Final Exam: IERG 6154

April 17, 2025

Instructions

- The final exam is due: April 24th, 11:59 P.M.
- Please provide complete reasoning for your solutions.
- You are allowed to consult the textbook and lectures notes only. You are **not allowed** to use the internet
- Please use your time wisely (do not spend too much time on one question).
- Some questions are relatively easier than others.
- All questions are worth 20 points.

Questions

1. (Information Inequalities) Prove the following inequalities:

   (a) (3 points) Let X, Y, Z be mutually independent finite valued random variables taking values in integers. Prove that
   $$H(X) + H(Y+Z) \leq H(X+Y) + H(X+Z)$$

   (b) (5 points) Let $X_1, ..., X_n$ be mutually independent finite valued random variables taking values in reals. Prove that
   $$(n-1)H(X_1 + X_2 + \cdots + X_n) \leq \sum_{S: S \subseteq [1:n], |S| = n-1} H(\sum_{i \in S} X_i).$$

   (c) (12 points) Let $T_{Y|X}$ be a channel with binary input alphabet and quarternary output alphabet described by $T(Y=2|X=1) = T(Y=-2|X=-1) = p, T(Y=1|X=1) = T(Y=-1|X=-1) = q, T(Y=-1|X=1) = T(Y=1|X=-1) = r, T(Y=-2|X=1) = T(Y=2|X=-1) = s$, where p, q, r, s are non-negative reals such that $p+q+r+s = 1$. Define $C = \max_{p_x} I(X;Y)$ to be the point-to-point capacity of the channel T. Let $\hat{T}_{Z|X}$ be a binary symmetric channel with crossover probability d such that $H_2(d) = 1 - C$. Let $U \to X \to Y, Z$ be a joint distribution given by $p_{U|X} p_X T_{Y|X} \hat{T}_{Z|X}$, and further assume that p_X is the uniform distribution. Show that
   $$I(U;Z) \geq I(U;Y).$$

2. (a) (10 points) Let Y_1, Y_2, Y_3 be the three outputs of a degraded broadcast channel in order of degradation. Assume that the input to the channel is given by X. Show that
   $$\mathcal{C}_{p_X}[I(X;Y_1) - 2I(X;Y_2) + I(X;Y_3)]$$
   is sub-additive in p_X.

(b) (10 points) Show using the above and the rotation idea that, subject to

$$\mathbb{E}(XX^T) \leq K_X,$$

the quantity

$$\frac{d^2}{dt^2} h(X + \sqrt{t}Z),$$

where Z is an independent Gaussian $\mathcal{N}(0, I)$, is maximized when X is Gaussian, for $t > 0$. (You do not need to show the existence of the maximizer).

3. (a) (10 points) Consider a 3-receiver degraded interference channel $T_{Y_1,Y_2,Y_3|X_1,X_2,X_3}$ that decomposes as

$$T_{Y_1,Y_2,Y_3|X_1,X_2,X_3} = T_{Y_1|X_1,X_2,X_3} \hat{T}_{Y_2|Y_1} T^{\dagger}_{Y_3|Y_2}.$$

Find the sum capacity, i.e. $\max R_1 + R_2 + R_3$, for this setting.

(b) (10 points) A two-receiver Z-interference channel (with all input and output alphabets binary) is defined by the conditional distributions $T_1(y_1|x_1)$ and $T_2(y_2|x_1, x_2)$ as follows:

$$T_1(Y_1 = 0|X_1 = 0) = \frac{1}{2}$$
$$T_1(Y_1 = 1|X_1 = 1) = 1,$$

and $Y_2 = X_1 \oplus X_2 \oplus Z$ where $Z \sim \text{Ber}(p)$, where $\frac{5-\sqrt{5}}{10} \leq p \leq \frac{1}{2}$, and $\oplus$ denotes addition (modulo 2). Find the capacity region for this setting. It may be easier to determine all the weighted sum-rate hyperplanes.

4. *One Source 2-helper source coding problem* In both these parts, we have the following setting: (X^n, Y^n, Z^n) are i.i.d. $p(x, y, z)$. X^n is coded into $M_X \in [1 : 2^{nR_X}]$, Y^n is coded into $M_Y \in [1 : 2^{nR_Y}]$, and Z^n is coded into $M_Z \in [1 : 2^{nR_Z}]$. The decoder wishes to recover Z^n with high probability.

(a) (10 points) Let X, Y, Z be finite valued random variables such that $H(Z|X, Y) = 0$, and X and Y are independent. Find the achievable (optimal) rate region for this setting. Sketch proof of achievability as well as a converse.

(b) (10 points) Let X, Y, Z be finite random variables satisfying $H(Z|X, Y) = 0$, $H(X|Y, Z) = 0$, and $H(Y|X, Z) = 0$. Further, the channel given by $p_{Z|X}$ is more-capable than that given by $p_{Y|X}$. If $R_z = 0$, find the achievable (optimal) rate region for this setting.