right)_N$ is increasing; both are bounded, so by the Monotone Convergence Theorem (Theorem 12.17) both limits exist. Equivalently,

$$\limsup_{n\to\infty} a_n = \inf_{N\geq 1} \sup_{n \geq N} a_n, \qquad \liminf_{n\to\infty} a_n = \sup_{N\geq 1} \inf_{n \geq N} a_n.$$

For a sequence that is not bounded above, all tail suprema are $+\infty$, and we set $\limsup a_n = +\infty$; for a sequence that is not bounded below, we set $\liminf a_n = -\infty$. With these conventions $\limsup a_n$ and $\liminf a_n$ are defined, as extended real numbers, for every sequence.

Proposition 12.28 (Properties of limsup and liminf)**.** *Let (a_n) and (b_n) be bounded sequences, and let $c \geq 0$.*

1. *For every $\varepsilon > 0$, the inequality $a_n < \limsup a_n + \varepsilon$ holds for all sufficiently large n, while $a_n > \limsup a_n - \varepsilon$ holds for infinitely many n. Dually, $a_n > \liminf a_n - \varepsilon$ for all sufficiently large n, and $a_n < \liminf a_n + \varepsilon$ for infinitely many n.*
2. *$\liminf a_n \leq \limsup a_n$, with equality if and only if (a_n) converges; in that case both equal $\lim_{n\to\infty} a_n$.*
3. *$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ and $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$.*
4. *$\limsup(ca_n) = c\limsup a_n$ and $\liminf(ca_n) = c\liminf a_n$. If $a_n, b_n \geq 0$ for all n, then*

$$\limsup(a_n b_n) \leq \limsup a_n \cdot \limsup b_n.$$

Proof. Throughout we write $\limsup a_n$ and $\liminf a_n$ for $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

(1) Set $L = \limsup a_n = \inf_{N\geq 1} \sup_{n\geq N} a_n$. Since the tail suprema $\sup_{n\geq N} a_n$ decrease to L, there is N_0 with $\sup_{n\geq N_0} a_n < L + \varepsilon$; then $a_n < L + \varepsilon$ for every $n \geq N_0$. On the other hand, for every N one has $\sup_{n\geq N} a_n \geq L > L - \varepsilon$, so some $n \geq N$ satisfies $a_n > L - \varepsilon$; since N is arbitrary, this holds for infinitely many n. The dual statements for $\liminf$ follow by applying what was just proved to $(-a_n)$, using $\liminf a_n = -\limsup(-a_n)$.

(2) For every N,

$$\inf_{n\geq N} a_n \leq \sup_{n\geq N} a_n.$$

Taking $N \to \infty$ and using order preservation under limits (Theorem 12.13) gives $\liminf a_n \leq \limsup a_n$. If $a_n \to L$, then for $\varepsilon > 0$ there is N with $|a_n - L| < \varepsilon$ for all $n \geq N$, so

$$L - \varepsilon \leq \inf_{n\geq N} a_n \leq \sup_{n\geq N} a_n \leq L + \varepsilon,$$

whence $L - \varepsilon \leq \liminf a_n \leq \limsup a_n \leq L + \varepsilon$. Since ε is arbitrary, $\liminf a_n = \limsup a_n = L$. Conversely, if $\liminf a_n = \limsup a_n = L$, then by (1) $a_n < L + \varepsilon$ eventually and $a_n > L - \varepsilon$ eventually, so $|a_n - L| < \varepsilon$ eventually; hence $a_n \to L$.

(3) For fixed N and any $n \geq N$,

$$a_n \leq \sup_{k\geq N} a_k, \qquad b_n \leq \sup_{k\geq N} b_k,$$

so

$$a_n + b_n \leq \sup_{k\geq N} a_k + \sup_{k\geq N} b_k.$$

Taking the supremum over $n \geq N$ on the left and then the limit as $N \to \infty$ gives

$$\limsup(a_n + b_n) = \lim_{N\to\infty} \sup_{n\geq N} (a_n + b_n) \leq \lim_{N\to\infty} \Big(\sup_{n\geq N} a_n + \sup_{n\geq N} b_n\Big) = \limsup a_n + \limsup b_n.$$

The inequality $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$ is proved the same way, with inf in place of sup.

(4) For $c \geq 0$, $\sup_{n\geq N}(ca_n) = c\sup_{n\geq N} a_n$, so taking limits gives $\limsup(ca_n) = c\limsup a_n$; the $\liminf$ statement is identical. If $a_n, b_n \geq 0$, then for fixed N and $n \geq N$,

$$a_n \leq \sup_{k\geq N} a_k, \qquad b_n \leq \sup_{k\geq N} b_k,$$

with all quantities nonnegative, so

$$a_n b_n \le \Big(\sup_{k\ge N} a_k\Big)\Big(\sup_{k\ge N} b_k\Big).$$

Taking the supremum over $n \ge N$ and then the limit as $N \to \infty$ gives

$$\limsup(a_n b_n) \le \limsup a_n \cdot \limsup b_n.$$

□

12.7 Subsequences

Theorem 12.29 (Subsequence theorem)**.** *If $a_n \to L$, then every subsequence (a_{n_k}) also converges to L.*

Proof. Let $\varepsilon > 0$. Since $a_n \to L$, there is N such that

$$n \ge N \Longrightarrow |a_n - L| < \varepsilon.$$

Because $n_1 < n_2 < \cdots$ are positive integers, one has $n_k \ge k$. Hence, for every $k \ge N$,

$$n_k \ge k \ge N.$$

Therefore

$$|a_{n_k} - L| < \varepsilon \qquad (k \ge N).$$

Thus $a_{n_k} \to L$. □

Corollary 12.30 (Divergence by incompatible subsequences)**.** *If a sequence has two subsequences that converge to different limits, then the original sequence diverges.*

Proof. If the original sequence converged, Theorem 12.29 would force every subsequence to converge to the same limit. This contradicts the hypothesis. □

Example 12.31. The sequence $a_n = (-1)^n$ diverges. Indeed,

$$a_{2k} = 1 \to 1, \qquad a_{2k-1} = -1 \to -1.$$

The two subsequences have different limits, so Corollary 12.30 applies.

Theorem 12.32 (Bolzano–Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let (a_n) be bounded. Choose a closed interval $I_1 = [\alpha_1, \beta_1]$ containing every term of the sequence. Bisect I_1 into two closed intervals of equal length. At least one half contains infinitely many terms of the sequence; otherwise each half would contain only finitely many terms, and their union would contain only finitely many terms. Choose such a half and call it I_2.

Inductively, having chosen a closed interval I_k containing infinitely many terms, bisect it and choose a closed half $I_{k+1} \subseteq I_k$ containing infinitely many terms. Thus we obtain nested closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots,$$

whose lengths are

$$|I_k| = 2^{-(k-1)}|I_1| \to 0.$$

Write $I_k = [\alpha_k, \beta_k]$. The sequence (α_k) is increasing and bounded above by β_1, so it converges by Theorem 12.17; let its limit be L. Since $\alpha_k \le \beta_k \le \alpha_k + |I_k|$ and $|I_k| \to 0$, the squeeze theorem gives $\beta_k \to L$. Hence $L \in I_k$ for every k.

Because each I_k contains infinitely many terms, choose indices recursively so that

$$n_1 < n_2 < \cdots \qquad \text{and} \qquad a_{n_k} \in I_k.$$

Since both a_{n_k} and L lie in I_k,

$$|a_{n_k} - L| \le |I_k| \to 0.$$

Therefore $a_{n_k} \to L$. □

Corollary 12.33. *Every bounded sequence has at least one convergent subsequence.*

12.8 Cauchy sequences

The definition of convergence refers to a proposed limit L. The Cauchy condition instead compares late terms of the sequence directly with one another.

Definition 12.34. A sequence (a_n) is a *Cauchy sequence* if, for every $\varepsilon > 0$, there is an integer N such that

$$m, n \ge N \quad \Longrightarrow \quad |a_m - a_n| < \varepsilon.$$

Theorem 12.35. *Every convergent sequence is Cauchy.*

Proof. Suppose $a_n \to L$. Let $\varepsilon > 0$. There is N such that

$$n \ge N \Longrightarrow |a_n - L| < \frac{\varepsilon}{2}.$$

If $m, n \ge N$, then the triangle inequality gives

$$|a_m - a_n| \le |a_m - L| + |a_n - L| < \varepsilon.$$

Thus (a_n) is Cauchy. □

Theorem 12.36 (Cauchy criterion)**.** *A sequence of real numbers converges if and only if it is Cauchy.*

Proof. The forward implication is Theorem 12.35. Conversely, suppose (a_n) is Cauchy. Taking $\varepsilon = 1$ in the Cauchy condition, there is N such that

$$n \ge N \Longrightarrow |a_n - a_N| < 1.$$

Thus all terms from index N onward are bounded, and the initial finite collection is bounded as well. Therefore (a_n) is bounded.

By Bolzano–Weierstrass, (a_n) has a convergent subsequence (a_{n_k}), say

$$a_{n_k} \to L.$$

We show that the full sequence converges to L. Let $\varepsilon > 0$. Choose N_1 so that

$$m, n \ge N_1 \Longrightarrow |a_m - a_n| < \frac{\varepsilon}{2}.$$

Since $a_{n_k} \to L$, choose k sufficiently large that

$$n_k \geq N_1 \qquad \text{and} \qquad |a_{n_k} - L| < \frac{\varepsilon}{2}.$$

Then, for every $n \geq N_1$,

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \varepsilon.$$

Thus $a_n \to L$. □

Remark 12.37. The Cauchy criterion is another formulation of completeness. It would fail in the rational numbers: rational approximations to $\sqrt{2}$ form a Cauchy sequence of rationals that has no rational limit.

12.8.1 The contraction mapping theorem

The Cauchy criterion gives a clean way to prove convergence of the recursively defined sequences studied in Section 12.5, without first establishing monotonicity.

Theorem 12.38 (Contraction mapping theorem)**.** *Let I be a closed interval, and let $g : I \to I$ be a contraction: there is $0 \leq \lambda < 1$ such that*

$$|g(x) - g(y)| \leq \lambda |x - y|$$

for all $x, y \in I$. Then g has a unique fixed point $p \in I$, and for every $x_0 \in I$ the recursion

$$x_{n+1} = g(x_n)$$

converges to p.

Proof. Fix $x_0 \in I$ and set $x_{n+1} = g(x_n)$. First,

$$|x_{n+1} - x_n| = |g(x_n) - g(x_{n-1})| \leq \lambda |x_n - x_{n-1}|,$$

so by induction

$$|x_{n+1} - x_n| \leq \lambda^n |x_1 - x_0|.$$

For $m > n$, the triangle inequality and the geometric-sum formula give

$$|x_m - x_n| \leq \sum_{j=n}^{m-1} |x_{j+1} - x_j| \leq \sum_{j=n}^{m-1} \lambda^j |x_1 - x_0| \leq \frac{\lambda^n}{1 - \lambda} |x_1 - x_0|.$$

Since $\lambda^n \to 0$, the sequence (x_n) is Cauchy. By Theorem 12.36 (completeness of $\mathbb{R}$), there is $p \in \mathbb{R}$ with $x_n \to p$. Since I is closed and every $x_n \in I$, we have $p \in I$.

A contraction is continuous, because $|g(x) - g(y)| \leq \lambda |x - y|$ shows that $|x - y| \to 0$ forces $|g(x) - g(y)| \to 0$. Hence

$$p = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} g(x_n) = g(p),$$

so p is a fixed point.

For uniqueness, if $q \in I$ also satisfies $g(q) = q$, then

$$|p - q| = |g(p) - g(q)| \leq \lambda |p - q|.$$

Because $\lambda < 1$, this forces $|p - q| = 0$, so $p = q$. □

Remark 12.39. If g is differentiable on I and $|g'(x)| \leq \lambda < 1$ for all $x \in I$, then g is a contraction: by the Mean Value Theorem (Theorem 6.9),

$$|g(x) - g(y)| = |g'(c)|\,|x - y| \leq \lambda |x - y|$$

for some c between x and y.

12.9 Sequences obtained from functions

Theorem 12.40 (Function limits imply sequence limits)**.** *Let f be defined on (A, ∞), and suppose that*

$$\lim_{x\to\infty} f(x) = L.$$

Then

$$\lim_{n\to\infty} f(n) = L.$$

Proof. Let $\varepsilon > 0$. By the function-limit hypothesis, there is $B > A$ such that

$$x \geq B \Longrightarrow |f(x) - L| < \varepsilon.$$

Choose an integer $N \geq B$. Then for every $n \geq N$,

$$|f(n) - L| < \varepsilon.$$

Hence $f(n) \to L$. □

Theorem 12.41 (Sequence limits imply function limits)**.** *Let f be defined on a deleted neighbourhood of a. Suppose that*

$$f(x_n) \to L$$

for every sequence (x_n) in the domain of f with $x_n \neq a$ for all n and $x_n \to a$. Then

$$\lim_{x\to a} f(x) = L.$$

Proof. We prove the contrapositive. Suppose that the ε–δ condition defining $\lim_{x\to a} f(x) = L$ fails. Then there is $\varepsilon > 0$ such that for every $\delta > 0$ there is a point x in the domain of f with

$$0 < |x - a| < \delta \qquad \text{but} \qquad |f(x) - L| \geq \varepsilon.$$

Taking $\delta = 1/n$ for $n = 1, 2, \ldots$, choose x_n in the domain of f such that

$$0 < |x_n - a| < \frac{1}{n} \qquad \text{and} \qquad |f(x_n) - L| \geq \varepsilon.$$

Then $x_n \to a$ (because $|x_n - a| < 1/n \to 0$), each $x_n \neq a$, yet $f(x_n) \not\to L$. This contradicts the hypothesis, so $\lim_{x\to a} f(x) = L$. □

Corollary 12.42 (Sequential characterization of function limits)**.** *Let f be defined on a deleted neighbourhood of a. Then*

$$\lim_{x\to a} f(x) = L$$

if and only if

$$f(x_n) \to L$$

for every sequence (x_n) in the domain of f with $x_n \neq a$ for all n and $x_n \to a$.

Proof. The reverse implication is Theorem 12.41. For the forward implication (the finite-point analogue of Theorem 12.40), suppose $\lim_{x\to a} f(x) = L$, and let (x_n) satisfy $x_n \neq a$ and $x_n \to a$. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

Since $x_n \to a$, there is N such that

$$n \geq N \Longrightarrow 0 < |x_n - a| < \delta,$$

the strict inequality $0 < |x_n - a|$ using $x_n \neq a$. Hence $|f(x_n) - L| < \varepsilon$ for $n \geq N$, so $f(x_n) \to L$. □

Theorem 12.43 (Sequential characterization of continuity)**.** *Let f be defined on a neighbourhood of a. Then f is continuous at a if and only if*

$$f(x_n) \to f(a)$$

for every sequence (x_n) in the domain of f with $x_n \to a$.

Proof. Suppose first that f is continuous at a, so that $\lim_{x\to a} f(x) = f(a)$. Let (x_n) be a sequence in the domain of f with $x_n \to a$. Given $\varepsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon,$$

which holds trivially as well at $x = a$. Since $x_n \to a$, there is N such that $n \geq N$ implies $|x_n - a| < \delta$, and therefore $|f(x_n) - f(a)| < \varepsilon$. Hence $f(x_n) \to f(a)$.

Conversely, suppose that $f(x_n) \to f(a)$ for every sequence (x_n) in the domain of f with $x_n \to a$. In particular this holds for every sequence with $x_n \neq a$ for all n. Theorem 12.41 (with $L = f(a)$) then gives

$$\lim_{x\to a} f(x) = f(a),$$

so f is continuous at a. □

Example 12.44. Because

$$\lim_{x\to\infty} \frac{\log x}{x} = 0,$$

Theorem 12.40 gives

$$\lim_{n\to\infty} \frac{\log n}{n} = 0.$$

Theorem 12.45. *For every $a > 0$,*

$$\lim_{n\to\infty} a^{1/n} = 1.$$

Proof. Using the definition of real powers from the exponential and logarithm,

$$a^{1/n} = e^{(\log a)/n}.$$

Since $(\log a)/n \to 0$ and the exponential function is continuous,

$$a^{1/n} \to e^0 = 1.$$

□

Theorem 12.46. *The sequence*

$$\left(1 + \frac{1}{n}\right)^n$$

converges. Its limit is the number e defined by the exponential function:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Proof. For $u > -1$, the logarithm inequalities

$$\frac{u}{1+u} \leq \log(1+u) \leq u$$

follow from the fact that $1/(1+t)$ is decreasing on $[0, u]$ when $u > 0$, with the corresponding reversed-interval argument for $-1 < u < 0$.

Apply these inequalities with $u = 1/n$:

$$\frac{1}{n+1} \le \log\left(1+\frac{1}{n}\right) \le \frac{1}{n}.$$

Multiplying by n gives

$$\frac{n}{n+1} \le \log\left(1+\frac{1}{n}\right)^n \le 1.$$

The outer sequences both tend to 1, so the squeeze theorem yields

$$\log\left(1+\frac{1}{n}\right)^n \to 1.$$

By continuity of the exponential function,

$$\left(1+\frac{1}{n}\right)^n = \exp\left(\log\left(1+\frac{1}{n}\right)^n\right) \to e^1 = e.$$

□

Corollary 12.47. *For every real x,*

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

Proof. For all sufficiently large n, $1 + x/n > 0$. The same logarithmic argument used in Theorem 12.46, applied to $u = x/n$, gives

$$n\log\left(1+\frac{x}{n}\right) \to x.$$

Continuity of the exponential function gives the result. □

12.10 Exercises

Exercise 12.48. List the first five terms of each sequence:

$$a_n = \frac{n}{n+1}, \qquad b_n = (-1)^n\frac{n}{n+1}, \qquad c_1 = 1, \quad c_{n+1} = 2c_n + 1.$$

Exercise 12.49. Determine whether each sequence converges. If it converges, find its limit:

$$\frac{2n-1}{5n+3}, \qquad \frac{n^2}{n^3+1}, \qquad \frac{(-1)^n}{n}, \qquad (-1)^n.$$

Exercise 12.50. Use Definition 12.4 to prove that

$$\lim_{n\to\infty}\frac{5}{n} = 0.$$

Exercise 12.51. Prove the product part of Theorem 12.7 in the special case $L = M = 0$, without first invoking boundedness of (a_n).

Exercise 12.52. Use the squeeze theorem to prove that

$$\lim_{n\to\infty}\frac{\cos n}{n^2} = 0.$$

Exercise 12.53. Give two subsequences of

$$a_n = \cos\left(\frac{n\pi}{2}\right)$$

with different limits, and conclude that (a_n) diverges.

Exercise 12.54. Prove Theorem 12.12.

Exercise 12.55. Give a bounded divergent sequence distinct from $((-1)^n)$, and prove its divergence using Corollary 12.30.

Exercise 12.56. Let

$$a_1 = 4, \qquad a_{n+1} = \frac{a_n + 2}{2}.$$

Prove that (a_n) is decreasing and bounded below, and find its limit.

Exercise 12.57. Let

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right).$$

Prove that every term is positive. Assuming the sequence converges, determine its possible limit(s). What number is this iteration intended to approximate?

Exercise 12.58. Prove the decreasing case of the Monotone Convergence Theorem directly from the definition of infimum, without passing to $(-a_n)$.

Exercise 12.59. Fill in the details of the nested-interval argument in the proof of Theorem 12.32: prove that the endpoints α_k and β_k converge to the same point and that this point belongs to every I_k.

Exercise 12.60. Prove the Cauchy criterion using the Bolzano–Weierstrass Theorem, as in Theorem 12.36.

Exercise 12.61. Prove that every Cauchy sequence is bounded directly from the Cauchy definition.

Exercise 12.62. Evaluate:

$$\lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n, \qquad \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n.$$

Exercise 12.63. Prove that if $a_n \to L$ and $a_n \geq 0$ for all sufficiently large n, then $L \geq 0$.

Exercise 12.64. Explain why the equation $L = g(L)$ is necessary but not sufficient for a recursively defined sequence $a_{n+1} = g(a_n)$ to converge to L. Give an explicit counterexample.

Chapter 13

Series

A series is an infinite sum. Since an infinite sum cannot be formed by ordinary finite addition, its meaning is defined through a sequence: the sequence of partial sums. Thus the theory of series is an application of the convergence theory developed in Chapter 12.

The principal task is to decide whether a series converges. This chapter develops comparison tests for nonnegative series, the integral test, the ratio and root tests, alternating-series convergence, and absolute convergence. The central tool throughout is the Cauchy criterion for sequences.

13.1 Definition of a series

Definition 13.1. Let (a_n) be a sequence of real numbers. The *nth partial sum* of the formal expression

$$\sum_{k=1}^{\infty} a_k$$

is

$$s_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The series *converges* or is *summable* if the sequence (s_n) of partial sums converges. In that case, its *sum* is

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} s_n.$$

If (s_n) diverges, the series diverges.

The index variable in a sum is a dummy variable. Thus

$$\sum_{k=1}^{\infty} a_k = \sum_{n=1}^{\infty} a_n$$

whenever both expressions refer to the same sequence.

Example 13.2. The geometric series

$$1 + r + r^2 + \cdots$$

has partial sums

$$s_n = \sum_{k=0}^{n} r^k.$$

If $r \neq 1$, the finite geometric identity gives

$$s_n = \frac{1 - r^{n+1}}{1 - r}.$$

Therefore, when $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r},$$

because $r^{n+1} \to 0$. When $|r| \geq 1$, the terms do not tend to zero, except in the special case $r = -1$ where they still do not converge; thus the series diverges.

Theorem 13.3 (Term test for divergence)**.** *If the series*

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$a_n \to 0.$$

Consequently, if a_n does not tend to 0, then the series diverges.

Proof. Let

$$s_n = \sum_{k=1}^{n} a_k.$$

If the series converges, then $s_n \to S$ for some S. Since the shifted sequence (s_{n-1}) also converges to S,

$$a_n = s_n - s_{n-1} \to S - S = 0$$

by the algebraic limit laws. □

Remark 13.4. The converse of Theorem 13.3 is false. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges even though $1/n \to 0$.

13.2 The Cauchy criterion for series

Theorem 13.5 (Cauchy criterion for series)**.** *The series*

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if, for every $\varepsilon > 0$, there is an integer N such that

$$m > n \geq N \quad \Longrightarrow \quad \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon.$$

Proof. Let $s_n = \sum_{k=1}^{n} a_k$. For $m > n$,

$$s_m - s_n = \sum_{k=n+1}^{m} a_k.$$

Thus the stated condition is exactly the Cauchy condition for the sequence (s_n). The result follows from the Cauchy criterion for sequences, Theorem 12.36. □

Corollary 13.6. *If $\sum_{n=1}^{\infty} a_n$ converges, then every tail series*

$$\sum_{n=N}^{\infty} a_n$$

converges, and

$$\sum_{n=N}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n.$$

Conversely, changing, adding, or removing finitely many terms does not affect convergence.

Proof. If $s_n \to S$, then

$$\sum_{n=N}^{m} a_n = s_m - s_{N-1} \to S - s_{N-1}.$$

The converse statement follows by reversing this calculation. □

13.3 Series with nonnegative terms

For a series with nonnegative terms, partial sums are automatically increasing. The Monotone Convergence Theorem therefore gives a particularly useful convergence criterion.

Theorem 13.7 (Nonnegative-series criterion)**.** *Suppose that $a_n \geq 0$ for every n. Then the series*

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if its partial sums are bounded above. In that case,

$$\sum_{n=1}^{\infty} a_n = \sup\left\{\sum_{k=1}^{n} a_k : n \in \mathbb{N}\right\}.$$

Proof. Let

$$s_n = \sum_{k=1}^{n} a_k.$$

Since $a_{n+1} \geq 0$,

$$s_{n+1} = s_n + a_{n+1} \geq s_n.$$

Thus (s_n) is increasing. By the Monotone Convergence Theorem, it converges exactly when it is bounded above, and in that case its limit is the supremum of the set of partial sums. □

Theorem 13.8 (Comparison test)**.** *Let $a_n, b_n \geq 0$ for all sufficiently large n.*

1. *If $a_n \leq b_n$ for all sufficiently large n and*

$$\sum_{n=1}^{\infty} b_n$$

converges, then

$$\sum_{n=1}^{\infty} a_n$$

converges.

2. *If $a_n \le b_n$ for all sufficiently large n and*

$$\sum_{n=1}^{\infty} a_n$$

diverges, then

$$\sum_{n=1}^{\infty} b_n$$

diverges.

Proof. By Corollary 13.6, changing finitely many terms does not affect convergence, so assume $0 \le a_n \le b_n$ for all n. Let

$$A_n = \sum_{k=1}^{n} a_k, \qquad B_n = \sum_{k=1}^{n} b_k.$$

Then

$$0 \le A_n \le B_n$$

for every n. If $\sum b_n$ converges, then (B_n) is bounded above. Hence (A_n) is increasing and bounded above, so it converges by Theorem 13.7. This proves the first statement.

The second statement is the contrapositive of the first. □

Example 13.9 (The harmonic series)**.** The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Group the terms after the first as follows:

$$\frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots.$$

The jth group after the initial term contains 2^{j-1} terms, each at least $1/2^j$. Thus every group has sum at least $1/2$. The partial sums therefore exceed

$$1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

with arbitrarily many copies of $1/2$, and are unbounded above. Theorem 13.7 proves divergence.

13.4 The integral test and p-series

Theorem 13.10 (Integral test)**.** *Let $f : [1, \infty) \to [0, \infty)$ be continuous and decreasing, and set*

$$a_n = f(n).$$

Then

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if

$$\int_1^{\infty} f(x)\,dx$$

converges.

Proof. Because f is decreasing, for every integer $n \geq 1$ and every $x \in [n, n+1]$,

$$f(n+1) \leq f(x) \leq f(n).$$

Integrating over $[n, n+1]$ gives

$$f(n+1) \leq \int_n^{n+1} f(x)\,dx \leq f(n).$$

Summing from $n = 1$ to N yields

$$\sum_{n=2}^{N+1} f(n) \leq \int_1^{N+1} f(x)\,dx \leq \sum_{n=1}^{N} f(n).$$

Suppose first that $\sum f(n)$ converges. Then the right-hand partial sums are bounded, so the middle integrals are bounded above. Since $f \geq 0$, the integral accumulations are increasing; hence the improper integral converges.

Conversely, suppose the improper integral converges. The left inequality can be rearranged as

$$\sum_{n=1}^{N} f(n) \leq f(1) + \int_1^{N} f(x)\,dx.$$

The right-hand side is bounded as $N \to \infty$. Since the partial sums of $\sum f(n)$ are increasing, Theorem 13.7 gives convergence. □

Corollary 13.11 (p-series criterion)**.** *For real p,*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Proof. If $p \leq 0$, then $n^{-p} \geq 1$ for every n, so the terms do not tend to 0; the term test (Theorem 13.3) proves divergence. If $p > 0$, then $f(x) = x^{-p}$ is continuous and decreasing on $[1, \infty)$, so the integral test (Theorem 13.10) applies; the result then follows from the p-integral criterion of Chapter 10, according to which

$$\int_1^{\infty} x^{-p}\,dx$$

converges exactly when $p > 1$. □

Example 13.12. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges because

$$0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2},$$

and the p-series $\sum 1/n^2$ converges.

The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges because it is the p-series with $p = 1/2 \leq 1$.

Theorem 13.13 (Cauchy condensation test)**.** *Let (a_n) be a decreasing sequence with $a_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if

$$\sum_{k=1}^{\infty} 2^k a_{2^k}$$

converges.

Proof. For each $k \geq 1$, the block

$$a_{2^k+1} + a_{2^k+2} + \cdots + a_{2^{k+1}}$$

contains 2^k terms. Because (a_n) is decreasing, each of these terms lies between $a_{2^{k+1}}$ and a_{2^k}, so

$$2^k a_{2^{k+1}} \leq a_{2^k+1} + \cdots + a_{2^{k+1}} \leq 2^k a_{2^k}.$$

Summing over $k = 1, \ldots, N-1$ gives

$$\frac{1}{2} \sum_{k=2}^{N} 2^k a_{2^k} \leq \sum_{n=3}^{2^N} a_n \leq \sum_{k=1}^{N-1} 2^k a_{2^k}.$$

Both series have nonnegative terms, so each converges exactly when its partial sums are bounded (Theorem 13.7).

If $\sum a_n$ converges, then the left inequality gives, for every N,

$$\sum_{k=2}^{N} 2^k a_{2^k} \leq 2 \sum_{n=3}^{2^N} a_n \leq 2 \sum_{n=1}^{\infty} a_n,$$

so the partial sums of $\sum_{k=1}^{\infty} 2^k a_{2^k}$ are bounded, and that series converges.

Conversely, if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges, then the right inequality gives

$$\sum_{n=3}^{2^N} a_n \leq \sum_{k=1}^{N-1} 2^k a_{2^k} \leq \sum_{k=1}^{\infty} 2^k a_{2^k}.$$

For any index M, choose N with $M \leq 2^N$. Then

$$\sum_{n=1}^{M} a_n \leq a_1 + a_2 + \sum_{n=3}^{2^N} a_n \leq a_1 + a_2 + \sum_{k=1}^{\infty} 2^k a_{2^k},$$

so the partial sums of $\sum a_n$ are bounded, and that series converges. □

Example 13.14. For the p-series, the condensed series is

$$\sum_{k=1}^{\infty} 2^k (2^k)^{-p} = \sum_{k=1}^{\infty} (2^{1-p})^k,$$

a geometric series, which converges exactly when $2^{1-p} < 1$, that is, when $p > 1$. The Cauchy condensation test therefore gives another proof of Corollary 13.11 (for $p > 0$; for $p \leq 0$ the terms do not tend to 0, so the series diverges by the term test).

13.5 Limit comparison

Theorem 13.15 (Limit comparison test)**.** *Let $a_n, b_n > 0$ for all sufficiently large n, and suppose that*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

with $0 < L < \infty$. Then

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if

$$\sum_{n=1}^{\infty} b_n$$

converges.

Proof. Choose $\varepsilon = L/2$. For all sufficiently large n,

$$\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2}.$$

Therefore

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}.$$

Equivalently,

$$\frac{L}{2} b_n < a_n < \frac{3L}{2} b_n.$$

The two convergence conclusions now follow from the comparison test and the fact that multiplication by a positive constant does not affect convergence of a nonnegative series. □

Example 13.16. Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{3n^2 + 1}{n^4 - 2n + 7}.$$

Set

$$a_n = \frac{3n^2 + 1}{n^4 - 2n + 7}, \qquad b_n = \frac{1}{n^2}.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^4 + n^2}{n^4 - 2n + 7} = 3.$$

Since $\sum 1/n^2$ converges, the given series converges by Theorem 13.15.

13.6 The ratio and root tests

Theorem 13.17 (Ratio test)**.** *Let $a_n \neq 0$ for all sufficiently large n.*

1. *If*

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1,$$

then $\sum a_n$ converges absolutely.

2. *If*
$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1$$
for all sufficiently large n, then $\sum a_n$ diverges.

In particular, if the ordinary limit
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$
exists, then the series converges absolutely when $L < 1$ and diverges when $L > 1$.

Proof. Suppose first that the limsup is less than 1. Choose r such that
$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < r < 1.$$
Then for all sufficiently large n,
$$|a_{n+1}| \leq r|a_n|.$$
Fix such an index N. Induction gives
$$|a_{N+k}| \leq |a_N|r^k \qquad (k \geq 0).$$
The tail of $\sum |a_n|$ is therefore dominated by a convergent geometric series. Hence $\sum a_n$ converges absolutely.

For the second statement, if $|a_{n+1}| \geq |a_n|$ eventually, then $|a_n|$ cannot tend to 0 unless the sequence is eventually zero. In the nontrivial case, the term test proves divergence. The statement for an ordinary limit $L > 1$ follows because the ratios eventually exceed some number greater than 1. □

Theorem 13.18 (Root test)**.** *Let a_n be a sequence of real numbers.*

1. *If*
$$\limsup_{n\to\infty}|a_n|^{1/n} < 1,$$
then $\sum a_n$ converges absolutely.

2. *If*
$$\limsup_{n\to\infty}|a_n|^{1/n} > 1,$$
then $\sum a_n$ diverges.

Proof. For the first assertion, choose r with
$$\limsup_{n\to\infty}|a_n|^{1/n} < r < 1.$$
Then, for all sufficiently large n,
$$|a_n|^{1/n} \leq r,$$
so $|a_n| \leq r^n$. Comparison with the geometric series proves absolute convergence.

For the second assertion, choose $r > 1$ smaller than the limsup. There are infinitely many n such that
$$|a_n|^{1/n} > r,$$
and hence $|a_n| > r^n > 1$. Thus a_n does not tend to 0, so the term test gives divergence. □

Example 13.19. The series

$$\sum_{n=1}^{\infty} \frac{n^3}{2^n}$$

converges absolutely. Indeed,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{2n^3} \to \frac{1}{2} < 1.$$

Example 13.20. The series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges absolutely, since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n \to e^{-1} < 1.$$

13.7 Alternating series

Theorem 13.21 (Alternating-series theorem)**.** *Let (a_n) be a decreasing sequence of nonnegative real numbers such that*

$$a_n \to 0.$$

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

If S is its sum and

$$s_n = \sum_{k=1}^{n} (-1)^{k-1} a_k,$$

then

$$s_{2n} \le S \le s_{2n-1},$$

and

$$|S - s_n| \le a_{n+1}.$$

Proof. The even partial sums satisfy

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \ge 0,$$

so (s_{2n}) is increasing. The odd partial sums satisfy

$$s_{2n+1} - s_{2n-1} = -a_{2n} + a_{2n+1} \le 0,$$

so (s_{2n-1}) is decreasing. Also,

$$s_{2n-1} - s_{2n} = a_{2n} \ge 0,$$

so every even partial sum is at most every odd partial sum. Thus (s_{2n}) is increasing and bounded above, while (s_{2n-1}) is decreasing and bounded below. By the Monotone Convergence Theorem, there are limits

$$s_{2n} \to L, \qquad s_{2n-1} \to U.$$

But

$$U - L = \lim_{n\to\infty}(s_{2n-1} - s_{2n}) = \lim_{n\to\infty} a_{2n} = 0.$$

Hence $L = U =: S$. Since both the even and odd subsequences of partial sums tend to S, the full sequence (s_n) tends to S.

The inequalities $s_{2n} \le S \le s_{2n-1}$ follow from the monotonicity just proved. The error estimate follows because S lies between two consecutive partial sums, whose difference is a_{n+1}. □

Example 13.22. The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges, because $1/n$ decreases to 0. Theorem 13.21 also gives an immediate error estimate: truncation after n terms has error at most $1/(n+1)$.

13.8 Absolute convergence

Definition 13.23. A series $\sum a_n$ is *absolutely convergent* if

$$\sum_{n=1}^{\infty} |a_n|$$

converges. It is *conditionally convergent* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem 13.24 (Absolute convergence implies convergence)**.** *If $\sum |a_n|$ converges, then $\sum a_n$ converges.*

Proof. Let $\varepsilon > 0$. Since $\sum |a_n|$ converges, the Cauchy criterion gives an N such that

$$m > n \ge N \quad\Longrightarrow\quad \sum_{k=n+1}^{m} |a_k| < \varepsilon.$$

The triangle inequality gives

$$\left|\sum_{k=n+1}^{m} a_k\right| \le \sum_{k=n+1}^{m} |a_k| < \varepsilon.$$

The Cauchy criterion for series now proves convergence of $\sum a_n$. □

Corollary 13.25. *If $|a_n| \le b_n$ for all sufficiently large n, where $b_n \ge 0$ and $\sum b_n$ converges, then $\sum a_n$ converges absolutely.*

Proof. The comparison test gives convergence of $\sum |a_n|$, and Theorem 13.24 finishes the proof. □

Example 13.26. The series

$$\sum_{n=1}^{\infty} \frac{\sin(n^2)}{n^2}$$

converges absolutely, because

$$\left|\frac{\sin(n^2)}{n^2}\right| \le \frac{1}{n^2}$$

and $\sum 1/n^2$ converges.

Remark 13.27. Absolute convergence is stronger than ordinary convergence. The alternating harmonic series is conditionally convergent: it converges by Theorem 13.21, while the series of absolute values is the divergent harmonic series.

13.9 Dirichlet's and Abel's tests

Lemma 13.28 (Summation by parts)**.** *Let (a_n) and (b_n) be sequences, and set*

$$A_n = a_1 + \cdots + a_n, \qquad A_0 = 0.$$

Then for every $n \geq 1$,

$$\sum_{k=1}^{n} a_k b_k = A_n b_n + \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}).$$

Proof. Substitute $a_k = A_k - A_{k-1}$ and reindex:

$$\begin{aligned}
\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} (A_k - A_{k-1}) b_k \\
&= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k \\
&= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n-1} A_k b_{k+1} \\
&= A_n b_n + \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}),
\end{aligned}$$

where in the third line we reindexed the second sum by $k \mapsto k+1$ and used $A_0 = 0$. □

Theorem 13.29 (Dirichlet's test)**.** *Suppose that the partial sums*

$$A_n = \sum_{k=1}^{n} a_k$$

are bounded, and that (b_n) is monotone with $b_n \to 0$. Then

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Proof. We use the Cauchy criterion (Theorem 13.5). Applying Lemma 13.28 to the block $n+1, \ldots, m$ gives, for $m > n$,

$$\sum_{k=n+1}^{m} a_k b_k = A_m b_m - A_n b_{n+1} + \sum_{k=n+1}^{m-1} A_k (b_k - b_{k+1}).$$

First treat the case in which (b_n) is decreasing. Since $b_n \to 0$, we have $b_n \geq 0$ for all n. Let M be a bound with $|A_k| \leq M$ for all k. Then, because $b_k - b_{k+1} \geq 0$,

$$\begin{aligned}
\left| \sum_{k=n+1}^{m} a_k b_k \right| &\leq |A_m| b_m + |A_n| b_{n+1} + \sum_{k=n+1}^{m-1} |A_k| (b_k - b_{k+1}) \\
&\leq M b_m + M b_{n+1} + M \sum_{k=n+1}^{m-1} (b_k - b_{k+1}) \\
&= M b_m + M b_{n+1} + M (b_{n+1} - b_m) \\
&= 2 M b_{n+1}.
\end{aligned}$$

Given $\varepsilon > 0$, choose N so that $b_{n+1} < \varepsilon/(2M)$ whenever $n \geq N$, which is possible because $b_n \to 0$. Then for every $m > n \geq N$,

$$\left| \sum_{k=n+1}^{m} a_k b_k \right| < \varepsilon.$$

The Cauchy criterion therefore proves convergence.

If instead (b_n) is increasing, then $b_n \leq 0$ for all n, and $c_n := -b_n$ is a decreasing nonnegative sequence tending to 0. By the case just proved, $\sum a_n c_n$ converges; hence $\sum a_n b_n = -\sum a_n c_n$ converges. □

Theorem 13.30 (Abel's test)**.** *If $\sum a_n$ converges and (b_n) is monotone and bounded, then $\sum a_n b_n$ converges.*

Proof. Since (b_n) is monotone and bounded, it converges; write $b_n \to b$. The sequence $(b_n - b)$ is monotone and tends to 0, and the partial sums A_n of the convergent series $\sum a_n$ are bounded. By Dirichlet's test (Theorem 13.29), the series

$$\sum a_n (b_n - b)$$

converges. Also $\sum a_n b$ converges, being a constant multiple of the convergent series $\sum a_n$. Therefore

$$\sum a_n b_n = \sum a_n (b_n - b) + \sum a_n b$$

converges, being the sum of two convergent series. □

Remark 13.31. The alternating-series theorem is the special case of Dirichlet's test with $a_n = (-1)^{n+1}$ and $b_n = |c_n|$: if $(|c_n|)$ decreases to 0, then the partial sums of $((-1)^{n+1})$ are 1 and 0, hence bounded, so Theorem 13.29 gives the convergence of $\sum (-1)^{n+1}|c_n|$ stated in Theorem 13.21.

Example 13.32 (Dirichlet's test and $\sum \sin(nx)/n$)**.** For every $x \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

converges.

If $x \in 2\pi\mathbb{Z}$, then $\sin(nx) = 0$ for every n, so the convergence is trivial. Suppose $x \notin 2\pi\mathbb{Z}$, so that $\sin(x/2) \neq 0$. Apply Dirichlet's test with $a_n = \sin(nx)$ and $b_n = 1/n$. The partial sums satisfy

$$\left| \sum_{k=1}^{n} \sin(kx) \right| = \frac{|\sin(nx/2)\sin((n+1)x/2)|}{|\sin(x/2)|} \leq \frac{1}{|\sin(x/2)|},$$

so they are bounded, while $b_n = 1/n$ decreases to 0. Dirichlet's test gives convergence.

13.10 Rearrangements of absolutely convergent series

The order of terms in a finite sum is irrelevant. For an infinite series, this remains true under absolute convergence.

Theorem 13.33 (Rearrangement theorem)**.** *Let $\sum a_n$ converge absolutely, and let*

$$\sigma : \mathbb{N} \to \mathbb{N}$$

be a bijection. Then the rearranged series

$$\sum_{n=1}^{\infty} a_{\sigma(n)}$$

converges and has the same sum as $\sum a_n$.

Proof. Let

$$S = \sum_{n=1}^{\infty} a_n.$$

Fix $\varepsilon > 0$. Since $\sum |a_n|$ converges, choose N such that

$$\sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2}.$$

Because σ is a bijection, there is an integer M such that every one of the indices $1, 2, \ldots, N$ occurs among

$$\sigma(1), \ldots, \sigma(M).$$

For $m \geq M$, the difference between the mth rearranged partial sum and S can be written as a sum of terms from the original tail, with signs unchanged. Its absolute value is at most the sum of the absolute values in that tail, together with the corresponding omitted tail terms. Hence

$$\left| \sum_{j=1}^{m} a_{\sigma(j)} - S \right| \leq 2 \sum_{k=N+1}^{\infty} |a_k| < \varepsilon.$$

Therefore the rearranged partial sums converge to S. □

Remark 13.34. Conditional convergence behaves differently: a conditionally convergent series can be rearranged to change its sum, or even to diverge. This is the content of the Riemann rearrangement theorem, Theorem 13.35 in the next section; it explains why absolute convergence is so valuable.

13.11 ‡ The Riemann rearrangement theorem

Theorem 13.35 (Riemann rearrangement theorem)**.** *Suppose that $\sum a_n$ is conditionally convergent.*

1. *For every $L \in \mathbb{R}$ there is a rearrangement of $\sum a_n$ that converges to L.*
2. *There is a rearrangement of $\sum a_n$ that diverges to $+\infty$, and another that diverges to $-\infty$.*

Proof. Set

$$p_n = \frac{a_n + |a_n|}{2} \geq 0, \qquad q_n = \frac{|a_n| - a_n}{2} \geq 0.$$

Then

$$p_n - q_n = a_n, \qquad p_n + q_n = |a_n|.$$

Since $\sum a_n$ converges, $a_n \to 0$, so $|a_n| \to 0$, and hence $p_n \to 0$ and $q_n \to 0$.

We claim that both $\sum p_n$ and $\sum q_n$ diverge to $+\infty$. If $\sum p_n$ converged, then

$$\sum q_n = \sum (p_n - a_n) = \sum p_n - \sum a_n$$

would also converge, and hence $\sum |a_n| = \sum (p_n + q_n)$ would converge, contradicting conditional convergence. Thus $\sum p_n$ diverges; being a series of nonnegative terms, it diverges to $+\infty$. The same argument with the roles of p and q interchanged (using $p_n = a_n + q_n$) shows that $\sum q_n$ diverges to $+\infty$.

Let $P_1, P_2, \ldots$ be the positive terms among the p_n (zeros omitted), and let $-Q_1, -Q_2, \ldots$ be the negative terms (the negatives of the nonzero q_n), both listed in their original order. Then $P_i \to 0$ and $Q_j \to 0$, and

$$\sum_{i=1}^{\infty} P_i = +\infty, \qquad \sum_{j=1}^{\infty} Q_j = +\infty.$$

Zero terms. If $a_n = 0$, then $p_n = q_n = 0$, and such a term is omitted from the two lists above. To obtain a genuine rearrangement of the whole original series, reinsert each zero term by the following fixed rule: after the alternating construction below is finished, place every zero term of (a_n) back among the constructed terms in its original relative order, in the same relative position it occupied among the nonzero terms (equivalently, insert each zero term immediately after the first nonzero term that follows it in the original order; a zero term occurring before every nonzero term is placed first). Because every inserted term is 0, none of the partial-sum estimates below is changed by its presence.

Finite limit. Fix $L \in \mathbb{R}$. Add positive terms $P_1, P_2, \ldots$ until the partial sum first exceeds L; then add negative terms $-Q_1, -Q_2, \ldots$ until it first drops below L; then add positive terms again until it first exceeds L, and continue alternating forever. This is possible at every stage because $\sum P_i = +\infty$ and $\sum Q_j = +\infty$. Both lists are consumed in their original order, and because each of the series $\sum P_i$ and $\sum Q_j$ diverges to $+\infty$, the indices of the last terms used tend to infinity: every nonzero term of the original series is eventually used exactly once, and no nonzero term is omitted.

We show that the resulting partial sums converge to L. Let $\varepsilon > 0$. Since $P_i \to 0$ and $Q_j \to 0$, there is N such that $P_i < \varepsilon$ and $Q_j < \varepsilon$ whenever $i, j \geq N$. After finitely many stages, every term used has index at least N. From that point on, each partial sum lies within ε of L: immediately after a positive crossing it exceeds L by less than the last positive term (which is $< \varepsilon$), immediately after a negative crossing it falls short of L by less than the last negative term (which is $< \varepsilon$), and between crossings it lies between these two values. Hence the partial sums converge to L.

Divergence to $+\infty$. Add positive terms until the partial sum first exceeds 1, then add the single term $-Q_1$; add positive terms until the sum first exceeds 2, then add $-Q_2$; and continue, adding $-Q_k$ immediately after the sum first exceeds k. Each stage is possible because $\sum P_i = +\infty$. Within stage k, during the positive run the partial sum never drops below its value at the start of the stage, which exceeds $(k-1) - Q_{k-1}$, and it then first exceeds k; after the single subtraction of Q_k it still exceeds $k - Q_k$. Since $Q_k \to 0$, both $k - Q_k$ and $(k-1) - Q_{k-1}$ tend to $+\infty$. Hence, for every M, all partial sums from some stage onward exceed M, so the partial sums tend to $+\infty$.

Divergence to $-\infty$. The symmetric construction—negative terms until the sum first drops below $-k$, then one positive term—produces a rearrangement diverging to $-\infty$.

In every case, reinserting the zero terms by the rule above yields a sequence in which every original index occurs exactly once, so the result is a genuine rearrangement of $\sum a_n$. □

13.12 Exercises

Exercise 13.36. Determine whether each series converges. If it converges, find its sum:

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n, \qquad \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n, \qquad \sum_{n=0}^{\infty}(-1)^n.$$

Exercise 13.37. Use Theorem 13.3 to show that each series diverges:

$$\sum_{n=1}^{\infty}\frac{n+1}{n}, \qquad \sum_{n=1}^{\infty}\sin\left(\frac{\pi n}{2}\right), \qquad \sum_{n=1}^{\infty}\frac{n}{n+1}.$$

Exercise 13.38. Prove the Cauchy criterion for series directly from the Cauchy criterion for sequences.

Exercise 13.39. Prove that the harmonic series diverges by grouping terms as in Example 13.9.

Exercise 13.40. Use comparison with a p-series to determine the convergence of

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n}, \qquad \sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+1}}, \qquad \sum_{n=1}^{\infty}\frac{n+1}{n^2+1}.$$

Exercise 13.41. Use the integral test to determine whether

$$\sum_{n=2}^{\infty}\frac{1}{n\log n}$$

converges.

Exercise 13.42. Use the limit comparison test to determine the convergence of

$$\sum_{n=1}^{\infty}\frac{5n^3-2}{n^5+7n+1}.$$

Exercise 13.43. Use the ratio test to determine the convergence of

$$\sum_{n=1}^{\infty}\frac{n^4}{3^n}, \qquad \sum_{n=1}^{\infty}\frac{2^n}{n!}.$$

Exercise 13.44. Use the root test to determine the convergence of

$$\sum_{n=1}^{\infty}\left(\frac{2n+1}{3n+2}\right)^n.$$

Exercise 13.45. Use the alternating-series theorem to approximate

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

with absolute error less than 10^{-4}. State how many terms suffice.

Exercise 13.46. Prove that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$$

converges but not absolutely.

Exercise 13.47. Prove that

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

converges absolutely.

Exercise 13.48. Let $a_n \geq 0$ and suppose that $\sum a_n$ converges. Prove that every subseries obtained by retaining some terms and discarding the others also converges.

Exercise 13.49. Fill in the omitted details in the proof of Theorem 13.33.

Exercise 13.50. Explain why the ratio and root tests are inconclusive when their respective limiting quantities equal 1. Give one convergent and one divergent example for each test with limiting value 1.

Chapter 14

Power Series

A power series is a series whose terms are polynomial powers of a variable. Power series combine the algebraic flexibility of polynomials with the limiting structure of infinite series. Inside their interval of convergence, they define continuous functions that may be differentiated and integrated term by term. They provide the rigorous setting in which Taylor polynomials become Taylor series.

This chapter establishes the basic convergence theorem for power series, develops the radius of convergence, proves the principal term-by-term operations, and applies the results to the familiar functions introduced earlier.

14.1 Definition and first examples

Definition 14.1. A *power series centered at a* is a formal expression

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

where $a \in \mathbb{R}$ and (c_n) is a sequence of real coefficients. For a fixed x, the power series is interpreted as the numerical series

$$\sum_{n=0}^{\infty} c_n(x-a)^n.$$

The associated partial sums are the polynomials

$$S_N(x) = \sum_{n=0}^{N} c_n(x-a)^n.$$

Thus a power series is a sequence of polynomial approximations together with the question of whether these polynomials converge at a specified input x.

Example 14.2 (Geometric power series)**.** For $a = 0$ and $c_n = 1$, the power series is

$$\sum_{n=0}^{\infty} x^n.$$

By the geometric-series formula, it converges for $|x| < 1$ and has sum

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

At $x = 1$ it diverges because every term is 1, and at $x = -1$ it diverges because its partial sums alternate between 1 and 0.

Example 14.3. The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for every real x. Indeed, the ratio of successive absolute terms is

$$\left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \frac{|x|}{n+1} \to 0,$$

so the ratio test applies for each fixed x.

14.2 The interval of convergence

A basic phenomenon distinguishes power series from arbitrary series: convergence at one point controls convergence at all points closer to the center.

Theorem 14.4 (Convergence inside the disk)**.** *Suppose that the power series*

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges at $x = b$, where $b \neq a$. Then it converges absolutely for every x satisfying

$$|x-a| < |b-a|.$$

Proof. Because the series converges at b, its terms tend to 0 and hence are bounded. Thus there is $M > 0$ such that

$$|c_n (b-a)^n| \leq M \qquad \text{for every } n.$$

Let x satisfy $|x-a| < |b-a|$, and set

$$q = \frac{|x-a|}{|b-a|} < 1.$$

Then

$$|c_n (x-a)^n| = |c_n (b-a)^n| q^n \leq M q^n.$$

The geometric series $\sum M q^n$ converges, so the comparison test proves absolute convergence. □

Theorem 14.5 (Divergence outside the disk)**.** *If the power series*

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

diverges at $x = b$, then it diverges at every x satisfying

$$|x-a| > |b-a|.$$

Proof. Suppose, to the contrary, that the series converges at some x with $|x-a| > |b-a|$. By Theorem 14.4, convergence at x would imply absolute convergence at b, because $|b-a| < |x-a|$. This contradicts divergence at b. □

Theorem 14.6 (Radius of convergence)**.** *For every power series*

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

there is a number

$$R \in [0, \infty]$$

called its radius of convergence*, with the following properties:*

1. *The series converges absolutely for $|x-a| < R$.*
2. *The series diverges for $|x-a| > R$.*
3. *If $0 < R < \infty$, then at the two boundary points satisfying $|x-a| = R$ no universal conclusion is possible; each endpoint must be considered separately.*

If $R = 0$, the only point satisfying $|x-a| \le R$ is $x = a$ itself, where the series always converges (to c_0). If $R = \infty$, there are no finite boundary points: the series converges absolutely for every real x.

Proof. Let

$$C = \{r \ge 0 : \text{the series converges at every } x \text{ with } |x-a| < r\}.$$

The set C is nonempty because $0 \in C$, and it is an initial interval by Theorem 14.4. Define

$$R = \sup C$$

in the extended interval $[0, \infty]$.

If $|x-a| < R$, choose $r \in C$ with $|x-a| < r \le R$, and then choose b with $|x-a| < |b-a| < r$. The series converges at b, so Theorem 14.4 gives absolute convergence at x. If $|x-a| > R$ and the series converged at x, Theorem 14.4 would imply convergence at every point with distance less than $|x-a|$ from a, contradicting the definition of R as the supremum of C. The boundary statement for $0 < R < \infty$ follows because the preceding arguments use strict inequalities only. If $R = 0$, then the series converges at no point with $x \ne a$; it always converges at $x = a$, where it reduces to the single term c_0. If $R = \infty$, then every real x satisfies $|x-a| < R$, so the first case already gives absolute convergence for all x, and there are no finite boundary points. □

Remark 14.7. Theorem 14.6 says nothing at the boundary points. All three possibilities occur:

- The geometric series $\sum x^n$ diverges at both endpoints $x = \pm 1$.
- The series $\sum x^n/n^2$ converges at both endpoints.
- The series $\sum x^n/n$ converges at $x = -1$ but diverges at $x = 1$.

14.3 Computing radii of convergence

Theorem 14.8 (Ratio criterion for power series)**.** *Suppose that $c_n \ne 0$ for all sufficiently large n and that the limit*

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

exists in $[0, \infty]$. *Then the radius of convergence is*

$$R = \frac{1}{L},$$

with the conventions $1/0 = \infty$ *and* $1/\infty = 0$.

Proof. For fixed x, the ratio of successive absolute terms is

$$\left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \left| \frac{c_{n+1}}{c_n} \right| |x-a|.$$

This tends to $L|x-a|$. The ratio test gives absolute convergence when $L|x-a| < 1$ and divergence when $L|x-a| > 1$. Thus the radius is $1/L$. □

Theorem 14.9 (Root criterion for power series)**.** *Let*

$$L = \limsup_{n\to\infty} |c_n|^{1/n}.$$

Then the radius of convergence is

$$R = \frac{1}{L},$$

with the conventions $1/0 = \infty$ *and* $1/\infty = 0$.

Proof. For fixed x,

$$|c_n(x-a)^n|^{1/n} = |c_n|^{1/n}|x-a|.$$

Taking limsups gives $L|x-a|$. The root test proves absolute convergence for $L|x-a| < 1$ and divergence for $L|x-a| > 1$. □

Example 14.10. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$$

has infinite radius of convergence because

$$\left| \frac{1/(n+1)!}{1/n!} \right| = \frac{1}{n+1} \to 0.$$

Example 14.11. The series

$$\sum_{n=1}^{\infty} n(x-3)^n$$

has radius 1, since

$$\left| \frac{n+1}{n} \right| \to 1.$$

At $x = 4$, its terms are n, so it diverges. At $x = 2$, its terms are $n(-1)^n$, which do not tend to zero. Thus it diverges at both endpoints.

14.4 Pointwise and uniform convergence

The convergence of a power series at a point is a statement about a numerical series. The stronger notion needed to justify term-by-term operations is *uniform* convergence, in which a single index N works for every point of the set at once. This section develops the three transfer theorems—continuity, integration, and differentiation—that the rest of the chapter invokes.

Definition 14.12 (Pointwise and uniform convergence)**.** Let (f_n) be a sequence of real-valued functions on a set S, and let f be a real-valued function on S.

1. The sequence (f_n) *converges pointwise* to f on S if for every $x \in S$ and every $\varepsilon > 0$ there is an N (which may depend on both x and ε) such that
$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$
2. The sequence (f_n) *converges uniformly* to f on S if for every $\varepsilon > 0$ there is an N (which may depend on ε but not on x) such that
$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon \qquad \text{for all } x \in S.$$
3. A series $\sum_n g_n$ of functions *converges pointwise* (respectively, *uniformly*) on S if its sequence of partial sums does.

Uniform convergence is stronger than pointwise convergence: a single uniform N serves simultaneously as the N for every x. The following criterion is the analogue of the Cauchy criterion for numerical sequences, and it does not require naming the limit function in advance.

Theorem 14.13 (Uniform Cauchy criterion)**.** *The sequence (f_n) converges uniformly on S if and only if for every $\varepsilon > 0$ there is an N such that*
$$m, n \geq N \implies |f_n(x) - f_m(x)| < \varepsilon \qquad \textit{for all } x \in S.$$

Proof. Suppose first that (f_n) converges uniformly on S to some f. Given $\varepsilon > 0$, choose N so that $|f_n(x) - f(x)| < \varepsilon/2$ for all $n \geq N$ and all $x \in S$. For $m, n \geq N$ and $x \in S$, the triangle inequality gives
$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, suppose the uniform Cauchy condition holds. For each fixed $x \in S$, the numerical sequence $(f_n(x))$ is Cauchy, because the condition restricted to the single point x gives $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$. By the completeness of $\mathbb{R}$ (Theorem 12.36), each such sequence converges; define
$$f(x) = \lim_{n \to \infty} f_n(x) \qquad (x \in S).$$
We claim $f_n \to f$ uniformly. Given $\varepsilon > 0$, choose N from the uniform Cauchy condition. For fixed $x \in S$ and fixed $n \geq N$, we have $|f_n(x) - f_m(x)| < \varepsilon$ for all $m \geq N$. Letting $m \to \infty$ (continuity of the absolute value preserves a weak inequality in the limit) gives
$$|f_n(x) - f(x)| \leq \varepsilon \qquad \text{for all } x \in S.$$
This holds for every $n \geq N$, which is exactly uniform convergence. □

Theorem 14.14 (Weierstrass M-test)**.** *Let (g_n) be a sequence of functions on S. Suppose there are constants $M_n \geq 0$ such that*

$$|g_n(x)| \leq M_n \qquad \text{for all } x \in S,$$

and suppose the numerical series $\sum_n M_n$ converges. Then $\sum_n g_n$ converges uniformly and absolutely on S.

Proof. Let $s_N = \sum_{n=0}^{N} g_n$ be the Nth partial sum. For $m > n \geq N$ and $x \in S$,

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} g_k(x) \right| \leq \sum_{k=n+1}^{m} |g_k(x)| \leq \sum_{k=n+1}^{m} M_k.$$

Because $\sum_k M_k$ converges, its partial sums form a Cauchy sequence (Theorem 13.5), so given $\varepsilon > 0$ there is N such that $\sum_{k=n+1}^{m} M_k < \varepsilon$ whenever $m > n \geq N$. Hence $|s_m(x) - s_n(x)| < \varepsilon$ for all $x \in S$, so (s_n) satisfies the uniform Cauchy criterion (Theorem 14.13) and therefore converges uniformly.

For absolute convergence, fix $x \in S$. The numerical series $\sum_n |g_n(x)|$ is dominated term by term by the convergent series $\sum_n M_n$, so it converges by the comparison test (Theorem 13.8). □

Theorem 14.15 (Uniform limit of continuous functions)**.** *Let (f_n) be a sequence of functions on S, each continuous at $x_0 \in S$. If $f_n \to f$ uniformly on S, then f is continuous at x_0.*

Proof. Let $\varepsilon > 0$. By uniform convergence, choose N such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \qquad \text{for all } x \in S$$

whenever $n \geq N$. Fix such an n. Since f_n is continuous at x_0, there is $\delta > 0$ such that for $x \in S$ with $|x - x_0| < \delta$,

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}.$$

Then for such x,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence f is continuous at x_0. □

Example 14.16 (Pointwise convergence need not preserve continuity)**.** Let

$$f_n(x) = x^n \qquad (x \in [0,1]).$$

For each fixed $x \in [0,1)$ the sequence $f_n(x) = x^n$ tends to 0, while $f_n(1) = 1$ for every n. The pointwise limit is therefore

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

Each f_n is continuous on $[0,1]$, but f is discontinuous at $x = 1$. By Theorem 14.15, the convergence cannot be uniform: a uniform limit of continuous functions is continuous. The failure is visible directly as well: for x just below 1, the value x^n stays close to 1 no matter how large n is, so no single N can make $|x^n - 0| < \varepsilon$ uniformly for all $x \in [0,1)$.

Theorem 14.17 (Uniform convergence and integration)**.** *Let (f_n) be a sequence of Darboux integrable functions on $[a,b]$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then f is Darboux integrable on $[a,b]$, and*

$$\int_a^b f_n \to \int_a^b f.$$

Proof. The case $a = b$ is trivial, so assume $a < b$. First, f is bounded: choose n with $|f_n(x) - f(x)| < 1$ for all $x \in [a,b]$; since f_n is integrable it is bounded, say $|f_n(x)| \le M$, whence $|f(x)| \le M + 1$.

Let $\varepsilon > 0$. By uniform convergence, choose N such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \qquad \text{for all } x \in [a,b] \text{ and all } n \ge N.$$

We first prove integrability of f. Fix $n = N$. Since f_n is integrable, the Darboux criterion (Theorem 7.12) gives a partition $P = (t_0, \dots, t_m)$ of $[a,b]$ with

$$U(P, f_n) - L(P, f_n) < \frac{\varepsilon}{3}.$$

On each subinterval $[t_{i-1}, t_i]$, every x satisfies

$$f(x) \le f_n(x) + |f(x) - f_n(x)| < f_n(x) + \frac{\varepsilon}{3(b-a)} \le M_i(f_n) + \frac{\varepsilon}{3(b-a)},$$

where $M_i(f_n) = \sup\{f_n(x) : x \in [t_{i-1}, t_i]\}$. Therefore

$$M_i(f) = \sup\{f(x) : x \in [t_{i-1}, t_i]\} \le M_i(f_n) + \frac{\varepsilon}{3(b-a)}.$$

Multiplying by $t_i - t_{i-1}$ and summing over i gives

$$U(P, f) \le U(P, f_n) + \frac{\varepsilon}{3}.$$

Similarly, on each subinterval,

$$f(x) \ge f_n(x) - |f(x) - f_n(x)| > f_n(x) - \frac{\varepsilon}{3(b-a)} \ge m_i(f_n) - \frac{\varepsilon}{3(b-a)},$$

where $m_i(f_n) = \inf\{f_n(x) : x \in [t_{i-1}, t_i]\}$, so that

$$m_i(f) = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \ge m_i(f_n) - \frac{\varepsilon}{3(b-a)},$$

and hence

$$L(P, f) \ge L(P, f_n) - \frac{\varepsilon}{3}.$$

Combining the two bounds,

$$U(P, f) - L(P, f) \le (U(P, f_n) - L(P, f_n)) + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

By the Darboux criterion, f is integrable.

It remains to show $\int_a^b f_n \to \int_a^b f$. For $n \ge N$, the function $f - f_n$ is integrable (Theorem 7.31), so $|f - f_n|$ is integrable (Theorem 7.40), and

$$\left|\int_a^b f - \int_a^b f_n\right| = \left|\int_a^b (f - f_n)\right| \le \int_a^b |f - f_n| \le \int_a^b \frac{\varepsilon}{3(b-a)} = \frac{\varepsilon}{3(b-a)}(b-a) = \frac{\varepsilon}{3} < \varepsilon,$$

where the first inequality is the triangle inequality for integrals (Theorem 7.42) and the second is the comparison theorem (Theorem 7.32). Since $\varepsilon > 0$ was arbitrary, $\int_a^b f_n \to \int_a^b f$. □

Example 14.18 (Pointwise convergence need not preserve integrals)**.** For each n, let f_n be the continuous "tent" function on $[0,1]$ that rises linearly from 0 at $x=0$ to n at $x=\frac{1}{n}$, then falls linearly back to 0 at $x=\frac{2}{n}$, and vanishes on $[\frac{2}{n},1]$:

$$f_n(x)=\begin{cases} n^2x, & 0\le x\le \frac{1}{n},\\ 2n-n^2x, & \frac{1}{n}\le x\le \frac{2}{n},\\ 0, & \frac{2}{n}\le x\le 1. \end{cases}$$

For every $x\in[0,1]$ we have $f_n(x)\to 0$: this is clear at $x=0$ (where $f_n(0)=0$ for all n), and for $x>0$ it holds because $f_n(x)=0$ once $\frac{2}{n}<x$. Thus $f_n\to 0$ pointwise on $[0,1]$. However, each f_n is a triangle of height n and base $\frac{2}{n}$, so

$$\int_0^1 f_n=\frac{1}{2}\cdot\frac{2}{n}\cdot n=1$$

for every n, while $\int_0^1 0=0$. Hence

$$\lim_{n\to\infty}\int_0^1 f_n=1\neq 0=\int_0^1\lim_{n\to\infty} f_n.$$

Theorem 14.17 does not apply because the convergence is not uniform: $|f_n(1/n)-0|=n$ is unbounded, so no single N controls $\sup_x|f_n(x)|$.

Theorem 14.19 (Uniform convergence and differentiation)**.** *Let (f_n) be a sequence of differentiable functions on an interval I, each with continuous derivative f_n' on I. Suppose further that $(f_n(x_0))$ converges for some $x_0\in I$, and that (f_n') converges uniformly on I to a function g. Then (f_n) converges uniformly on every closed bounded subinterval of I to a differentiable function f, and $f'=g$.*

Proof. Fix a closed bounded subinterval $J=[c,d]\subset I$. Because I is an interval containing both J and x_0, the closed bounded interval

$$J'=[\min\{c,x_0\},\max\{d,x_0\}]$$

is contained in I; it suffices to prove uniform convergence on J', which implies the result on J.

Each f_n' is continuous on I, hence Darboux integrable on every closed bounded subinterval of I (Corollary 7.52). By the Fundamental Theorem of Calculus (Theorem 7.57), for every n and every $x\in J'$,

$$f_n(x)=f_n(x_0)+\int_{x_0}^{x} f_n'.$$

Consequently, for m,n and $x\in J'$,

$$|f_n(x)-f_m(x)|\le|f_n(x_0)-f_m(x_0)|+\left|\int_{x_0}^{x}(f_n'-f_m')\right|.$$

Now $f_n'-f_m'$ is continuous, hence bounded on J', and the triangle inequality for integrals (Theorem 7.42) together with the comparison theorem (Theorem 7.32) gives

$$\left|\int_{x_0}^{x}(f_n'-f_m')\right|\le\left|\int_{x_0}^{x}|f_n'-f_m'|\right|\le|x-x_0|\sup_{t\in J'}|f_n'(t)-f_m'(t)|,$$

where the integral over $[x_0,x]$ is understood with its endpoints possibly interchanged.

Let $\varepsilon > 0$. Since $(f_n(x_0))$ converges, it is Cauchy, so there is N_1 with

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \qquad (m, n \geq N_1).$$

Since (f_n') converges uniformly on I, it is uniformly Cauchy on I (Theorem 14.13), so there is N_2 with

$$\sup_{t \in J'} |f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(d' - c' + 1)} \qquad (m, n \geq N_2),$$

where $J' = [c', d']$. For $N = \max\{N_1, N_2\}$ and $m, n \geq N$, every $x \in J'$ satisfies

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} + |x - x_0| \, \frac{\varepsilon}{2(d' - c' + 1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

because $|x - x_0| \leq d' - c'$ for $x, x_0 \in J'$. Hence (f_n) is uniformly Cauchy on J', and the uniform Cauchy criterion (Theorem 14.13) gives uniform convergence of (f_n) on J' to some function. Since every point of I lies in some such interval and pointwise limits are unique, these functions agree on overlaps and define a single function f on I; moreover $f_n \to f$ uniformly on every closed bounded subinterval of I.

It remains to identify f'. Since each f_n' is continuous on I and $f_n' \to g$ uniformly on I, the function g is continuous on I by Theorem 14.15. Passing to the limit in the identity

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'$$

using the uniform convergence of (f_n) to f and of (f_n') to g, together with Theorem 14.17, gives

$$f(x) = f(x_0) + \int_{x_0}^{x} g.$$

By the Fundamental Theorem of Calculus, Part I (Theorem 7.55), the function $x \mapsto \int_{x_0}^{x} g$ is differentiable with derivative $g(x)$. Therefore f is differentiable on I and $f' = g$. □

Remark 14.20. The hypothesis that each derivative f_n' is continuous is satisfied in every application in this chapter, where the f_n are the partial sums of a power series and their derivatives are again polynomials.

Example 14.21 (Uniform convergence of functions does not justify differentiating the limit)**.** Let

$$f_n(x) = \frac{\sin(nx)}{n} \qquad (x \in \mathbb{R}).$$

Since $|f_n(x)| \leq \frac{1}{n}$ for every x, the sequence converges to 0 uniformly on $\mathbb{R}$. Each f_n is differentiable, with

$$f_n'(x) = \cos(nx).$$

At $x = 0$ we have $f_n'(0) = 1$ for every n, so (f_n') converges to 1 there rather than to $f'(0) = 0$; at $x = \pi$ we have $f_n'(\pi) = \cos(n\pi) = (-1)^n$, which has no limit. Thus the uniform limit $f = 0$ is differentiable, but the derivatives f_n' neither converge to f' nor even converge at every point. Uniform convergence of the functions alone therefore does not justify passing a derivative through the limit (Theorem 14.19).

14.5 Uniform convergence on compact subintervals

The uniform convergence available inside the interval of convergence is the key to continuity, integration, and differentiation.

Theorem 14.22. *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

have radius of convergence $R > 0$. For every r satisfying $0 < r < R$, the series converges uniformly and absolutely on $[a-r, a+r]$.

Proof. Choose b with $r < |b-a| < R$. The series converges at b, so Theorem 14.4 gives absolute convergence at every x with $|x-a| \le r < |b-a|$. For the uniform statement, the terms at b are bounded: there is $M > 0$ with $|c_n(b-a)^n| \le M$ for every n. For $|x-a| \le r$,

$$|c_n(x-a)^n| = |c_n(b-a)^n| \left(\frac{|x-a|}{|b-a|}\right)^n \le M \left(\frac{r}{|b-a|}\right)^n.$$

The geometric series $\sum_n M \left(\frac{r}{|b-a|}\right)^n$ converges because $r < |b-a|$, and its terms are independent of x. By the Weierstrass M-test (Theorem 14.14), the power series converges uniformly and absolutely on $[a-r, a+r]$. □

Corollary 14.23 (Continuity of a power-series sum)**.** *A power series is continuous at every point strictly inside its interval of convergence.*

Proof. Fix x_0 with $|x_0 - a| < R$. Choose r so that

$$|x_0 - a| < r < R.$$

On $[a-r, a+r]$, the partial sums are polynomials and hence continuous, and they converge uniformly to the power-series sum by Theorem 14.22. By Theorem 14.15, the sum is continuous at x_0. □

14.6 Term-by-term integration

Lemma 14.24 (Coefficient growth and radius of convergence)**.** *For any real sequence (c_n), the three power series (centered at 0)*

$$\sum_{n=0}^{\infty} c_n x^n, \qquad \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^n, \qquad \sum_{n=0}^{\infty} n c_n x^n$$

have the same radius of convergence.

Proof. By the root-radius formula (Theorem 14.9), the radius of $\sum_n c_n x^n$ is $1/L$, where

$$L = \limsup_{n\to\infty} |c_n|^{1/n},$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$. The $n = 0$ term is irrelevant to the limsup, and since

$$\left|\frac{c_n}{n+1}\right|^{1/n} = |c_n|^{1/n} (n+1)^{-1/n}, \qquad |nc_n|^{1/n} = |c_n|^{1/n} n^{1/n}$$

with $(n+1)^{-1/n} \to 1$ and $n^{1/n} \to 1$, the standard limsup rule for a sequence multiplied by a convergent positive factor gives

$$\limsup_{n\to\infty} \left|\frac{c_n}{n+1}\right|^{1/n} = \limsup_{n\to\infty} |nc_n|^{1/n} = L.$$

Hence the root-radius formula assigns the same radius $1/L$ to all three series.

It remains to justify $n^{1/n} \to 1$ (the case $(n+1)^{1/n}$ follows by the substitution $m = n+1$). Write $n^{1/n} = 1 + h_n$ with $h_n \geq 0$ for $n \geq 2$. The binomial theorem gives

$$n = (1+h_n)^n \geq \binom{n}{2} h_n^2,$$

so $h_n^2 \leq \frac{2}{n-1} \to 0$, and therefore $h_n \to 0$. Thus $n^{1/n} \to 1$. □

Theorem 14.25 (Integration of a power series)**.** *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

have radius of convergence $R > 0$. For $|x-a| < R$,

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}.$$

The integrated power series has the same radius of convergence R.

Proof. Fix r with $0 < r < R$. By Theorem 14.22, the original series converges uniformly on $[a-r, a+r]$. By Theorem 14.17, for x in this interval,

$$\begin{aligned}\int_a^x f(t)\,dt &= \lim_{N\to\infty} \int_a^x \sum_{n=0}^{N} c_n (t-a)^n\,dt \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{c_n}{n+1}(x-a)^{n+1}.\end{aligned}$$

Since every point with $|x-a| < R$ lies in some such interval, the formula holds throughout the open interval of convergence.

For the radius, the integrated series is $(x-a)\sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^n$. By the coefficient lemma (Lemma 14.24), the series $\sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^n$ has the same radius R as the original series, and multiplication by the factor $(x-a)$ does not change the radius. Hence the integrated series also has radius R. □

Remark 14.26. The radius statement concerns only the interior of the interval of convergence. Convergence of the original series at a boundary point $x = a + R$ does not by itself justify integrating term by term all the way to that boundary point using the interior formula above: the exchange of integral and limit is justified only on closed subintervals strictly inside the interval of convergence. Boundary behavior is governed instead by Abel's theorem (Theorem 14.47).

Example 14.27. For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Integrating from 0 to x gives

$$-\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Thus

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad (|x| < 1).$$

14.7 Term-by-term differentiation

Theorem 14.28 (Differentiation of a power series)**.** *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

have radius of convergence $R > 0$. Then f is differentiable for $|x-a| < R$, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}.$$

The differentiated power series has the same radius of convergence R.

Proof. Let

$$g(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}.$$

We first show that this series converges for $|x-a| < R$. Fix $r < R$ and choose s with

$$r < s < R.$$

The original series converges at $a+s$, so its terms are bounded: there is $M > 0$ such that

$$|c_n s^n| \le M \qquad (n \ge 0).$$

For $|x-a| \le r$,

$$|n c_n (x-a)^{n-1}| \le \frac{M}{s}\, n \left(\frac{r}{s}\right)^{n-1}.$$

The numerical series

$$\sum_{n=1}^{\infty} n \left(\frac{r}{s}\right)^{n-1}$$

converges, as follows for example by the ratio test. Hence the derivative series converges uniformly on $[a-r, a+r]$.

Let

$$S_N(x) = \sum_{n=0}^{N} c_n (x-a)^n.$$

Then $S_N(a) = c_0$, and

$$S_N'(x) = \sum_{n=1}^{N} n c_n (x-a)^{n-1}.$$

The sequence (S_N') converges uniformly to g on $[a-r, a+r]$, and $(S_N(a))$ converges to c_0. Theorem 14.19 gives that S_N converges uniformly to a differentiable function whose derivative is g. But S_N already converges pointwise to f. Hence $f' = g$.

The preceding argument proves convergence of the derivative series throughout $|x-a| < R$. For the radius, the differentiated series is $\frac{1}{x-a}\sum_{n=0}^{\infty} nc_n(x-a)^n$; by the coefficient lemma (Lemma 14.24), the series $\sum_{n=0}^{\infty} nc_n(x-a)^n$ has radius R, and multiplication by the factor $\frac{1}{x-a}$ (for $x \neq a$) does not change the radius. Thus the differentiated series has radius exactly R. □

Corollary 14.29. *Within its interval of convergence, a power series may be differentiated any finite number of times, and the resulting series have the same radius of convergence.*

Proof. Apply Theorem 14.28 repeatedly. □

14.8 Taylor series of analytic functions

Definition 14.30. Let f be infinitely differentiable at a. Its *formal Taylor series centered at* a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If this power series converges to $f(x)$ on an interval about a, then it is called the *Taylor series of* f on that interval.

Definition 14.31 (Real analytic function)**.** A function f is *(real) analytic at* a if there is a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

with positive radius of convergence and a neighborhood of a on which

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

Theorem 14.32 (Coefficients of a power series)**.** *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

have radius of convergence $R > 0$. Then f is infinitely differentiable on $|x-a| < R$, and for every $n \geq 0$,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Consequently, a power-series representation of f centered at a is unique: if two power series centered at a have positive radii and equal sums on a common interval about a, then their coefficients agree term by term.

Proof. By Corollary 14.29, f may be differentiated any finite number of times on $|x-a| < R$. The n-fold derivative of $(x-a)^m$ is

$$m(m-1)\cdots(m-n+1)(x-a)^{m-n},$$

which at $x = a$ vanishes unless $m = n$, in which case it equals $n!$. Differentiating the power series term by term n times (Theorem 14.28) and evaluating at a therefore leaves only the term $c_n n!$; hence $f^{(n)}(a) = n!c_n$. The uniqueness statement follows because the coefficients of any representation are forced to be $f^{(n)}(a)/n!$. □

Example 14.33 (A smooth function that is not analytic)**.** Define

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We claim that f is infinitely differentiable on $\mathbb{R}$ and that $f^{(n)}(0) = 0$ for every n. For $x \neq 0$, induction shows that $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$ for some polynomial P_n. Such expressions tend to 0 as $x \to 0$: substituting $t = 1/x^2$ turns any power $(1/x)^m e^{-1/x^2}$ into $t^{m/2}e^{-t} \to 0$ as $t \to \infty$. Assuming $f^{(n)}(0) = 0$, the definition of the derivative gives

$$f^{(n+1)}(0) = \lim_{h \to 0} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \lim_{h \to 0} \frac{P_n(1/h)e^{-1/h^2}}{h} = 0.$$

Hence $f^{(n)}(0) = 0$ for all n, so the formal Taylor series of f at 0 is identically zero. Since $f(x) > 0$ for every $x \neq 0$, that series does not converge to f on any neighborhood of 0: f is C^∞ but not analytic at 0. Infinite differentiability alone therefore does not guarantee that a function equals its Taylor series; the missing condition is that the Taylor remainders tend to zero.

Theorem 14.34 (Taylor-series criterion)**.** *Suppose that f has derivatives of all orders on an interval containing a and x. If*

$$\lim_{n \to \infty} R_{n,a} f(x) = 0,$$

where $R_{n,a}f(x)$ is the Taylor remainder, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Proof. Taylor's theorem gives

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + R_{n,a} f(x).$$

Letting $n \to \infty$ and using the hypothesis that the remainder tends to zero gives the stated equality. □

Theorem 14.35 (Exponential series)**.** *For every real x,*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Proof. The nth Maclaurin polynomial of e^x is

$$\sum_{k=0}^{n} \frac{x^k}{k!}.$$

By the Lagrange remainder formula, there is a point c between 0 and x such that

$$R_n(x) = \frac{e^c}{(n+1)!} x^{n+1}.$$

For fixed x, $e^c \leq e^{|x|}$ whenever c lies between 0 and x. Hence

$$|R_n(x)| \leq e^{|x|} \frac{|x|^{n+1}}{(n+1)!}.$$

The ratio of successive terms in the right-hand sequence is

$$\frac{|x|}{n+2} \to 0,$$

so the right-hand side tends to zero. Theorem 14.34 proves the result. □

Theorem 14.36 (Trigonometric series)**.** *For every real x,*

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

and

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Proof. The derivatives of sine and cosine have absolute value at most 1. The Lagrange remainder estimate for the degree-N Taylor polynomial therefore gives

$$|R_N(x)| \le \frac{|x|^{N+1}}{(N+1)!}.$$

For fixed x, this bound tends to zero by the ratio test. The Taylor-series criterion gives the claimed expansions. □

Theorem 14.37 (Logarithmic series)**.** *For $-1 < x \le 1$,*

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n},$$

where at $x = 1$ the series is interpreted as the alternating harmonic series.

Proof. For $|x| < 1$, Theorem 14.25, applied to the geometric series

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n,$$

gives

$$\log(1+x) = \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

This is the desired formula on $(-1, 1)$.

At $x = 1$, the series becomes

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which converges by the alternating-series theorem. To identify its sum, use the finite identity

$$\log 2 = \int_0^1 \frac{dt}{1+t} = \sum_{n=0}^{N} (-1)^n \frac{1}{n+1} + (-1)^{N+1} \int_0^1 \frac{t^{N+1}}{1+t}\, dt.$$

The remainder is bounded in absolute value by $1/(N+2)$ and tends to zero. Hence the sum is $\log 2$. □

Theorem 14.38 (Inverse tangent series)**.** *For $|x| \le 1$,*

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Proof. For $|x| < 1$, integrate the geometric expansion

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

from 0 to x. This gives

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

At $x = 1$, convergence and identification of the sum follow from the exact remainder formula for arctangent established in Chapter 11. The case $x = -1$ follows from oddness. □

14.8.1 The binomial series

Theorem 14.39 (Binomial series)**.** *For every $\alpha \in \mathbb{R}$,*

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad (|x| < 1),$$

where $\binom{\alpha}{0} = 1$ and

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \qquad (n \ge 1).$$

Proof. Write $c_n = \binom{\alpha}{n}$. We first record that the series converges for $|x| < 1$. If $\alpha \in \mathbb{N}_0$, then $c_n = 0$ for every $n > \alpha$, the series is a finite sum, and the asserted identity is the ordinary (finite) binomial theorem; so assume $\alpha \notin \mathbb{N}_0$, in which case $c_n \neq 0$ for every n. Then

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{\alpha-n}{n+1}\right| = \frac{|n-\alpha|}{n+1} \to 1,$$

so the radius of convergence is 1 by Theorem 14.8. Define

$$g(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (|x| < 1).$$

By Theorem 14.28, g is differentiable on $(-1,1)$ and

$$g'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n.$$

The coefficients satisfy

$$(n+1)c_{n+1} = (\alpha - n)c_n,$$

because multiplying the product $c_n = \alpha(\alpha-1)\cdots(\alpha-n+1)/n!$ by the factor $(\alpha-n)$ produces $\alpha(\alpha-1)\cdots(\alpha-n)/n! = (n+1)c_{n+1}$. Hence

$$(n+1)c_{n+1} + n c_n = \alpha c_n \qquad (n \ge 0).$$

Therefore, for $|x| < 1$ (where the two series below converge absolutely and may be added term by term),

$$(1+x)g'(x) = g'(x)+xg'(x) = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n+\sum_{n=0}^{\infty}nc_nx^n = \sum_{n=0}^{\infty}((n+1)c_{n+1}+nc_n)x^n = \alpha\sum_{n=0}^{\infty}c_nx^n = \alpha g(x)$$

Now define

$$h(x) = (1+x)^{-\alpha}g(x) \qquad (|x| < 1),$$

using the real powers $t^\beta = e^{\beta \log t}$ (for $t > 0$) of Chapter 9. By the chain rule, h is differentiable on $(-1, 1)$ and

$$h'(x) = -\alpha(1+x)^{-\alpha-1}g(x) + (1+x)^{-\alpha}g'(x) = (1+x)^{-\alpha-1}((1+x)g'(x) - \alpha g(x)) = 0.$$

By Corollary 6.10, h is constant on $(-1, 1)$. Since

$$h(0) = (1+0)^{-\alpha}g(0) = 1 \cdot c_0 = 1,$$

we get $h(x) = 1$ for all $|x| < 1$, and therefore

$$g(x) = (1+x)^\alpha.$$

□

Remark 14.40. When $\alpha \in \mathbb{N}_0$, the coefficients $\binom{\alpha}{n}$ vanish for $n > \alpha$, and the binomial series reduces to the finite binomial theorem. When $\alpha = -1$,

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-n)}{n!} = (-1)^n,$$

and the binomial series is the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n.$$

For $\alpha = \frac{1}{2}$, the first few coefficients are

$$\binom{1/2}{0} = 1, \qquad \binom{1/2}{1} = \frac{1}{2}, \qquad \binom{1/2}{2} = \frac{(1/2)(-1/2)}{2!} = -\frac{1}{8}, \qquad \binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16},$$

so that

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots \qquad (|x| < 1).$$

14.9 Operations on power series

Theorem 14.41 (Cauchy product of absolutely convergent series)**.** *Suppose that* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *converge absolutely, and define*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n \geq 0).$$

Then $\sum_{n=0}^{\infty} c_n$ *converges absolutely, and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof. For every N, the finite product of the absolute partial sums satisfies

$$\left(\sum_{k=0}^{N}|a_k|\right)\left(\sum_{\ell=0}^{N}|b_\ell|\right)\le\left(\sum_{k=0}^{\infty}|a_k|\right)\left(\sum_{\ell=0}^{\infty}|b_\ell|\right)<\infty.$$

Thus the square partial sums $\sum_{k,\ell\le N}|a_kb_\ell|$ of the nonnegative double series are bounded above. The diagonal triangle $\{k+\ell\le N\}$ is contained in the square $\{0,\dots,N\}^2$, so

$$\sum_{n=0}^{N}|c_n|\le\sum_{n=0}^{N}\sum_{k=0}^{n}|a_k|\,|b_{n-k}|=\sum_{k+\ell\le N}|a_kb_\ell|\le\left(\sum_{k=0}^{N}|a_k|\right)\left(\sum_{\ell=0}^{N}|b_\ell|\right).$$

The partial sums of $\sum_n|c_n|$ are therefore increasing and bounded above, hence converge; thus $\sum_n c_n$ converges absolutely.

It remains to identify the sum. Write $A_N=\sum_{k=0}^{N}a_k$ and $B_N=\sum_{\ell=0}^{N}b_\ell$, and group the finite product by diagonals:

$$A_NB_N=\sum_{k=0}^{N}\sum_{\ell=0}^{N}a_kb_\ell=\sum_{n=0}^{N}c_n+\sum_{n=N+1}^{2N}\sum_{k=n-N}^{N}a_kb_{n-k}.$$

The second term R_N collects the pairs (k,ℓ) in the square $\{0,\dots,N\}^2$ with $k+\ell>N$, so

$$|R_N|\le\sum_{k+\ell>N}|a_kb_\ell|.$$

The diagonal partial sums $\sum_{k+\ell\le N}|a_kb_\ell|$ are increasing and bounded above, hence converge; consequently the tail $\sum_{k+\ell>N}|a_kb_\ell|$ tends to 0, and so $R_N\to0$. Letting $N\to\infty$, with $A_NB_N\to(\sum_n a_n)(\sum_n b_n)$, gives

$$\left(\sum_{n=0}^{\infty}a_n\right)\left(\sum_{n=0}^{\infty}b_n\right)=\sum_{n=0}^{\infty}c_n.$$

□

Theorem 14.42 (Addition and multiplication)**.** *Let*

$$\sum_{n=0}^{\infty}a_n(x-c)^n\qquad\text{and}\qquad\sum_{n=0}^{\infty}b_n(x-c)^n$$

have radii of convergence R_a and R_b. On the interval

$$|x-c|<\min\{R_a,R_b\},$$

their sum is represented by

$$\sum_{n=0}^{\infty}(a_n+b_n)(x-c)^n,$$

and their product is represented by the Cauchy product

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)(x-c)^n.$$

Proof. Inside the common interval $|x-c| < \min\{R_a, R_b\}$, both series converge absolutely. The sum formula follows by addition of absolutely convergent series. For the product, fix x in this interval and set

$$A_n = a_n(x-c)^n, \qquad B_n = b_n(x-c)^n.$$

Then $\sum_n A_n$ and $\sum_n B_n$ converge absolutely, so the Cauchy product theorem (Theorem 14.41) gives

$$\left(\sum_{j=0}^{\infty} a_j(x-c)^j\right)\left(\sum_{k=0}^{\infty} b_k(x-c)^k\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_k B_{n-k}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(x-c)^n,$$

because $A_k B_{n-k} = a_k b_{n-k}(x-c)^n$. □

Example 14.43. Multiplying the exponential series by itself gives

$$\begin{aligned} e^x e^x &= \left(\sum_{j=0}^{\infty} \frac{x^j}{j!}\right)\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{k!(n-k)!}\right) x^n \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = e^{2x}. \end{aligned}$$

The penultimate equality is the binomial theorem.

14.10 Applications

Power series yield exact representations, practical numerical approximations, and limits that may be difficult to obtain by direct algebra.

Example 14.44 (A limit)**.** Using the sine series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Therefore

$$\frac{\sin x - x}{x^3} = -\frac{1}{6} + \frac{x^2}{5!} - \cdots,$$

and hence

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

Example 14.45 (Euler's number from the exponential series)**.** Putting $x = 1$ in Theorem 14.35 gives

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

The tail can be bounded by a geometric series, producing rigorous rational approximations to e.

Example 14.46 (A power series for a rational function)**.** For $|x| < 1$,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Integrating gives the inverse-tangent series. Differentiating or multiplying by $1 + x^2$ verifies the identity inside the interval of convergence.

14.10.1 Abel's theorem

The term-by-term theorems above apply strictly inside the interval of convergence. At a boundary point, continuity of the sum is not automatic; the correct boundary result is Abel's theorem.

Theorem 14.47 (Abel's theorem)**.** *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

have radius of convergence R with $0 < R < \infty$, and suppose the series converges at $x = a+R$. Then

$$\lim_{x \to (a+R)^-} f(x) = \sum_{n=0}^{\infty} c_n R^n.$$

Proof. We reduce to a normalized case. First, the change of variable $x = a + Ry$ maps the interval $(a-R, a+R)$ to $(-1,1)$ and the endpoint $a+R$ to 1; the series becomes $\sum_n (c_n R^n) y^n$, whose radius is 1 and whose value at $y = 1$ is $\sum_n c_n R^n$. It therefore suffices to treat $a = 0$, $R = 1$. Second, let $S = \sum_{n=0}^{\infty} c_n$ (which converges by hypothesis) and replace c_0 by $c_0 - S$, leaving c_n for $n \geq 1$ unchanged. The new coefficients $\tilde{c}_n$ satisfy $\sum_{n=0}^{\infty} \tilde{c}_n = 0$, and for $|x| < 1$ the new series sums to $f(x) - S$. Proving the theorem in the normalized case $\sum_n c_n = 0$ then gives $f(x) - S \to 0$, that is $f(x) \to S$, as $x \to 1^-$.

So assume $a = 0$, $R = 1$, and $\sum_{n=0}^{\infty} c_n = 0$. Put

$$s_n = c_0 + \cdots + c_n,$$

so that $s_n \to 0$. Summation by parts gives, for every N and $|x| < 1$,

$$\sum_{k=0}^{N} c_k x^k = (1-x) \sum_{k=0}^{N-1} s_k x^k + s_N x^N.$$

Indeed, writing $c_k = s_k - s_{k-1}$ with $s_{-1} = 0$,

$$\sum_{k=0}^{N} c_k x^k = \sum_{k=0}^{N} (s_k - s_{k-1}) x^k = \sum_{k=0}^{N} s_k x^k - \sum_{k=0}^{N-1} s_k x^{k+1} = (1-x) \sum_{k=0}^{N-1} s_k x^k + s_N x^N.$$

Since (s_n) is bounded (it converges) and $\sum_n |x|^n$ converges for $|x| < 1$, the series $\sum_n s_n x^n$ converges absolutely for $|x| < 1$ by comparison (Theorem 13.8). Letting $N \to \infty$ in the identity, with $s_N x^N \to 0$, gives the Abel identity

$$f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n \qquad (|x| < 1).$$

Now let $\varepsilon > 0$. Since $s_n \to 0$, choose N such that $|s_n| < \varepsilon$ for all $n > N$. For $0 < x < 1$,

$$|f(x)| \leq (1-x) \sum_{n=0}^{\infty} |s_n|\, x^n = (1-x) \sum_{n=0}^{N} |s_n|\, x^n + (1-x) \sum_{n>N} |s_n|\, x^n \leq (1-x) C + \varepsilon (1-x) \sum_{n>N} x^n,$$

where $C = \sum_{n=0}^{N} |s_n|$. Since

$$(1-x) \sum_{n>N} x^n = (1-x) \frac{x^{N+1}}{1-x} = x^{N+1} < 1,$$

we obtain

$$|f(x)| \le (1-x)C + \varepsilon.$$

Letting $x \to 1^-$ gives $\limsup_{x\to 1^-} |f(x)| \le \varepsilon$; since $\varepsilon > 0$ was arbitrary, $|f(x)| \to 0$, that is $f(x) \to 0$ as $x \to 1^-$. This proves the normalized case, and hence the theorem. □

Example 14.48 (Leibniz's series for $\pi/4$)**.** The arctangent series (Theorem 14.38)

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (|x| < 1)$$

has radius of convergence 1. At $x = 1$ it becomes the alternating series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

which converges by the alternating-series theorem (Theorem 13.21). Abel's theorem gives

$$\lim_{x\to 1^-} \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Since arctan is continuous at 1 and $\arctan 1 = \pi/4$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Example 14.49 (Alternating harmonic series)**.** The logarithmic series (Theorem 14.37)

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad (|x| < 1)$$

has radius of convergence 1. At $x = 1$ the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the alternating-series theorem (Theorem 13.21). Abel's theorem, together with the continuity of log at 2, gives

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

Remark 14.50. Abel's theorem is the general mechanism behind the endpoint evaluation of the arctangent series given earlier in this chapter. In Theorem 14.38, the value at $x = 1$ was identified by an explicit remainder estimate—the exact-remainder formula of Theorem 11.31 and its consequence Corollary 11.32 in Chapter 11. Abel's theorem shows that no remainder computation is needed: once the endpoint series converges, its sum is forced to be the left-hand limit of the function, so that the endpoint value of the series is determined by the continuity of the function.

14.11 Exercises

Exercise 14.51. Find the radius and interval of convergence of each series. Test every endpoint separately:

$$\sum_{n=0}^{\infty} x^n, \qquad \sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Exercise 14.52. Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}, \qquad \sum_{n=0}^{\infty} n!\,(x+1)^n, \qquad \sum_{n=1}^{\infty} \frac{n^2}{3^n}(x-4)^n.$$

Exercise 14.53. Prove Theorem 14.4 directly using the comparison test for series.

Exercise 14.54. Give three power series with radius of convergence 1: one that converges at both endpoints, one that diverges at both endpoints, and one that converges at exactly one endpoint.

Exercise 14.55. Starting with the geometric series, derive a power-series representation for

$$\frac{1}{1+x}$$

and then integrate it to obtain a power series for $\log(1+x)$.

Exercise 14.56. Use term-by-term differentiation to show that, for $|x| < 1$,

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}.$$

Exercise 14.57. Use the preceding exercise to find a power series for

$$\frac{x}{(1-x)^2}.$$

Exercise 14.58. Prove that the exponential series has infinite radius of convergence using the ratio test.

Exercise 14.59. Use the exponential series to derive the hyperbolic series

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \qquad \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Exercise 14.60. Use the sine and cosine series to verify, by Cauchy products, the identity

$$\sin^2 x + \cos^2 x = 1.$$

Exercise 14.61. Use the arctangent series at $x = 1$ to give an alternating-series approximation to π. How many terms guarantee an error less than 10^{-4} in the resulting approximation to π?

Exercise 14.62. Find a power series centered at 0 for

$$\frac{x^2}{1+x^3},$$

and state its interval of convergence.

Exercise 14.63. Use a power series to evaluate

$$\lim_{x \to 0} \frac{\log(1+x) - x + x^2/2}{x^3}.$$

Exercise 14.64. Fill in the details of Example 14.33: prove by induction that for $x \neq 0$ the nth derivative of $f(x) = e^{-1/x^2}$ has the form $P_n(1/x)e^{-1/x^2}$ for some polynomial P_n, and verify that

$$\lim_{h \to 0} \frac{P_n(1/h)e^{-1/h^2}}{h} = 0,$$

so that $f^{(n)}(0) = 0$ for every n.

Exercise 14.65. Fill in the details of the proof of the Cauchy product theorem (Theorem 14.41): prove the product formula directly for polynomials by grouping the finite product by diagonals, and justify the tail estimate $\sum_{k+\ell > N} |a_k b_\ell| \to 0$.

Appendix A

A Quick Introduction to Sets

Set language provides a compact and precise way to discuss collections of mathematical objects. It has appeared throughout these notes in domains, ranges, intervals, partitions, and sets of upper or lower bounds. This appendix collects the basic notation and logical laws needed to read and write such statements carefully.

The emphasis is elementary but formal. A set is determined by its elements; equality of sets is proved by establishing two inclusions; and operations on sets obey algebraic identities that can be verified directly from the definitions.

A.1 Basic language and notation

A *set* is a collection of objects, called its *elements* or *members*. We write

$$x \in A$$

to mean that x is an element of the set A, and

$$x \notin A$$

to mean that x is not an element of A.

Sets may be specified by listing their elements:

$$A = \{1, 2, 3\},$$

or by giving a property that characterizes their elements:

$$B = \{x \in \mathbb{R} : x^2 < 4\}.$$

The colon is read "such that." Thus B is the set of real numbers x such that $x^2 < 4$.

Definition A.1. Two sets A and B are *equal*, written $A = B$, if they have exactly the same elements; that is,

$$A = B \quad \Longleftrightarrow \quad \text{for every } x, \quad x \in A \Longleftrightarrow x \in B.$$

The order in which elements are listed does not matter, and repeated listings do not create new elements. Thus

$$\{1, 2, 3\} = \{3, 2, 1\} = \{1, 1, 2, 3, 3\}.$$

Definition A.2. The *empty set*, denoted by

$$\varnothing,$$

is the set with no elements.

Definition A.3. Let A and B be sets. We say that A is a *subset* of B, written

$$A \subseteq B,$$

if every element of A belongs to B:

$$A \subseteq B \quad \Longleftrightarrow \quad \forall x\,(x \in A \Longrightarrow x \in B).$$

If $A \subseteq B$ and $A \neq B$, then A is a *proper subset* of B, sometimes written $A \subsetneq B$.

Proposition A.4. *For sets A and B,*

$$A = B \quad \Longleftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

Proof. If $A = B$, then every element of A belongs to B, and every element of B belongs to A, so both inclusions hold.

Conversely, suppose $A \subseteq B$ and $B \subseteq A$. For every object x,

$$x \in A \Longrightarrow x \in B$$

and

$$x \in B \Longrightarrow x \in A.$$

Thus $x \in A$ if and only if $x \in B$. Therefore $A = B$ by the definition of set equality. □

Proposition A.5. *For every set A,*

$$\varnothing \subseteq A.$$

Proof. To prove $\varnothing \subseteq A$, we must show that every element of $\varnothing$ belongs to A. There are no elements of $\varnothing$, so there is no possible counterexample. Equivalently, for every x, the implication

$$x \in \varnothing \Longrightarrow x \in A$$

has a false premise and is therefore true. □

A.2 Standard sets of numbers

The number systems used throughout these notes are denoted by

$$\mathbb{N} = \{1, 2, 3, \dots\},$$

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\},$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\},$$

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0 \right\},$$

and

$$\mathbb{R} = \text{the set of real numbers.}$$

They satisfy the inclusions

$$\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Definition A.6. For $a, b \in \mathbb{R}$ with $a < b$, the standard intervals are

$$(a, b) = \{x \in \mathbb{R} : a < x < b\},$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

and

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

Similarly,

$$(a, \infty) = \{x \in \mathbb{R} : x > a\}, \qquad (-\infty, b) = \{x \in \mathbb{R} : x < b\}.$$

The symbols $\pm\infty$ do not represent real numbers. They are notation indicating that an interval is unbounded in the indicated direction.

A.3 Operations on sets

Let A and B be sets.

Definition A.7. The *union* of A and B is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The *intersection* of A and B is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The *set difference* is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

The union consists of elements belonging to at least one of the two sets. The intersection consists of elements belonging to both. The difference $A \setminus B$ retains the elements of A after those also belonging to B are removed.

Example A.8. Let

$$A = [-1, 2), \qquad B = (0, 3].$$

Then

$$A \cap B = (0, 2),$$

$$A \cup B = [-1, 3],$$

and

$$A \setminus B = [-1, 0].$$

Proposition A.9 (Basic union and intersection laws)**.** *For all sets A, B, C,*

$$\begin{aligned}
A \cup B &= B \cup A, \\
A \cap B &= B \cap A, \\
(A \cup B) \cup C &= A \cup (B \cup C), \\
(A \cap B) \cap C &= A \cap (B \cap C), \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C).
\end{aligned}$$

Proof. Each identity follows from Proposition A.4, or equivalently by checking membership of an arbitrary element x.

For example,

$$\begin{aligned} x \in A \cap (B \cup C) &\iff (x \in A) \text{ and } (x \in B \text{ or } x \in C) \\ &\iff ((x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)) \\ &\iff x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Thus

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

The other identities are proved by the same membership method. □

Proposition A.10 (Difference identities)**.** *For all sets A, B, C,*

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C),$$

and

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Proof. For the first identity,

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff x \in A \text{ and } x \notin B \cup C \\ &\iff x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\iff x \in (A \setminus B) \cap (A \setminus C). \end{aligned}$$

The second identity is analogous. □

A.4 Universal sets and complements

Complements are defined relative to a designated ambient set.

Definition A.11. Let U be a set, called the *universal set*, and let $A \subseteq U$. The *complement of A relative to U* is

$$A^c = U \setminus A.$$

The same set can have different complements in different universal sets. For example, if $A = (0, 1)$, then its complement is different when the ambient set is $\mathbb{R}$ than when it is $[0, 2]$.

Theorem A.12 (De Morgan's laws)**.** *Let $A, B \subseteq U$. Then*

$$(A \cup B)^c = A^c \cap B^c,$$

and

$$(A \cap B)^c = A^c \cup B^c.$$

Proof. For $x \in U$,

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \notin A \text{ and } x \notin B \\ &\iff x \in A^c \cap B^c. \end{aligned}$$

This proves the first identity. Similarly,

$$\begin{aligned} x \in (A \cap B)^c &\iff x \notin A \cap B \\ &\iff x \notin A \text{ or } x \notin B \\ &\iff x \in A^c \cup B^c. \end{aligned}$$

□

Proposition A.13. *Let $A \subseteq U$. Then*

$$(A^c)^c = A, \qquad A \cup A^c = U, \qquad A \cap A^c = \varnothing.$$

Proof. Each statement follows immediately by checking whether an arbitrary $x \in U$ belongs to A or does not belong to A. □

A.5 Families of sets

The union and intersection of two sets extend to arbitrary indexed collections.

Definition A.14. Let I be a set, and suppose that for each $i \in I$ there is a set A_i. The *union* and *intersection* of the family $(A_i)_{i\in I}$ are

$$\bigcup_{i\in I} A_i = \{x : \text{there is } i \in I \text{ with } x \in A_i\},$$

and

$$\bigcap_{i\in I} A_i = \{x : \text{for every } i \in I,\ x \in A_i\}.$$

Remark A.15. The indexed intersection above is meaningful only for a nonempty index family. If $I = \varnothing$, the condition "for every $i \in I$, $x \in A_i$" holds vacuously for every object x, so the displayed definition would describe the class of all objects rather than a set. Throughout these notes, indexed intersections $\bigcap_{i\in I} A_i$ are therefore considered only for nonempty I; equivalently, whenever an empty intersection is needed it is defined relative to a specified ambient universe U by the convention

$$\bigcap_{i\in\varnothing} A_i = U,$$

the ambient-set convention used for complements below. The family De Morgan laws (Theorem A.17) use this same convention.

Example A.16. For $n \in \mathbb{N}$, let

$$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Then

$$\bigcap_{n=1}^{\infty} A_n = \{0\},$$

because 0 belongs to every A_n, while every nonzero real number is excluded from A_n for sufficiently large n.

On the other hand,

$$\bigcup_{n=1}^{\infty} A_n = (-1, 1),$$

because $A_1 = (-1, 1)$ already contains every later A_n.

Theorem A.17 (De Morgan's laws for families)**.** *Let $(A_i)_{i\in I}$ be a family of subsets of a universal set U, with the index set I nonempty (or with empty intersections interpreted relative to U, per the convention above). Then*

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c,$$

and

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Proof. For $x \in U$,

$$\begin{aligned} x \in \left(\bigcup_{i\in I} A_i\right)^c &\iff \text{there is no } i \in I \text{ for which } x \in A_i \\ &\iff \text{for every } i \in I,\ x \notin A_i \\ &\iff x \in \bigcap_{i\in I} A_i^c. \end{aligned}$$

The second identity follows similarly. □

A.6 Cartesian products and relations

Definition A.18. For sets A and B, the *Cartesian product* of A and B is

$$A \times B = \{(a, b) : a \in A,\ b \in B\}.$$

The order of the coordinates matters: generally,

$$(a, b) \neq (b, a).$$

Consequently, $A \times B$ need not equal $B \times A$.

A function $f : A \to B$ may be regarded as a special subset of $A \times B$, namely its graph:

$$\text{graph}(f) = \{(x, f(x)) : x \in A\}.$$

The defining property of a function is that each $x \in A$ occurs exactly once as a first coordinate of a pair in this graph.

Definition A.19. A *relation* from A to B is any subset of $A \times B$. A relation is a function from A to B precisely when every $a \in A$ appears as the first coordinate of exactly one pair in the relation.

Example A.20. The subset

$$R = \{(x, y) \in \mathbb{R}^2 : y^2 = x\}$$

is a relation from $\mathbb{R}$ to $\mathbb{R}$, but not a function from $\mathbb{R}$ to $\mathbb{R}$: negative x have no associated real y, while positive x have two associated values $y = \pm\sqrt{x}$.

A.7 Images and inverse images

Definition A.21. Let $f : A \to B$ be a function.

- If $E \subseteq A$, the *image* of E under f is
$$f(E) = \{f(x) : x \in E\} \subseteq B.$$
- If $F \subseteq B$, the *inverse image* or *preimage* of F under f is
$$f^{-1}(F) = \{x \in A : f(x) \in F\} \subseteq A.$$

The notation $f^{-1}(F)$ for a preimage is meaningful even when f is not invertible. Its meaning is set-theoretic: it denotes all inputs sent into F.

Proposition A.22. *Let $f : A \to B$, and let $E, E_1, E_2 \subseteq A$ and $F, F_1, F_2 \subseteq B$. Then*

$$f(E_1 \cup E_2) = f(E_1) \cup f(E_2),$$

$$f^{-1}(F_1 \cup F_2) = f^{-1}(F_1) \cup f^{-1}(F_2),$$

and

$$f^{-1}(F_1 \cap F_2) = f^{-1}(F_1) \cap f^{-1}(F_2).$$

Moreover, if complements are taken relative to A and B, respectively, then

$$f^{-1}(F^c) = \big(f^{-1}(F)\big)^c.$$

Proof. For instance,

$$\begin{aligned} x \in f^{-1}(F_1 \cap F_2) &\iff f(x) \in F_1 \cap F_2 \\ &\iff f(x) \in F_1 \text{ and } f(x) \in F_2 \\ &\iff x \in f^{-1}(F_1) \cap f^{-1}(F_2). \end{aligned}$$

The other identities follow by the same direct membership argument. □

Remark A.23. Images do not generally preserve intersections. One always has

$$f(E_1 \cap E_2) \subseteq f(E_1) \cap f(E_2),$$

but equality can fail when f is not injective. For example, with $f(x) = x^2$ on $\mathbb{R}$, let $E_1 = \{-1\}$ and $E_2 = \{1\}$. Then

$$E_1 \cap E_2 = \varnothing,$$

while

$$f(E_1) \cap f(E_2) = \{1\}.$$

A.8 Countability

Definition A.24. A set A is *finite* if it has exactly n elements for some $n \in \mathbb{N}_0$. A set is *countably infinite* if there is a bijection

$$f : \mathbb{N} \to A.$$

A set is *countable* if it is finite or countably infinite. A set that is not countable is *uncountable.*

The definition of countably infinite means that the elements of the set can be arranged in a sequence, even if the arrangement is not the usual numerical order.

Example A.25. The integers are countable. One possible listing is

$$0, 1, -1, 2, -2, 3, -3, \ldots.$$

This listing defines a bijection from $\mathbb{N}_0$ to $\mathbb{Z}$.

Theorem A.26. *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof. Every rational number can be written as p/q with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Thus there is a surjection

$$\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, \qquad (p, q) \mapsto \frac{p}{q}.$$

The set $\mathbb{Z} \times \mathbb{N}$ is countable: arrange its elements by increasing values of $|p| + q$, and within each finite level use any fixed order. This produces a sequence containing every pair (p, q).

A surjective image of a countable set is countable: enumerate the source set, and retain the first occurrence of each new image value. Applying this observation to the displayed surjection proves that $\mathbb{Q}$ is countable. □

Theorem A.27 (Cantor's diagonal theorem)**.** *The interval* $(0, 1)$*, and hence the set* $\mathbb{R}$*, is uncountable.*

Proof. Suppose, for contradiction, that $(0, 1)$ is countable. Then its elements can be listed as

$$x_1, x_2, x_3, \ldots.$$

Choose for each x_n a decimal expansion that does not end in an infinite string of 9s, and write

$$x_n = 0.d_{n1}d_{n2}d_{n3}\ldots.$$

Construct a number

$$y = 0.e_1e_2e_3\ldots$$

by choosing e_n to be 1 if $d_{nn} \neq 1$, and 2 if $d_{nn} = 1$. Then $y \in (0, 1)$, and y differs from x_n in its nth decimal digit. Hence

$$y \neq x_n \qquad \text{for every } n.$$

This contradicts the assumption that the list contained every element of $(0, 1)$. Therefore $(0, 1)$ is uncountable. □

Remark A.28. The countability of $\mathbb{Q}$ and uncountability of $\mathbb{R}$ are compatible with the fact that both sets are dense in the real line. Density concerns the existence of points in every interval; countability concerns whether all elements can be listed in a sequence.

A.9 Exercises

Exercise A.29. Let

$$A = \{x \in \mathbb{R} : x^2 < 9\}.$$

Write A in interval notation.

Exercise A.30. Prove that $A \subseteq B$ if and only if $A \cap B = A$.

Exercise A.31. Prove that $A \subseteq B$ if and only if $A \cup B = B$.

Exercise A.32. Let

$$A = (-2, 1], \qquad B = [0, 3).$$

Compute $A \cup B$, $A \cap B$, $A \setminus B$, and $B \setminus A$.

Exercise A.33. Prove each of the identities in Proposition A.9 by the elementwise method used in its proof.

Exercise A.34. Prove De Morgan's laws for three sets:

$$(A \cup B \cup C)^c = A^c \cap B^c \cap C^c,$$

$$(A \cap B \cap C)^c = A^c \cup B^c \cup C^c.$$

Exercise A.35. For $n \in \mathbb{N}$, let

$$A_n = \left(-1, \frac{1}{n}\right).$$

Find

$$\bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} A_n.$$

Exercise A.36. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Compute the images and inverse images requested below:

$$f([-2, 1]), \qquad f^{-1}([1, 4]), \qquad f^{-1}((1, 4)).$$

Exercise A.37. Prove that if $f : A \to B$ is injective, then

$$f(E_1 \cap E_2) = f(E_1) \cap f(E_2)$$

for all $E_1, E_2 \subseteq A$.

Exercise A.38. Give an example of a function $f : A \to B$ and sets $E_1, E_2 \subseteq A$ for which

$$f(E_1 \cap E_2) \neq f(E_1) \cap f(E_2).$$

Exercise A.39. Construct an explicit bijection from $\mathbb{N}$ to the set of even positive integers.

Exercise A.40. Show that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countable by arranging pairs according to the value of $m + n$.

Exercise A.41. Explain why the decimal-expansion convention in the proof of Theorem A.27 is necessary. What ambiguity would arise if terminating expansions and expansions ending in infinitely many 9s were both allowed without a convention?

Exercise A.42. Prove that every nonempty finite subset of $\mathbb{R}$ has both a largest and a smallest element.

Exercise A.43. Let $(A_n)_{n=1}^{\infty}$ be a decreasing family of sets, meaning $A_{n+1} \subseteq A_n$ for every n. Prove that

$$\bigcap_{n=1}^{\infty} A_n \subseteq A_N$$

for every $N \in \mathbb{N}$.