MIDTERM EXAMINATION I

September 30, 2025

Question	Points	Score
System Properties	12	
LTI System	12	
Fourier Series	14	
Short Questions	2	
Question - compulsory for elite students; optional for others.	0	
Total:	40	

ALWAYS JUSTIFY YOUR ANSWERS.
NOTE: Describing your steps can help us give marks even if you make numerical errors.

Question 1 [12 points]: System Properties

Consider the following systems with $x(t)$ and $y(t)$ as the input and output of the system, respectively. Determine whether they are (1) Memoryless, (2) Time-invariant, (3) Linear, (4) Causal, (5) Stable, and (6) Invertible. (For invertibility, assume that the input space $\mathcal{S} = \{x[n] : x[n] = 0, n < 0\}$. That is, determine if two different signals from the above space can lead to the same output or not.)
Justify your answer for each property. (Marking: 1 mark for each property. If you do not justify your answers or if your justification is wrong, you will not get any mark for that property.)

(a) [6 points] System 1: A discrete-time system

$$y[n] = nx[n] - (n-1)x[n-1], n \geq 1.$$

Further

$$y[0] = x[0].$$

(b) [6 points] System 2: A continuous-time system.

$$y(t) = x(t)\frac{t}{1+(t+1)^2}, t \neq 0.$$

Further $y(0) = x(0)$.

Question 2 [12 points]: LTI System

An LTI system has the following impulse response.

$$h(t) = \begin{cases} (1-t^2), & -1 \le t < 1 \\ 0, & \text{otherwise} \end{cases}.$$

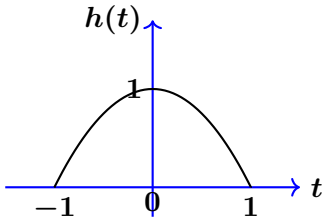


Figure 1: Impulse response $h(t)$

(a) [4 points] Compute the output of the system, $y(t)$, for the following input:

$$x(t) = \begin{cases} 1, & -\frac{1}{2} \le t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}.$$

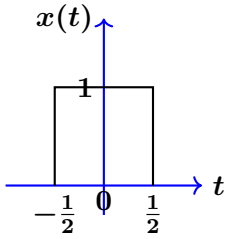


Figure 2: Input $x(t)$

(b) [4 points] Compute the input $x_2(t)$ when the output of the same system is

$$y_2(t) = \begin{cases} 1-t^2, & -1 \le t < 0 \\ 1-2t, & 0 \le t < 1 \\ t^2-2t, & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

(c) [4 points] Let us define a new notion of stability of LTI systems as follows: We say that an LTI system is *integral-stable* if $\int_{\mathbb{R}} |y(t)|dt < \infty$, whenever if $\int_{\mathbb{R}} |x(t)|dt < \infty$. Is it true that if an LTI system is BIBO stable (usual stability), then it is *integral-stable*? Prove it or find a counterexample.

Question 3 [14 points]: Fourier Series

Let $x(t)$ be the periodic extension (with $T_0 = 1$) of $x_p(t)$ where

$$x_p(t) = \begin{cases} e^{-(a+jb)t} & t \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}.$$

(a) [4 points] Compute, using integration, the (exponential) Fourier Series coefficients of $x(t)$, i.e, compute $\{\hat{X}[k]\}$ where

$$x(t) = \sum_k \hat{X}[k] e^{j2\pi kt}.$$

(b) [3 points] Recompute the Fourier Series coefficients of $x(t)$ using the differentiation property.

(c) [4 points] Using the above Fourier Series coefficients, argue that

$$\sum_{k\geq 1} \frac{(-1)^{(k-1)}}{a^2 + (4\pi^2 k^2)} = \frac{\frac{2\sinh\left(\frac{a}{2}\right)}{a} - 1}{4a\sinh\left(\frac{a}{2}\right)}.$$

Note: $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

(d) [3 points] Using the above Fourier Series coefficients, argue that

$$\sum_{k\geq 1} \frac{1}{a^2 + (4\pi^2 k^2)} = \frac{\frac{\sinh(a)}{a} - \frac{4\sinh^2\left(\frac{a}{2}\right)}{a^2}}{8\sinh^2\left(\frac{a}{2}\right)}.$$

Question 4 [2 points]: Short Questions

(a) [1 point] Is the cascade (putting one after another) of two invertible (not necessarily LTI) systems invertible?

(b) [1 point] Let a time-invariant system be invertible. Is the inverse system also necessarily time-invariant.?

Question 5 [0 points]: Question - compulsory for elite students; optional for others.

Let $x(t)$ be a continuous real-valued function on $[0, 1]$. You can show that this implies that $x(t)$ is bounded on the interval $[0, 1]$ and further that it is uniformly continuous. That is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $|t_1, t_2 \in [0, 1]$ and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.

Let us compute the Fourier series co-efficients of its periodic extension

$$\hat{X}(k) = \int_0^1 x(t) e^{-j2\pi kt} dt.$$

Show that

$$\lim_{k \to \infty} \hat{X}(k) = 0.$$