MIDTERM EXAMINATION I

September 26, 2024

Question	Points	Score
System Properties	12	
LTI System	12	
Fourier Series	13	
Short Questions	3	
Question - compulsory for elite students; optional for others.	0	
Total:	40	

You are not allowed to use resources and forums outside those given in Piazza for this class or your own notes from this class
ALWAYS JUSTIFY YOUR ANSWERS.
NOTE: Describing your steps can help us give marks even if you make numerical errors.

Question 1 [12 points]: System Properties

Consider the following systems with $x(t)$ and $y(t)$ as the input and output of the system, respectively. Determine whether they are (1) Memoryless, (2) Time-invariant, (3) Linear, (4) Causal, (5) Stable, and (6) Invertible. (For invertibility, assume that the input space $\mathcal{S} = \{x[n] : x[n] = 0, n < 0\}$. That is, determine if two different signals from the above space can lead to the same output or not.)
Justify your answer for each property. (Marking: 1 mark for each property. If you do not justify your answers or if your justification is wrong, you will not get any mark for that property.)

(a) [6 points] System 1: A discrete-time system

$$y[n] = \sum_{k=0}^{\infty} \frac{x[n-k]}{1+k^2}.$$

(b) [6 points] System 2: A continuous-time system.

$$\frac{d}{dt}y(t) = x(t) + t\frac{d}{dt}x(t).$$

In this problem, we assume that the input space is:
$\mathcal{S} = \{x(t) : x(t)$ is differentiable and the derivative is continuous, $x(t) = 0 \; \forall \; t < 0\}$.
Further, assume that $y(t) = 0$ for $t < 0$.

Question 2 [12 points]: LTI System

An LTI system has the following impulse response.

$$h(t) = \begin{cases} \sin(\pi t), & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}.$$

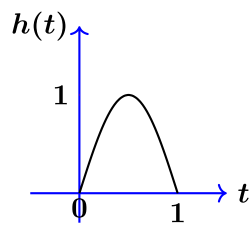


Figure 1: Impulse response $h(t)$

(a) [4 points] Compute the output of the system for the following input:

$$x(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}.$$

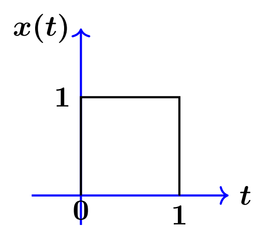


Figure 2: Input $x(t)$

(b) [4 points] Compute the output $y_2(t)$ when the input $x_2(t) = e^{-t}u(t)$.

(c) [4 points] Let $h(t)$, shown above, be the input to a new system $h_{new}(t)$, and let the produced output be $y_{new}(t)$., i.e. $y_{new}(t) = h(t) * h_{new}(t)$. Alice says that there exists constants a, b such that

$$h_{new}(t) = \sum_{k=0}^{\infty} (a y''_{new}(t-k) + b y_{new}(t-k)).$$

Is Alice correct in her assertion? If she is, determine the values of a, b.

Question 3 [13 points]: Fourier Series

Let $x(t)$ be the periodic extension (with $T_0 = 1$) of $x_p(t)$ where

$$x_p(t) = \begin{cases} e^{-|t|} & t \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}.$$

(a) [5 points] Compute, using integration, the (exponential) Fourier Series coefficients of $x(t)$, i.e, compute $\{\hat{X}[k]\}$ where

$$x(t) = \sum_k \hat{X}[k] e^{j2\pi kt}.$$

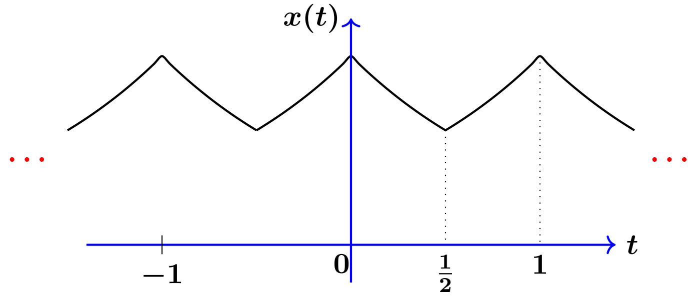


Figure 3: The signal $x(t)$

(b) [3 points] Recompute the Fourier Series coefficients of $x(t)$ using the differentiation property.

(c) [5 points] Using the above Fourier Series co-efficients., argue that

$$\sum_{k=1}^{\infty} \frac{1}{1+4\pi^2k^2} = \frac{3-e}{4(e-1)}.$$

Question 4 [3 points]: Short Questions

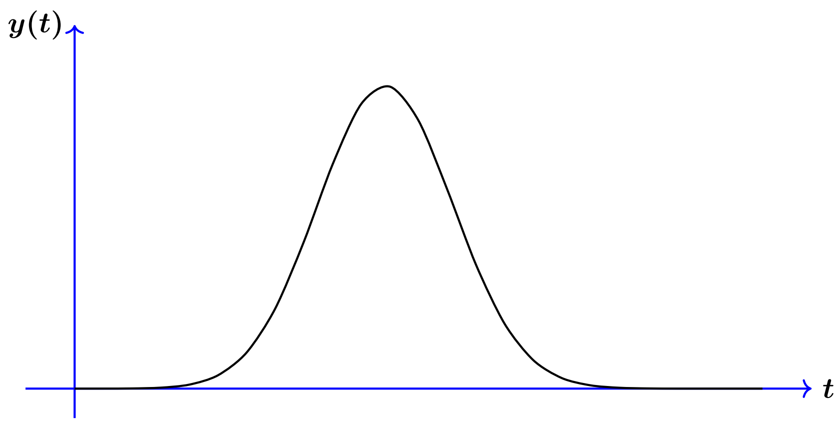
(a) [1 point] Is the cascade (putting one after another) of two stable (not necessarily LTI) systems stable?

(b) [2 points] Determine the fundamental period of $\sin(2\pi t) + \cos(3\pi t)$.

Question 5 [0 points]: Question - compulsory for elite students; optional for others.

Tom kept hearing his echo in an online conversation. He wanted to estimate the delay, d, so he could cancel it. He asked his friend to keep silent, transmitted a signal $x(t)$ and received the signal $x_r(t) = x(t-d)$. Tom computed $y(t) = x(-t) * x_r(t)$. The output, $y(t)$, is described below and is also plotted in the figure below.

$$y(t) = \begin{cases} te^{-(t-3)^2} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$



Compute the value of d (you have to justify your answer).