MIDTERM EXAMINATION II

Nov 7, 2023

Question	Points	Score
Fourier Transform	15	
Laplace Transform	12	
Discrete Fourier Transform	13	
Question - compulsory for elite students; optional for others.	0	
Total:	40	

This is a closed-book examination.
ALWAYS JUSTIFY YOUR ANSWERS.
NOTE: Describing your steps can help us give marks even if you make numerical errors.

Question 1 [15 points]: Fourier Transform

Consider the following function, $x(t)$, plotted in the following figure (Figure 1):

$$x(t) = \begin{cases} |\cos(2\pi t)| & t \in [-\frac{3}{4}, \frac{3}{4}] \\ 0 & \text{otherwise} \end{cases}$$

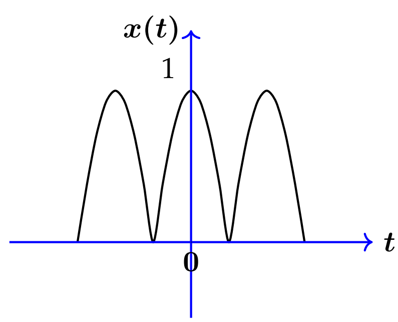

Figure 1: Signal $x(t)$

(a) [7 points] By direct integration, find the Fourier transform of $x(t)$.

(b) [2 points] Show that $x(t)$ satisfies a differential equation of the form

$$\frac{d^2}{dt^2}x(t) = Ax(t) + \sum_{n=1}^{k} c_n \delta(t - t_n).$$

Furthermore, determine the values of A, k, c_n and t_n.

(c) [3 points] Use the differentiation property of the Fourier Transform to rederive the Fourier Transform of $x(t)$ using the above differential equation.

(d) [3 points] Using the above parts, find the value of the infinite sum

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4k^2 - 1}.$$

Question 2 [12 points]: Laplace Transform

Consider a causal LTI system that satisfies the following differential equation:

$$5\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t) + 2x(t).$$

(a) [3 points] Determine the Lapalace Transform, $\hat{H}(s)$, of the above system and its corresponding Region of Convergence, ROC.

(b) [3 points] Invert the Laplace transform to determine the impulse response, $h(t)$.

(c) [3 points] We know that for an LTI system, the input $e^{j2\pi ft}$ produces an output given by $\hat{H}(f)e^{j2\pi ft}$. Let $|\hat{H}(f)|$ be called the frequency amplification of the system at frequency f. For the system in question, determine

$$v = \max_{f \in \mathbb{R}} |\hat{H}(f)|,$$

the largest value of the frequency amplification.

(d) [3 points] Compute the output of the LTI system when the input is $e^{-2t}u(t)$.

Question 3 [13 points]: Discrete Fourier Transform

(a) [4 points] Consider the following function:

$$x[n] = (n+1), 0 \leq n \leq N-1.$$

Compute the N-point DFT, $\hat{X}[k]$, of the function $\{x[n]\}$,

$$\hat{X}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}.$$

(Hint: Let $S = 1 + 2a + 3a^2 + \cdots + (m+1)a^m$. Then you can get an explicit formula for S, for $a \neq 1$, by considering $S - aS$.)

(b) [5 points] Let $|a| < 1$. Consider the following function:

$$y[n] = \frac{1 - a^N}{1 - ae^{-j2\pi n/N}}, 0 \leq n \leq N-1.$$

Compute the N-point DFT, $\hat{Y}[k]$, of the function $\{y[n]\}$,

$$\hat{Y}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y[n] e^{-j2\pi nk/N}.$$

(c) [4 points] Determine the space of vectors $x[n]$ of length $N = 3$ such that the DFT of $x[n]$ is itself.

Question 4 [0 points]: Question - compulsory for elite students; optional for others.

Let Q_N be the N-point unitary DFT matrix. Define its characteristic polynomial as

$$P_N(\lambda) = \det(Q_N - \lambda I_N).$$

Determine $P_{4\ell+2}(\lambda)$ for $\ell \in \mathbb{N}, \ell \geq 1$. (Hint: all the coefficients of the polynomial are integers and can be expressed in terms of ℓ)