# ESTR 1007: Single-Variable Calculus

## Homework 4: Continuity

*Due: 14th October 2026 Calculus*

**Instructions.** Give complete solutions. Unless a problem explicitly allows a theorem to be cited, justify each use of the definition of continuity or of a limit theorem. In an $\varepsilon$-$\delta$ proof, specify the choice of $\delta$ and verify the required inequalities. You may use results proved in Chapter 4, provided that you state clearly which result is being used.

Please acknowledge any use of AI tools or assistance from other people.

**1.** For each function below, determine every point at which it is continuous. At each point of discontinuity, state whether the discontinuity is removable, a jump discontinuity, or neither; if it is removable, give a value at which the function can be defined to make it continuous.

(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ for $x \neq 1$.

(b) $g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$

(c) $h(x) = \begin{cases} x^2, & x < 1, \\ 2 - x, & x \geq 1. \end{cases}$

(d) $k(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$

**2.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, and suppose that $f$ is continuous at 0. Prove that $f$ is continuous at every $a \in \mathbb{R}$. Your proof must use the $\varepsilon$-$\delta$ definition of continuity.

**3.** Let $g$ be defined in a deleted neighbourhood of $a$, and suppose that

$$\lim_{x \to a} g(x) = \ell.$$

Let $f$ be continuous at $\ell$.

(a) Prove directly from the definitions that

$$\lim_{x \to a} f(g(x)) = f(\ell).$$

(b) Give an example showing that the conclusion need not hold if $f$ is not continuous at $\ell$.

**4.** Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that

$$f(a) < 0 < f(b).$$

Prove that there exists $c \in (a, b)$ such that $f(c) = 0$.

You may use the least-upper-bound property of $\mathbb{R}$, but not the Intermediate Value Theorem. Define

$$S = \{x \in [a, b] : f(x) < 0\}, \qquad c = \sup S,$$

and prove that $f(c) = 0$.

**5.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that there is no such function which takes every real value exactly twice.

*Hint.* If $f(a) = f(b)$ with $a < b$, show first that either $f(x) > f(a)$ for every $x \in (a, b)$, or $f(x) < f(a)$ for every $x \in (a, b)$. Then use the Intermediate Value Theorem.

*End of Homework 4*