Final Exam: IERG 6154

May 18, 2023

Instructions

- The final exam is due: May 19th, 11:59 P.M.
- Please provide complete reasoning for your solutions.
- You are allowed to consult the textbook and lectures notes only.
- Please use your time wisely (do not spend too much time on one question).
- Some questions are relatively easier than others.
- All questions are worth 20 points.

Questions

1. (Hint: Last Question of Homework 1) Prove the following inequalities:
   (a) (10 points) Let $0 \leq \alpha \leq 1$, and $0 \leq \epsilon_i \leq 1$, for $i = 1, .., n$. Define $\hat{\epsilon}_S = \prod_{i \notin S} \epsilon_i$. Let $H_2(x) := -x \log(x) - (1-x)\log(1-x)$. Let β_S, $S \subseteq [1:n]$ be a fractional partition. Show that
   $$\sum_{S \subseteq [1:n]} \beta_S H_2\left(\frac{1-\alpha\hat{\epsilon}_S}{2}\right) \geq \left(\sum_{S \subseteq [1:n]} \beta_S - 1\right) H_2\left(\frac{1-\alpha\hat{\epsilon}_\emptyset}{2}\right) + H_2\left(\frac{1-\alpha}{2}\right)$$
   (b) (2 points) Using the above part, argue that
   $$\sum_{S \subseteq [1:n]} \beta_S \left(\hat{\epsilon}_S\right)^2 \leq \left(\sum_{S \subseteq [1:n]} \beta_S - 1\right) \left(\hat{\epsilon}_\emptyset\right)^2 + 1$$
   (c) (8 points) An exponential random variable with parameter λ has a probability density given by $\lambda e^{-\lambda x}, x \geq 0$, and is denoted by E_λ. Let $\lambda, \lambda_1, ..., \lambda_n$ be positive real numbers. Show that
   $$\sum_{S \subseteq [1:n]} \beta_S h\left(E_{\lambda + \sum_{i \notin S} \lambda_i}\right) \geq \left(\sum_{S \subseteq [1:n]} \beta_S - 1\right) h\left(E_{\lambda + \sum_{i=1}^n \lambda_i}\right) + h\left(E_\lambda\right).$$
   Here $h(E)$ denotes the differential entropy.
2. Let $Y, \hat{Y}, Z$ be the three outputs of a three-receiver broadcast channel whose input is X. Further assume that $W_{\hat{Y}|X}$ is less noisy than $W_{Y|X}$. Show that
   $$\mathcal{C}_{p_X}\left[\alpha_1 I(X;\hat{Y}) - (\alpha_1 + \alpha_0) I(X;Y) + \mathcal{C}_{p_X}[\alpha_3 I(X;\hat{Y}) - (\alpha_2 + \alpha_3) I(X;Z)]\right]$$
   is sub-additive, for any non-negative $\alpha_0, \alpha_1, \alpha_2, \alpha_3$.

3. Consider a broadcast channel characterized by $W_{Y_1|X}$ and $W_{Y_2|X}$. Assume that $W_{Y_2|X}$ is not more-capable than $W_{Y_1|X}$. Let

$$\lambda_* := \inf\{\lambda : I(X;Y_1) - \lambda I(X;Y_2) \leq 0,\ \forall p_X\}.$$

The broadcast channel is called an *aligned-broadcast-channel* if $\lambda_* < +\infty$ and a fixed distribution p_X^* maximizes the following three quantities:
$(i) : I(X;Y_1)$,
$(ii) : I(X;Y_2)$,
$(iii) : I(X;Y_1) - \lambda_* I(X;Y_2)$.

Find the capacity region of an *aligned-broadcast-channel*.

4. Let $X_1 \to X_2 \to X_3 \to X_4$ for a Markov chain. Further let $X_2, X_3 \in \{0,1\}$. Define $Y = X_2 \oplus X_3$.
   (a) (4 points) Prove that $I(X_1X_4;X_2X_3) + 2H(Y|X_1X_4) \geq H(X_2,X_3) + I(X_2;Y|X_1) - I(X_2;X_3|X_1)$
   (b) (8 points) For every $p_{X_3|X_2}$, one can either find a $\hat{p}_{\hat{Y}|X_3}$ (or $p^{\dagger}_{X_3^{\dagger}|Y}$) so that $p_{Y|X_2} = \hat{p}_{\hat{Y}|X_3} \odot p_{X_3|X_2}$ (or $p_{X_3|X_2} = p^{\dagger}_{X_3^{\dagger}|Y} \odot p_{Y|X_2}$). In other words, when X_2 is the input: we can either treat Y as a stochastically degraded version of X_3, or X_3 as a stochastically degraded version of Y.
   (c) (2 points) Argue that if X_3 is a stochastically degraded version of Y, then

$$I(X_1X_4;X_2X_3) + 2H(Y|X_1X_4) \geq H(X_2,X_3).$$

   (d) (4 points) Prove that $I(X_1X_4;X_2X_3) + 2H(Y|X_1X_4) \geq 2H(Y) - I(X_2;Y) - I(X_3;Y)$
   (e) (2 points) Identify the conditions on p_{X_2,X_3} such that X_2, Y and X_3, Y are pairwise independent.

5. Consider a binary-input-binary-output interference channel such that the input alphabets $\mathcal{X}_1 = \{0,1\}$ and $\mathcal{X}_2 = \{0,1\}$. Further, let $Y_1 = X_1 \oplus X_2 \oplus Z_1$, and $Y_2 = X_1 \oplus X_2 \oplus Z_2$, where Z_1 and Z_2 are Bernoulli variables such that $\mathrm{P}(Z_i = 1) = p_i$, $0 \leq p_1 \leq p_2 \leq \frac{1}{2}$. Find the capacity region of this interference channel.