ESTR 1007: Single-Variable Calculus

Homework 6: Applications of the Derivative

Due: 21st, Oct, 2026 Calculus

Instructions. Give complete solutions, and identify the theorems you use (for example the Extreme Value Theorem, the Mean Value Theorem, L'Hôpital's rule, or a convexity criterion). You may use results from Chapters 1–6, including the derivative formulas for the exponential, logarithmic, and trigonometric functions. Do not use Taylor polynomials, series, or results from later chapters. Material marked ‡ in the notes is not needed. Please acknowledge any use of AI tools or assistance from other people.

1. Optimization on an unbounded domain. Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$, and justify that your point really does give the global minimum distance.

2. Rolle's theorem and the Mean Value Theorem.

(a) Suppose that f is twice differentiable on $[a, b]$ and that f has three distinct zeros in $[a, b]$. Prove that f'' has a zero in (a, b).

(b) Prove that if f is differentiable on an interval I and $|f'(x)| \le M$ for all $x \in I$, then

$$|f(x) - f(y)| \le M|x - y| \qquad \text{for all } x, y \in I.$$

(c) Prove that $e^x > 1 + x$ for every $x \neq 0$.

3. L'Hôpital's rule. Evaluate each limit, verifying the hypotheses of the rule at each step.

(a) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2}$.

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

(c) $\displaystyle\lim_{x\to \infty} \frac{\log x}{\sqrt{x}}$.

(d) $\displaystyle\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}$.

In part (d), explain carefully why L'Hôpital's rule cannot be applied, and evaluate the limit by another method.

4. Roots of polynomials (adapted from Spivak). Let $n \ge 1$, and let $f(x) = x^n + ax + b$.

(a) Prove that f has at most two real roots when n is even.

(b) Prove that f has at most three real roots when n is odd.

Hint. Between two zeros of a differentiable function there is a zero of its derivative.

5. Convexity and Jensen's inequality.

(a) Prove directly from the definition that $f(x) = x^2$ is strictly convex on $\mathbb{R}$.

(b) Use convexity to prove that $e^x \geq 1 + x$ for every $x \in \mathbb{R}$, with equality only at $x = 0$.

(c) Let $x_1, \ldots, x_n$ be real numbers. Prove that

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^2 \leq \frac{x_1^2 + \cdots + x_n^2}{n},$$

with equality if and only if $x_1 = \cdots = x_n$. Interpret this inequality in words.

6. The higher-order derivative test. Let

$$f(x) = (x-1)^6 - (x-1)^4.$$

Find all critical points of f, and classify each as a local maximum, a local minimum, or neither. At the point(s) where the second-derivative test is inconclusive, use the higher-order derivative test. Then sketch the graph of f.

7. Newton's method.

(a) Write down the Newton iteration for solving $x^2 = a$, where $a > 0$, and show that it takes the form

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Use it with $x_0 = 2$ to approximate $\sqrt{5}$, computing x_1, x_2, x_3 and comparing with $\sqrt{5} \approx 2.236067977$.

(b) Let $a \neq 0$. Show that Newton's method applied to $f(x) = \frac{1}{x} - a$ produces a sequence converging to $1/a$ using only multiplication and subtraction (no division). What is the iteration formula?

8. Interior minima. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and suppose the one-sided derivatives at the endpoints exist with

$$f'(a) < 0 < f'(b).$$

Prove that f attains its minimum on $[a, b]$ at an interior point, and conclude that there exists $c \in (a, b)$ with

$$f'(c) = 0.$$

9. Boudns for binary entropy The binary entropy function is defined by

$$H_2(x) = -x \log_2 x - (1 - x) \log_2(1 - x).$$

Prove that, for every $x \in (0, \frac{1}{2})$,

$$4x(1-x) \leq H_2(x) \leq 2\sqrt{x(1-x)}.$$

End of Homework 6