ESTR 1007: Single-Variable Calculus

Homework 5: Differentiation

Due: To be announced Calculus

Instructions. Give complete solutions. In Question 1 use the definition of the derivative, not differentiation rules. Elsewhere you may use results from Chapters 1–5, provided that you identify the rules or theorems used. The formulas for the derivatives of sine, cosine, and the exponential function may be used. Do not use the Mean Value Theorem, L'Hôpital's rule, or results from later chapters. Please acknowledge any use of AI tools or assistance from other people.

1. **Derivatives from the definition.** Use difference quotients to compute the following derivatives. State the permitted values of a.

   (a) $f'(a)$ for $f(x) = x^3$.

   (b) $g'(a)$ for $g(x) = \sqrt{x}$, where $a > 0$.

   (c) $h'(a)$ for $h(x) = 1/x$.

2. **Differentiation rules.** Differentiate each function on its domain, showing the intermediate steps or clearly indicating the rules used.

   (a) $f(x) = \sin((x+1)^2(x+2))$.

   (b) $f(x) = \sin^3(x^2 + \sin x)$.

   (c) $f(x) = \sin\left(\dfrac{x^3}{\cos(x^3)}\right)$.

   (d) $f(x) = \big((x^2+x)^3 + x\big)^4$.

   (e) $f(x) = \dfrac{\sin(x^2)\sin^2 x}{1+\sin x}$.

3. **Joining differentiable functions.** Let

$$f(x) = \begin{cases} x^2, & x < 1, \\ Ax + B, & x \geq 1. \end{cases}$$

   (a) Determine all pairs (A, B) for which f is continuous at 1.

   (b) Determine all pairs for which f is differentiable at 1.

   (c) For the differentiable choice, determine whether f' is continuous at 1, and whether $f''(1)$ exists.

4. Oscillation and linear approximation.

(a) Define $F(0) = 0$ and $F(x) = x^2 \sin(1/x)$ for $x \neq 0$. Prove from the definition that $F'(0)$ exists. Compute $F'(x)$ for $x \neq 0$, and prove that F' is not continuous at 0. Give an argument that does not invoke a theorem about sequences.

(b) Let f be differentiable at a. Prove that its linearization $L(x) = f(a) + f'(a)(x-a)$ satisfies

$$\lim_{x \to a} \frac{f(x) - L(x)}{x - a} = 0.$$

5. Higher derivatives: Leibniz's formula. Let $n \geq 1$, and suppose that f and g are n times differentiable on an open interval I. Set $f^{(0)} = f$ and $g^{(0)} = g$.

(a) Prove by induction that, for every $x \in I$,

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

(b) Use this formula to compute the n-th derivative of $x^2 e^x$ for every integer $n \geq 1$.

6. Challenge: Absolute values and differentiability. Suppose f is differentiable at a, and let $G(x) = |f(x)|$.

(a) If $f(a) \neq 0$, prove that G is differentiable at a and find $G'(a)$.

(b) If $f(a) = 0$, prove that G is differentiable at a if and only if $f'(a) = 0$. Find $G'(a)$ whenever it exists.

(c) A student claims that existence of

$$\lim_{h \to 0} \frac{G(a+h) - G(a-h)}{2h}$$

implies differentiability at a. Disprove the claim using a function of the form $G = |f|$, with f differentiable everywhere.

End of Homework 5