Final Exam: IERG 6154

April 23, 2021

Instructions

- The final exam is due: April 25th, 11:59 P.M.
- Please provide complete reasoning for your solutions.
- You are allowed to consult the textbook and lectures notes only.
- Please use your time wisely (do not spend too much time on one question).
- Some questions are relatively easier than others.
- All questions are worth 20 points.

Questions

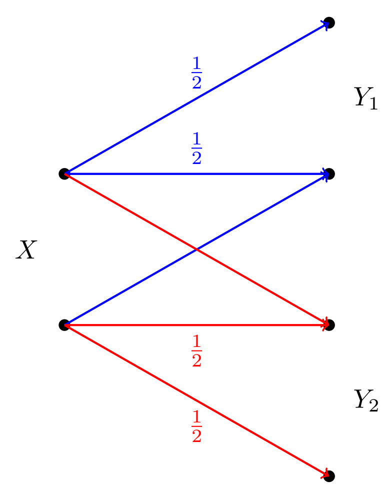
1. Let $X_1, .., X_n \sim p(x^n)$, where each X_i takes value in a finite set $\mathcal{X}$. Define the following function:

$$f(k) = \frac{\sum_{S \subseteq [1:n] : |S| = k} H(X_S)}{\binom{n}{k}}, \quad 1 \leq k \leq n.$$

   Show the following:

   1. $f(k)$ is increasing.
   2. $\frac{f(k)}{k}$ is decreasing.
   3. $f(k) - f(k-1) \geq f(k+1) - f(k), 2 \leq k \leq n-1$

2. A binary skew-symmetric broadcast channel is characterized by the following transition diagram.



Find the private-message capacity region for the broadcast channel subject to the constraint that the normalized hamming weights of the codewords must be at most 0.2, i.e. fraction of each 1s in each codeword must not exceed $\frac{1}{5}$. (Hint: Compute the concave envelope of $I(X;Y_2) - I(X;Y_1)$ explicitly to determine the significance of $P(X=1) = \frac{1}{5}$.)

3. Consider a physically degraded state-dependent Gaussian broadcast channel characterized by

$$Y_1 = X + S + Z_1$$
$$Y_2 = Y_1 + Z_2.$$

Here $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z_2 \sim (0, N_2)$. The state sequence $S^n \sim \prod_i p_S(s_i)$, and is known non-causally at the encoder. The decoder has no state information. As usual, there is a power constraint of P on X. Find the capacity region for this setting.

4. An discrete-memoryless interference channel is said to have very strong interference if

$$I(X_1;Y_2) \geq I(X_1;Y_1|X_2)$$
$$I(X_2;Y_1) \geq I(X_2;Y_2|X_1)$$

for all $p_1(x_1)p_2(x_2)$. Now consider a scalar Gaussian interference channel defined as

$$Y_1 = X_1 + bX_2 + Z_1, \quad Y_2 = X_2 + aX_1 + Z_2$$

where Z_1, Z_2 are standard Gaussians. Given power constraints P_1 and P_2 for X_1 and X_2 respectively, let us say that such a channel satisfies very strong interference condition if

$$I(X_1;Y_2|Q) \geq I(X_1;Y_1|X_2,Q)$$
$$I(X_2;Y_1|Q) \geq I(X_2;Y_2|X_1,Q)$$

for all $p_Q p_{X_1|Q} p_{X_2|Q}$ where X_1 and X_2 satisfy the power constraints.
Prove that the above condition for very strong interference never holds for any non-zero parameters P_1, P_2.

5. Consider the interference channel shown in the figure. The output $Y_2 = X_2$, so it belongs to clean Z-interference channels. Let C_1 denote the maximum rate that can be decoded by receiver Y_1, i.e. the rate pair $(C_1, 0)$ is an extreme point of the capacity region. Determine a finite λ that

$$\max_{(R_1,R_2)\in\mathcal{C}} \lambda R_1 + R_2 = \lambda C_1.$$

Or in other words the supporting hyperplane of the form $\lambda R_1 + R_2$ passes through the extreme point on the axis. (Hint: This might require some computer based calculation. Based on my previous calculations $\lambda \geq 12$ should work, but I guess takine $\lambda = 200$ should yield you a much easier situation to prove. Final hint: Etkin-Ordentlich outer bound may be helpful.).