ESTR 1007: Single-Variable Calculus

Homework 2: Functions

Due: _____ *Calculus*

Instructions. Give complete proofs and state clearly the domains on which functions and identities are being considered. When an assertion about functions is false, give a counterexample and explicitly identify an input at which the two sides differ, or explain why their domains differ. You may use results established in Chapter 2, provided that you cite them. In questions involving even and odd functions, assume that every function under discussion has a domain symmetric about 0.

Please acknowledge any use of AI tools or assistance from other people.

1. Let $f, g, h : \mathbb{R} \to \mathbb{R}$. For each assertion below, prove it if it is true for all such functions, or give a counterexample if it is false.

(a) $f \circ (g + h) = f \circ g + f \circ h$.

(b) $(g + h) \circ f = g \circ f + h \circ f$.

(c) $1/(f \circ g) = (1/f) \circ g$, wherever both sides are defined.

(d) $1/(f \circ g) = f \circ (1/g)$, wherever the expressions make sense.

2. Let $f : A \to B$ and $g : A \to C$ be functions.

(a) Suppose that $g = h \circ f$ for some function $h : f(A) \to C$. Prove that

$$f(x) = f(y) \quad \Longrightarrow \quad g(x) = g(y)$$

for all $x, y \in A$.

(b) Conversely, suppose that

$$f(x) = f(y) \quad \Longrightarrow \quad g(x) = g(y)$$

for all $x, y \in A$. Prove that there is a function $h : f(A) \to C$ such that

$$g = h \circ f.$$

3. A function f is called *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$, for every x in its domain.

(a) Complete the following table for sums. Enter "even," "odd," or "neither in general." Give proofs for the entries that are always true and counterexamples for the entries marked "neither in general."

f	g	$f + g$
even	even	
even	odd	
odd	even	
odd	odd	

(b) Complete the analogous table for products fg, with proof.

(c) Complete the analogous table for compositions $f \circ g$, assuming that all indicated compositions are defined.

(d) Prove that every even function $f : \mathbb{R} \to \mathbb{R}$ can be written in the form

$$f(x) = u(|x|)$$

for some function $u : [0, \infty) \to \mathbb{R}$. Explain why there are infinitely many functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) = g(|x|) \qquad \text{for every } x \in \mathbb{R}.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary. Define

$$E(x) := \frac{f(x) + f(-x)}{2}, \qquad O(x) := \frac{f(x) - f(-x)}{2}.$$

(a) Prove that E is even, O is odd, and

$$f = E + O.$$

(b) Prove that this decomposition is unique: if

$$f = E_1 + O_1,$$

where E_1 is even and O_1 is odd, then $E_1 = E$ and $O_1 = O$.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x + y) = f(x) + f(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

(a) Prove that, for every positive integer n,

$$f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n).$$

(b) Let $c = f(1)$. Prove that

$$f(q) = cq \qquad \text{for every } q \in \mathbb{Q}.$$

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is not identically zero and satisfies

$$f(x + y) = f(x) + f(y) \qquad \text{and} \qquad f(xy) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$. Prove that

$$f(x) = x \qquad \text{for every } x \in \mathbb{R}.$$

You may use the density of $\mathbb{Q}$ in $\mathbb{R}$.

End of Homework 2