 Student's Id: ______________

MIDTERM EXAMINATION II

Oct 29, 2024

Question	Points	Score
Fourier Transform	10	
Discrete-Time Fourier Transform and Discrete Fourier Transform	6	
Discrete-Time Fourier Transform and Fourier Transform	4	
Discrete Fourier Transform	7	
Fourier Transform and Circuits	13	
Question - compulsory for elite students; optional for others.	0	
Total:	40	

This is a closed-book examination.
ALWAYS JUSTIFY YOUR ANSWERS.
NOTE: Describing your steps can help us give marks even if you make numerical errors.

Question 1 [10 points]: Fourier Transform

Consider the following function, $x(t)$, plotted in the following figure (Figure 1):

$$x(t) = \begin{cases} 1 - t^2 & t \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

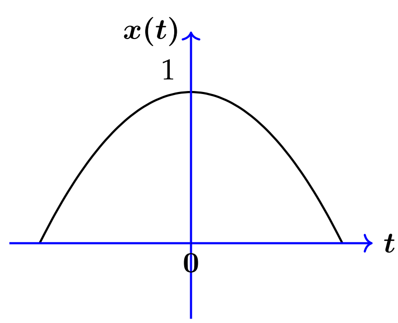


Figure 1: Signal $x(t)$

(a) [5 points] Find the Fourier transform of $x(t)$.

(b) [5 points] Let $x_1(t) \leftrightarrow \hat{X}_1(f)$ be a generic Fourier-Transform pair. For any $t_0 \in \mathbb{R}$, prove the Poisson summation formula:

$$\sum_{k\in\mathbb{Z}} x(kT - t_0) = \frac{1}{T}\sum_{k\in\mathbb{Z}} \hat{X}\left(\frac{k}{T}\right) e^{-j2\pi \frac{k}{T} t_0}.$$

(Note: to do this, you **are not allowed to** use the Poisson summation formula in the handouts)

Question 2 [6 points]: Discrete-Time Fourier Transform and Discrete Fourier Transform

(a) [4 points] Let $x[n]$ be a periodic sequence with period 3, with

$$x[n] = \begin{cases} -1 & n = -1 \\ 2 & n = 0 \\ 1 & n = 1 \end{cases}.$$

Find its Discrete-Time Fourier Transform.

(b) [2 points] Find the 3-pt DFT of the vector $(2, 1, -1)$, i.e. $x[0] = 2, x[1] = 1, x[2] = -1$.

Question 3 [4 points]: Discrete-Time Fourier Transform and Fourier Transform

Let $x(t) \leftrightarrow \hat{X}(f)$ be a generic Fourier-Transform pair.
Form a discrete time sequence $x_s[n] = x(n)$ for $\in \mathbb{Z}$. Let $\hat{X}_s(f)$ be the Discrete-time Fourier Transform of $x_s[n]$.
Express $X_s(f)$ in terms of $X(f)$.

Question 4 [7 points]: Discrete Fourier Transform

(a) [4 points] Consider the following function: for $0 \leq n \leq N-1$,

$$x[n] = \begin{cases} (-1)^n \binom{N}{n}, & n \equiv 0 \bmod 3 \\ 0 & o.w. \end{cases}$$

Compute the N-point DFT, $\hat{X}[k]$, of the function $\{x[n]\}$,

$$\hat{X}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}.$$

You should simplify your answer and not leave it as a sum of N terms trivially.

(b) [3 points] Let $N = 100$, where $x[n]$ is the same as above. Compute the value of $\hat{X}[50]$. If $\hat{X}[50]$ is a non-negative rational number expressed in the simplest form as $\frac{m}{n}$, with $gcd(m, n) = 1$, then compute the value of n.

Question 5 [13 points]: Fourier Transform and Circuits

Consider the electrical circuit below. The input to the system is a voltage source producing a voltage waveform given by $v_{in}(t)$. The output of the system is the voltage $v_{out}(t)$, measured across the resistor R_2.

Remarks:

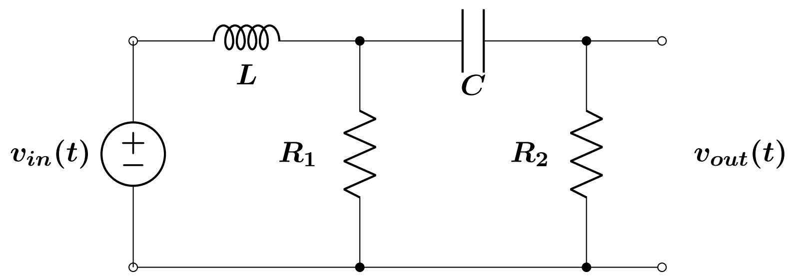
i. A *resistor* is a device where the voltage drop across the device is proportional to the current passing through it, in particular $v = iR$.

ii. A *capacitor* is a device where the current through the device is proportional to the rate of change of charge held in the device, in particular $i = C\frac{dv}{dt}$.

iii. An *inductor* is a device where the voltage drop across the device is proportional to the rate of change of current passing through it, in particular $v = L\frac{di}{dt}$.

iv. Kirchoff's current law states that the total current entering a node equals the total current leaving the node.

v. Kirchoff's voltage law states that the total voltage drop across a closed loop is zero.



(a) [6 points] Using properties of circuit elements and Kirchoff's current and voltage law's show that the following differential equation is satisfied by the system.

$$\frac{dv_{in}(t)}{dt} = \frac{1}{CR_2}v_{out}(t) + \left(\frac{L}{CR_1R_2} + 1\right)\frac{dv_{out}(t)}{dt} + \left(\frac{L}{R_1} + \frac{L}{R_2}\right)\frac{d^2v_{out}(t)}{dt^2}.$$

(b) [2 points] From the differential equation above, compute $\hat{H}(f)$, the Fourier Transform of the system given by the circuit.

(c) [3 points] Determine the value of f where $|\hat{H}(f)|$ is maximized. Determine also the corresponding maximum value of $|\hat{H}(f)|$.

(d) [2 points] Determine the value of $|\hat{H}(f)|$ at $f = 0$ and as $f \to \infty$. Thus, would you term the system: low-pass filter, band-pass filter, or high-pass filter.

Question 6 [0 points]: Question - compulsory for elite students; optional for others.

Heisenberg's uncertainty for Discrete-Time Fourier Transform.

Let $\{x[n]\} \leftrightarrow \hat{X}(f)$ be a generic discrete time Fourier Transform pair, with $\hat{X}(\frac{1}{2}) = 0$. Further let $\sum_n |x[n]|^2 = 1$. Then show that

$$\left(\sum_n n^2|x[n]|^2\right)\left(\int_{f=-\frac{1}{2}}^{\frac{1}{2}} f^2|\hat{X}(f)|^2 df\right) \geq \frac{1}{16\pi^2}.$$