ESTR 1007: Single-Variable Calculus

Homework 3: Limits

Due: 30th September 2026 Calculus

Instructions. Give complete solutions. For a limit calculation, show the algebraic simplification and cite the limit laws that you use. For a proof from the definition of limit, specify how δ depends on ε, and verify all required inequalities. You may use the limit laws and results proved in the limits chapter, provided that you cite them.

Please acknowledge any use of AI tools or assistance from other people.

1. Evaluate the following limits. In each case, justify every algebraic step that changes the displayed expression.

(a) $\displaystyle\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$.

(b) $\displaystyle\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x}$.

(c) $\displaystyle\lim_{x\to 0} \frac{1-\sqrt{1-x}}{x}$.

(d) $\displaystyle\lim_{x\to \infty} \left(\sqrt{x^2+50x}-x\right)$.

(e) $\displaystyle\lim_{x\to \infty} \frac{3x^3-2x+1}{5x^3+x^2-7}$.

2. Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

and

$$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0,$$

where $a_n \neq 0$, $b_m \neq 0$, and m, n are nonnegative integers. Determine

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

in each of the three cases $n < m$, $n = m$, and $n > m$. Give a proof using the limit laws.

3. Define

$$f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ -x, & x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that $\lim_{x\to 0} f(x) = 0$.

(b) Prove that $\lim_{x\to a} f(x)$ does not exist for every $a \neq 0$. You may use the density of $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$.

4. Let f be defined on (A, ∞), and suppose that

$$\lim_{x\to\infty} f(x) = 0.$$

Let g be defined on (A, ∞) and suppose that there is a constant $M \geq 0$ such that

$$|g(x)| \leq M$$

for all sufficiently large x. Prove that

$$\lim_{x\to\infty} f(x)g(x) = 0.$$

5. Suppose that

$$\lim_{x\to\infty} f(x) = a \qquad \text{and} \qquad \lim_{x\to\infty} g(x) = b.$$

Prove that

$$\lim_{x\to\infty} \max\{f(x), g(x)\} = \max\{a, b\}.$$

6. Let f and g be functions that are defined for all x satisfying

$$0 < |x - a| < r$$

for some $r > 0$.

(a) Suppose that

$$f(x) \leq g(x)$$

whenever

$$0 < |x - a| < r,$$

and that

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

Prove that $L \leq M$.

(b) Suppose instead that

$$f(x) < g(x)$$

whenever

$$0 < |x - a| < r.$$

Must $L < M$? Prove the assertion or give a counterexample.

7. Suppose that both

$$\lim_{x\to a^+} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = M$$

exist. Prove that

$$L = M.$$

End of Homework 3