MIDTERM EXAMINATION I

September 30, 2023

Question	Points	Score
System Properties	12	
LTI System	10	
A linear system	4	
Fourier Series	14	
Question - compulsory for elite students; optional for others.	0	
Total:	40	

You are not allowed to use resources and forums outside those given in Piazza for this class or your own notes from this class
ALWAYS JUSTIFY YOUR ANSWERS.
NOTE: Describing your steps can help us give marks even if you make numerical errors.

Question 1 [12 points]: System Properties

Consider the following systems with $x(t)$ and $y(t)$ being the input and output of the system, respectively. Determine whether they are (1) Memoryless, (2) Time-invariant, (3) Linear, (4) Causal, (5) Stable, and (6) Invertible. Justify your answer for each property. (Marking: 1 mark for each property. If you do not justify your answers or if your justification is wrong, you will not get any mark for that property.)

(a) [6 points] System 1: A discrete-time system

$$y[n] = \begin{cases} 0, & n < 0 \\ \sum_{m=0}^{n} \frac{1}{2^{m+|x[m]|}} & n \geq 0 \end{cases} .$$

(b) [6 points] System 2: A continuous time system

$$y(t) = \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\int_{t-k-1}^{t-k} x(\tau) d\tau \right) .$$

Question 2 [10 points]: LTI System

A **causal** LTI system produces the following input-output relationship shown in Figure 1. The signal

$$x(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 0, & \text{otherwise} \end{cases}, \qquad y(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 3 \\ t-4, & 3 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}.$$

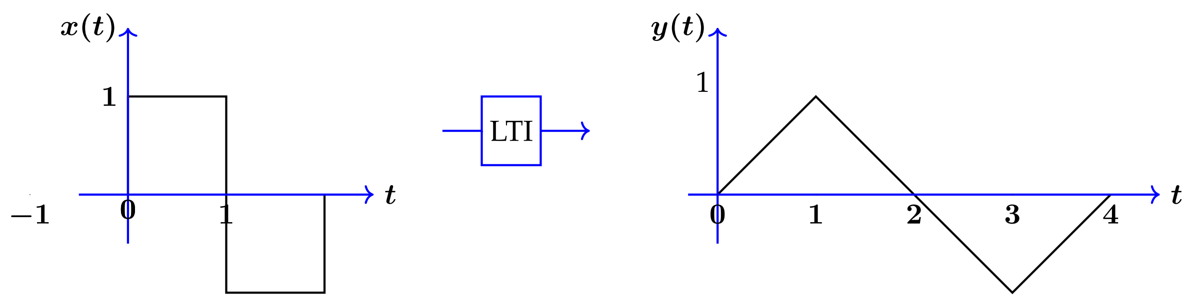


Figure 1: An input-output pair

(a) [4 points] Compute the output of the LTI system when the input is $x_1(t)$ shown below.

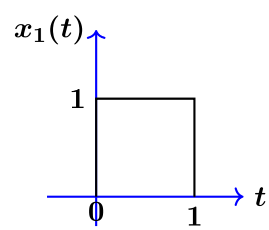


Figure 2: An input $x_1(t)$

$$x_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0, & o.w. \end{cases}.$$

(b) [3 points] Compute the output $y(t)$ when the input $x_2(t) = u(t)$, the unit step function.

(c) [2 points] Compute $h(t)$, the impulse response of this system.

(d) [1 point] Prove that the LTI system is a stable system.

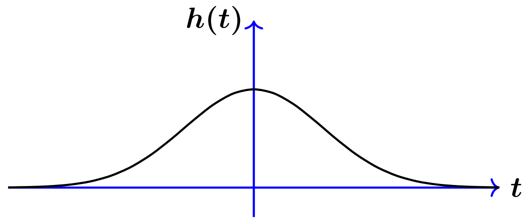
Question 3 [4 points]: A linear system

A linear memoryless system produces for the constant input signal

$$c(t) = 1, \forall t.$$

the following output:

$$h(t) = e^{-t^2}.$$



(a) [1 point] Express the output $y(t)$, of the above system, for a generic input, $x(t)$ in terms of $h(t)$.

(b) [1 point] Is this system stable?

(c) [2 points] If one puts k identical such systems in series, express the output $y_k(t)$ in terms of a generic input $x(t)$ and the signal $h(t)$ mentioned above.

Question 4 [14 points]: Fourier Series

Let $x(t)$ be the periodic extension (with $T_0 = 1$) of $x_p(t)$ where

$$x_p(t) = \begin{cases} \sin(\pi t) & t \in [0, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases} .$$

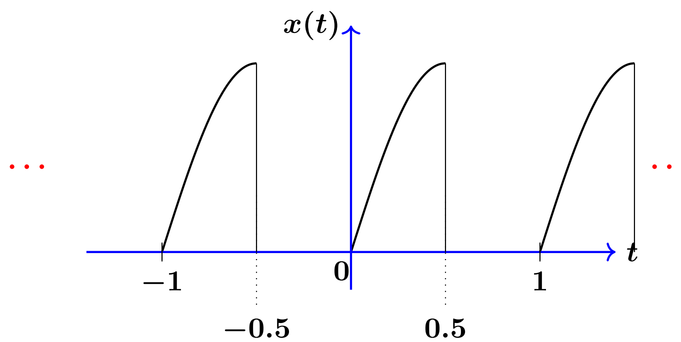


Figure 3: The signal $x(t)$

(a) [6 points] Compute, using integration, the (exponential) Fourier Series coefficients of $x(t)$, i.e, compute $\{\hat{X}[k]\}$ where

$$x(t) = \sum_k \hat{X}[k] e^{j2\pi kt}.$$

(b) [2 points] Compute $x_1(t)$ and $r(t)$

$$x'(t) = x_1(t) + \sum_{k \in \mathbb{Z}} \delta(t - k - \frac{1}{2})$$
$$x_1'(t) = -\pi^2 x(t) + r(t)$$

(c) [3 points] Use the differentiation property of the Fourier Series to re-derive the Fourier Series co-efficients $\{\hat{X}[k]\}$that you obtained in the first part.

(d) [3 points] Using the Fourier Series co-efficients computed above, determine the value of (justify your steps)

$$\sum_{k\geq 1} \frac{4k^2+1}{(4k^2-1)^2}.$$

Question 5 [0 points]: Question - compulsory for elite students; optional for others.

We know that for a linear shift invariant system that the eigen-basis depends only on N. Here we are considering $N = 2$. Let $[x[0],\ x[1]]$ be a vector such that $|x[0]|^2 + |x[1]|^2 = 1$. Assume that all scalars are complex numbers.

Define

$$\hat{X}[0] = \frac{1}{\sqrt{2}}(x[0] + x[1])$$
$$\hat{X}[1] = \frac{1}{\sqrt{2}}(x[0] - x[1])$$

(a) Show that $|\hat{X}[0]|^2 + |\hat{X}[1]|^2 = 1$.

(b) Define two probability vectors according to $p_0 = |x[0]|^2, p_1 = |x[1]|^2$ and $q_0 = |\hat{X}[0]|^2, q_1 = |\hat{X}[1]|^2$. Prove that

$$-p_0 \log p_0 - p_1 \log p_1 - q_0 \log q_0 - q_1 \log q_1 \geq \log 2.$$