IERG 5254: Network Information Theory
Homework 3

Due: February 24, 2025

1. Determine the capacity and the capacity achieving distribution of the Z channel depicted below

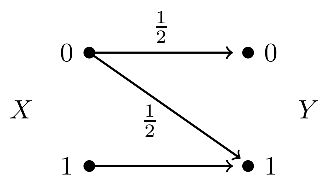

Figure 1: Z Channel

2. *Independently generated codebooks*: Consider a joint distribution $p(x,y)$, and let $p_X(x)$ and $p_Y(y)$ be its marginal distributions. Generate two codebooks $\mathcal{C}_1 = \{X_1^n, ..., X_{2^{nR_1}}^n\}$, and $\mathcal{C}_2 = \{Y_1^n, ..., Y_{2^{nR_2}}^n\}$ respectively by generating each X^n sequence independently according to $\prod_{i=1}^n p_X(x_i)$ and each Y^n sequence independently according to $\prod_{i=1}^n p_Y(y_i)$. Define the set
$$\mathcal{C} = \{(x^n, y^n) \in \mathcal{C}_1 \times \mathcal{C}_2 \text{ such that } (x^n, y^n) \in T_{\epsilon_n}(X,Y)\}.$$
Assume $R_1 < H(X_1), R_2 < H(X_2)$. Show that the expected size of $\mathcal{C}$ satisfies
$$\lim_{n\to\infty} \frac{1}{n} \log_2 \mathrm{E}(|\mathcal{C}|) = R_1 + R_2 - I(X;Y).$$

3. *Erasure degradation*: Consider a DMC $(\mathcal{X}, p(y|x), \mathcal{Y})$ and let Z be a noisy version of Y, $\mathcal{Z} = \mathcal{Y} \cup \{E\}$ (where $E \notin \mathcal{Y}$ is an erasure symbol), such that $\mathrm{P}(Z = y|Y = y) = 1 - e, \mathrm{P}(Z = E|Y = y) = e, \forall y$. Further assume that $X \to Y \to Z$ is Markov. Show that for all $p(x)$, we have
$$I(X;Z) = (1-e)I(X;Y).$$

4. Consider two DMCs, where the first channel is a $BSC(p)$ (binary symmetric channel) and the second channel is a $BEC(e)$. Let the output of the first channel be called Y and the output of the second channel be called Z. Assume both channels to have the same input X. (Therefore Y and Z are two different noisy versions of X.) Let $\mathrm{P}(X = 0) = x, 0 \le x \le 1$. Define the function $D(x) = I(X;Z) - I(X;Y)$. Show that

(a) $D(x)$ is concave in x *if and only if* $e \leq 4p(1-p)$.

(b) $D(x) \geq 0, \forall x \in [0,1]$ *if and only if* $e \leq H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$.

Convexity of $H(p * H^{-1}(u))$ in u may be useful.

5. Let $U, X, Y \sim p(u,x,y)$ be discrete random variables. Let $0 < \delta_n < \epsilon_n$ such that $\epsilon_n \to 0$ and $\delta_n \sqrt{n}, (\epsilon_n - \delta_n)\sqrt{n} \to \infty$. For any n, consider a fixed $u_0^n \in T_{\delta_n}(U)$. Let $\mathcal{A}_n = \{x^n : (u_0^n, x^n) \in T_{\epsilon_n}(U,X)\}$ and let $\mathcal{B}_n = \{y^n : (u_0^n, y^n) \in T_{\epsilon_n}(U,Y)\}$. Let $\mathcal{C}_n \subseteq \mathcal{A}_n \times \mathcal{B}_n$ be defined by $\mathcal{C}_n = \{(x^n, y^n) : (x^n, y^n) \in T_{\epsilon_n}(X,Y)\}$. Show that

(a) If X^n is generated according to $\prod_{i=1}^n p(x_i|u_{0i})$, then $\mathrm{P}(X^n \in \mathcal{A}_n) \to 1$. Similarly if Y^n is generated according to $\prod_{i=1}^n p(y_i|u_{0i})$, then $\mathrm{P}(Y^n \in \mathcal{B}_n) \to 1$.

(b) Show that

$$\frac{1}{n} \log \mathrm{P}\big((X^n, Y^n) \in \mathcal{C}_n | \{X^n \in \mathcal{A}_n, Y^n \in \mathcal{B}_n\}\big) \to - \min_{p(u,x,y) \in Q} I(X;Y|U),$$

where Q is the collection of distributions on (U,X,Y) such that $q(u,x) = p(u,x), q(u,y) = p(u,y)$, and $q(x,y) = p(x,y)$.