 Student's Id: __________

MIDTERM EXAMINATION II

Nov 3, 2022

Question	Points	Score
DFT and DTFT	13	
Fourier Transform	14	
Sampling Theorem	13	
Question - compulsory for elite students; optional for others.	0	
Total:	40	

This is a closed-book examination.
ALWAYS JUSTIFY YOUR ANSWERS.
NOTE: Describing your steps can help us give marks even if you make numerical errors.

Question 1 [13 points]: DFT and DTFT

Consider two discrete-time signals $x[n]$ and $h[n]$ defined as follows:

$$x[n] = \begin{cases} a^n & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} a^n & 0 \le n \le K-1 \\ 0 & \text{otherwise} \end{cases}$$

Note that the duration of the two signals are different. Assume that $K > L \ge 2$ and that $|a| < 1$.

Discrete-time Fourier Transform

(a) [2 points] Compute the DTFT of $x[n]$ and $h[n]$

(b) [4 points] Compute $y[n]$, where $y[n] = x[n] * h[n]$ (i.e., output of the convolution) (You may use DTFT properties or otherwise)

Discrete Fourier Transform Let $N \ge K$ be a fixed number.

(a) [2 points] Compute the N-point DFT of $x[n]$ and $h[n]$

(b) [2 points] Let $N = K + L - 1$. Compute

$$y_1[n] = x[n] \circledast_N h[n]$$

where $\circledast_N$ denotes the N-point periodic convolution of $x[n]$ and $h[n]$.
(Recall: You need to extend $x[n]$ and $h[n]$ periodically with period N to visualize this.)

(c) [3 points] Let $N = K + L - 2$. Compute

$$y_2[n] = x[n] \circledast_N h[n]$$

where $\circledast_N$ denotes the N-point periodic convolution of $x[n]$ and $h[n]$.

Question 2 [14 points]: Fourier Transform

(a) [5 points] Consider the following function, $x(t)$, plotted in the following figure (Figure 1):

$$x(t) = \begin{cases} 0 & t < 0 \\ 1 - 2\left|t - \frac{1}{2}\right| & 0 \le t \le 1 \\ \frac{1}{2}\left(1 - 2\left|t - \frac{3}{2}\right|\right) & 1 \le t \le 2 \\ \vdots & \vdots \\ \frac{1}{2^k}\left(1 - 2\left|t - \frac{2k+1}{2}\right|\right) & k \le t \le k+1 \\ \vdots & \vdots \end{cases}$$

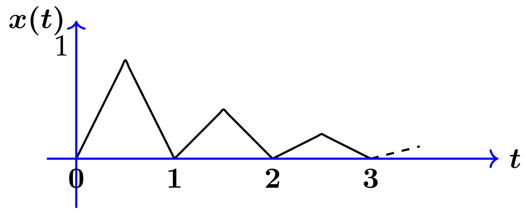

Figure 1: Signal $x(t)$

Find the Fourier transform of $x(t)$.

(b) [3 points] Given an input signal $x(t)$, for any $B > 0$, define

$$x_B(t) = \int_{-\infty}^{\infty} Bx(\tau)\text{sinc}(B(t-\tau))d\tau.$$

Define

$$A_B = \int_{-\infty}^{\infty} |x_B(t)|^2 dt.$$

Show that A_B is monotonically non-decreasing in B.

(c) A causal LTI system is characterized by the following differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = x(t).$$

i. [2 points] Determine $\hat{H}(f)$, the Fourier Transform of the impulse response, $h(t)$.

ii. [3 points] Hence determine the impulse response $h(t)$, of the LTI system

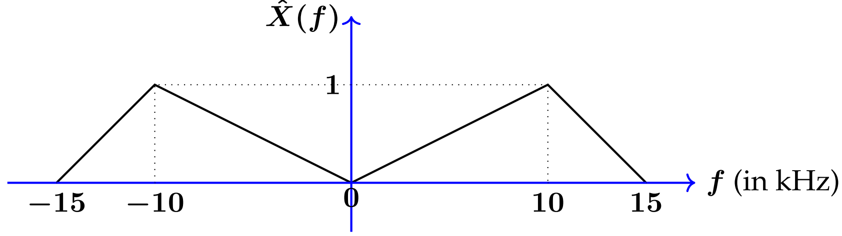

Figure 2: The Fourier Transform $\hat{X}(f)$

iii. [1 point] Determine if the LTI system is stable or not.

Question 3 [13 points]: Sampling Theorem

Let $x(t)$ be a continuous time signal, and $x_s[n] = x(nT_s), n \in \mathbb{Z}$ be its samples. The Fourier Transform, $\hat{X}(f)$, of the signal $x(t)$ is given in the figure (Figure 2) below.

The samples were stored on a disk. However there was a tragic disk-failure and every third of the samples were zeroed out. Now the samples on the disk satisfy

$$x_d[n] = \begin{cases} 0 & n \equiv 0(\text{mod} 3) \\ x_s[n] & n \not\equiv 0(\text{mod} 3) \end{cases}.$$

The signal is attempted to be reconstructed using the usual way, i.e. first generate

$$x_d(t) := T_s \sum_n x_d[n]\delta(t - nT_s)$$

and then $x_d(t)$ is passed through an ideal low-pass filter, $l(t)$, with

$$\hat{L}(f) = \begin{cases} 1 & |f| \leq 20kHz \\ 0 & |f| > 20kHz \end{cases}.$$

to obtain the reconstructed signal $x_1(t)$.

(a) [4 points] if $T_s = \frac{1}{90} * 10^{-3}$ (or equivalently $f_s = 90$ kHz), then calculate (and plot), the Fourier transform, $\hat{X}_1(f)$, of the reconstructed signal.

(b) [9 points] if $T_s = \frac{1}{45} * 10^{-3}$ (or equivalently $f_s = 45$ kHz), then calculate (and plot), the Fourier transform, $\hat{X}_1(f)$, of the reconstructed signal.

(In your solution, please explain how you work through this question. It will help yield marks for correct logical process even if you make numerical mistakes)

Question 4 [0 points]: Question - compulsory for elite students; optional for others.

Let Q_N be the N-point unitary DFT matrix. Define it's characteristic polynomial as

$$P_N(\lambda) = \det(Q_N - \lambda I_N).$$

Determine $P_{4\ell+2}(\lambda)$ for $\ell \in \mathbb{N}, \ell \geq 1$. (Hint: all the coefficients of the polynomial are integers and can be expressed in terms of ℓ)