ESTR 1007: Single-Variable Calculus

Homework 1: Axioms of the Real Numbers

Calculus

Instructions. Give complete proofs. This assignment is an exercise in using the field and order axioms of $\mathbb{R}$ carefully. When a step uses an axiom or a previously established fact, say so explicitly. In particular, identify uses of associativity, commutativity, distributivity, additive inverses, multiplicative inverses, and the order axioms. You may use a result from an earlier problem only when you cite it.

Please acknowledge any use of AI tools or assistance from other people.

1. Prove that, for every real number a,

$$a \cdot 0 = 0 \qquad \text{and} \qquad 0 \cdot a = 0.$$

2. Demonstrate that

$$(-a)(-b) = ab.$$

3. Prove that if $xy = 0$, then $x = 0$ or $y = 0$.

4. Prove that the product of two negative numbers is positive.

5. Prove that

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Indicate explicitly where distributivity and commutativity are used.

6. Show that

$$|ab| = |a||b|.$$

7. Prove the triangle inequality:

$$|a+b| \le |a| + |b|.$$

8. Prove the binomial theorem: for every nonnegative integer n,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.$$

Your proof should use mathematical induction. Indicate explicitly where you use the distributive law, associativity and commutativity of addition and multiplication, and Pascal's identity.

9. Show that if $x < y$ and n is an odd positive integer, then

$$x^n < y^n.$$

10. Use Problem 9 to show that if x and y are not both zero, then

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0.$$

11. If $|x - a| < \epsilon$ for all $\epsilon > 0$, prove that $x = a$.

12. Prove that, for $\epsilon > 0$, if

$$|x - x_0| \leq \frac{\epsilon}{2|y_0| + \sqrt{\epsilon}} \qquad \text{and} \qquad |y - y_0| \leq \frac{\epsilon}{2|x_0| + \sqrt{\epsilon}},$$

then

$$|xy - x_0y_0| \leq \epsilon.$$

13. Prove that if $y_0 \neq 0$, $\epsilon > 0$, and

$$|y - y_0| < |y_0| \frac{\epsilon |y_0|}{1 + \epsilon |y_0|},$$

then $y \neq 0$ and

$$\left| \frac{1}{y} - \frac{1}{y_0} \right| < \epsilon.$$

End of Homework 1