 Student's Id: _____

MIDTERM EXAMINATION II

Oct 20, 2025

Question	Points	Score
Fourier Transform	15	
Discrete-Time Fourier Transform	8	
Fourier Transform and Circuits	5	
Discrete Fourier Transform	12	
Question - compulsory for elite students; optional for others.	0	
Total:	40	

This is a closed-book examination.
ALWAYS JUSTIFY YOUR ANSWERS.
NOTE: Describing your steps can help us give marks even if you make numerical errors.

Question 1 [15 points]: Fourier Transform

Consider the following function, $x(t)$, plotted in the following figure (Figure 1):

$$x(t) = \begin{cases} e^{-|t|} & t \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

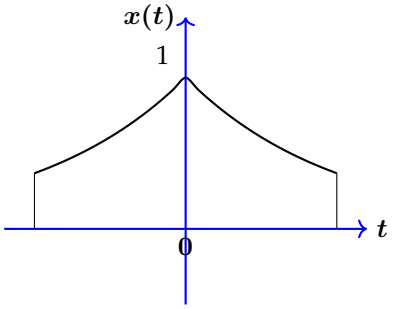


Figure 1: Signal $x(t)$

(a) [1 point] Argue that $\hat{X}(f)$ is real.

Solution: Observe that $\overline{\hat{X}(f)} \xrightarrow{IFT} \overline{x(-t)}$. Hoever since $x(t)$ is real and even, we have $\overline{\hat{X}(f)} = \hat{X}(f)$.

(b) [5 points] Find the Fourier transform of $x(t)$ by direct integration. Please write it as a real number.

Solution:

$$\begin{aligned}\hat{X}(f) &= \int_{-1}^{0} e^{t}e^{-j2\pi ft}dt + \int_{0}^{1} e^{-t}e^{-j2\pi ft}dt \\ &= \frac{e^{(1-j2\pi f)t}}{1-j2\pi f}\Big|_{-1}^{0} - \frac{e^{(-1-j2\pi f)t}}{1+j2\pi f}\Big|_{0}^{1} \\ &= \frac{1-e^{-1}e^{j2\pi f}}{1-j2\pi f} + \frac{1-e^{-1}e^{-j2\pi f}}{1+j2\pi f} \\ &= \frac{(1-e^{-1}e^{j2\pi f})(1+j2\pi f) + (1-e^{-1}e^{-j2\pi f})(1-j2\pi f)}{1+4\pi^2 f^2} \\ &= 2\left(\frac{1-e^{-1}\cos(2\pi f) + 2\pi f e^{-1}\sin(2\pi f)}{1+4\pi^2 f^2}\right)\end{aligned}$$

(c) [6 points] Recompute the Fourier Transform by using the differentiation property

Solution: Observe that $x'(t)$ satisfies, $x'(t) = e^{-1}\delta(t+1) - e^{-1}\delta(t-1) + x_1(t)$, where

$$x_1(t) = \begin{cases} e^{t} & t \in [-1,0] \\ -e^{-t} & t \in [-1,0] \\ 0 & \text{otherwise} \end{cases}$$

Differentiating $x_1(t)$, we obtain $x_1'(t) = e^{-1}\delta(t+1) + e^{-1}\delta(t-1) - 2\delta(t) + x(t)$.
Therefore, taking the Fourier transforms of the two differentiations we obtain

$$\begin{aligned} j2\pi f\hat{X}(f) &= e^{-1}e^{j2\pi f} - e^{-1}e^{-j2\pi f} + \hat{X}_1(f) \\ j2\pi f\hat{X}_1(f) &= e^{-1}e^{j2\pi f} + e^{-1}e^{-j2\pi f} - 2 + \hat{X}(f). \end{aligned}$$

Therefore

$$\begin{aligned} & -4\pi^2 f^2\hat{X}(f) = e^{-1}e^{j2\pi f}(1+j2\pi f) + e^{-1}e^{-j2\pi f}(1-j2\pi f) + \hat{X}(f) - 2 \\ \iff \quad & \hat{X}(f)(1+4\pi^2 f^2) = 2 - 2e^{-1}\cos(2\pi f) - j\pi f e^{-1}(e^{j2\pi f} - e^{-j2\pi f}) \\ \iff \quad & \hat{X}(f) = \frac{2 - 2e^{-1}\cos(2\pi f) + 4\pi f e^{-1}\sin(2\pi f)}{1+4\pi^2 f^2} \end{aligned}$$

(d) [1 point] Use the Poisson summation formula and the shifting property of the Fourier transform to argue that for a generic Fourier transform pair

$$\frac{\sum_k (x(k+t_0) + x(k-t_0)}{2} = \sum_k \hat{X}(k)\cos(2\pi k t_0).$$

Solution: Let $x_1(t) = x(t-t_0)$, then $\hat{X}_1(f) = \hat{X}(f)e^{-j2\pi ft_0}$. Now let

$$x_2(t) = \frac{x(t-t_0) + x(t+t_0)}{2} \xrightarrow{FT} \hat{X}(f)\cos(2\pi f t_0).$$

By the Poisson summation formula: $\sum_k x_2(k) = \sum_k \hat{X}_2(k)$, yielding

$$\frac{\sum_k (x(k+t_0) + x(k-t_0)}{2} = \sum_k \hat{X}(k)\cos(2\pi k t_0).$$

(e) [2 points] Taking $t_0 \to 0$ (to get around the discontinuity of the function in first parts at $t = -1, 1$), compute

$$\sum_{k=0}^{\infty} \frac{1}{1+4\pi^2 k^2}.$$

Solution: Let $t_0 \to 0$, we see that We know that $x(0) + \frac{1}{2}(x(1^+) + x(-1^+) + x(1^-) + x(-1^-)) = \sum_k \hat{X}(k)$. Therefore

$$1 + e^{-1} = 4(1-e^{-1})\left(\sum_{k=0}^{\infty} \frac{1}{1+4\pi^2 k^2}\right) - 2(1-e^{-1}).$$

Therefore

$$\sum_{k=0}^{\infty} \frac{1}{1+4\pi^2 k^2} = \frac{3-e^{-1}}{4(1-e^{-1})}.$$

Question 2 [8 points]: Discrete-Time Fourier Transform

Let $x_k[n], k \geq 1, k \in \mathbb{N}$ be defined as

$$x_k[n] = \frac{n^k}{k!} a^n u[n],$$

where $|a| < 1$. Show that its Discrete-Time Fourier Transform is

$$\sum_{\ell \in Z} \frac{1}{(-\log(a) + j2\pi(f-\ell))^{k+1}}.$$

Solution: Let $x_k(t) = \frac{t^k}{k!} e^{t\log a} u(t)$. For $k \geq 1$, if you differentiate it, you get

$$\frac{d}{dt} x_k(t) = \frac{t^{k-1}}{(k-1)!} e^{t\log a} u(t) + \frac{t^k}{k!}(\log a) e^{t\log a} u(t) + \frac{t^k}{k!} e^{t\log a}\delta(t).$$

Since, for $k \geq 1$, $\frac{t^k}{k!} e^{t\log a}\delta(t) = 0$, taking Fourier Transforms I get

$$j2\pi f \hat{X}_k(f) = \hat{X}_{k-1}(f) + (\log a)\hat{X}_k(f).$$

In other words,

$$\hat{X}_k(f) = \frac{\hat{X}_{k-1}(f)}{j2\pi f - \log(a)}.$$

Since $\hat{X}_0(f) = \frac{1}{j2\pi f + \log(a)}$, we obtain $\hat{X}_k(f) = \frac{1}{(j2\pi f - \log(a))^{k+1}}$.

Now, consider $x_k(t)\sum_n \delta(t-n) = \sum_n x[n]\delta(t-n)$. Taking Fourier Transform on both sides, we obtain

$$\sum_n \hat{X}_k((f-n)) = \sum_n x[n]e^{-j2\pi fn}.$$

Therefore, the DTFT is

$$\sum_n \frac{1}{(j2\pi(f-n)-\log(a))^{k+1}}$$

Question 3 [5 points]: Fourier Transform and Circuits

Consider the electrical circuit below. The input to the system is a voltage source producing a voltage waveform given by $v_{in}(t)$. The output of the system is the voltage $v_{out}(t)$, measured across the resistor R_2.

Remarks:

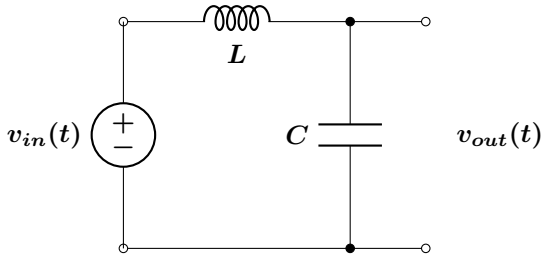
i. A *resistor* is a device where the voltage drop across the device is proportional to the current passing through it, in particular $v = iR$.

ii. A *capacitor* is a device where the current through the device is proportional to the rate of change of charge held in the device, in particular $i = C\frac{dv}{dt}$.

iii. An *inductor* is a device where the voltage drop across the device is proportional to the rate of change of current passing through it, in particular $v = L\frac{di}{dt}$.

iv. Kirchoff's current law states that the total current entering a node equals the total current leaving the node.

v. Kirchoff's voltage law states that the total voltage drop across a closed loop is zero.



(a) [2 points] Using properties of circuit elements and Kirchoff's current and voltage law's show that the following differential equation is satisfied by the system.

$$v_{in}(t) = LC\frac{d^2}{dt^2}v_{out}(t) + v_{out}(t).$$

Solution: Let $i(t)$ be a common current through the inductor and capacitor, and $v_L(t)$ be the voltage-drop across the inductor. We have

$$i(t) = C\frac{d}{dt}v_{out}(t), \quad v_L(t) = L\frac{d}{dt}v_{out}(t) = LC\frac{d^2}{dt^2}v_{out}(t), \quad v_{in}(t) = v_L(t) + v_{out}(t)$$

yielding the desired result.

(b) [1 point] From the differential equation above, compute $\hat{H}(f)$, the Fourier Transform of the system given by the circuit.

Solution:

$$\hat{V}_{in}(f) = \hat{V}_{out}(f)(1 - 4\pi^2 f^2 LC).$$

Therefore

$$\hat{H}(f) = \frac{1}{1 - 4\pi^2 f^2 LC}.$$

(c) [2 points] Is this system stable? (You have to justify your answer mathematically)

Solution: No. This is not stable. Assume otherwise. If the system was stable, then there is B such that $\int_{-\infty}^{\infty} |h(t)|dt < B$. This would imply that

$$|\hat{H}(f)| \leq \int_{-\infty}^{\infty} |h(t)e^{-j2\pi ft}|dt < B.$$

However, as $\lim_{f \to \frac{1}{2\pi\sqrt{LC}}} |H(f)| = \infty$, the above cannot hold..

Question 4 [12 points]: Discrete Fourier Transform

Find all unit vectors $\begin{bmatrix} x[0] & x[1] & x[2] & x[3] \end{bmatrix}$ in $\mathbb{C}^4$, and $x[0] \neq 0$, that satisfies the Heisenberg's inequality with equality, i.e. support(x)support$(\hat{X}) = 4$.

Solution: We divide into cases: support$(x) = 4$, support$(\hat{X}) = 1$; support$(x) = 2$, support$(\hat{X}) = 2$; and support$(x) = 1$, support$(\hat{X}) = 4$.

Cases 1 and 3 are relatively easy.

Case 1: $\hat{X}[k] = e^{j\theta}$ for some k and is zero otherwise. By IDFT

$$x[n] = \frac{1}{2}\hat{X}[k]e^{j\frac{2\pi nk}{4}} = \frac{1}{2}e^{j\left(\theta + \frac{\pi nk}{2}\right)}, 0 \leq n \leq 3.$$

Case 3: $x[0] = e^{j\theta}$ and is zero otherwise. By IDFT

$$\hat{X}[k] = \frac{1}{2}x[0] = \frac{1}{2}e^{j\theta}, 0 \leq k \leq 3.$$

Case 2: The non-zero values are (consecutive) $x[0], x[1]$, or $x[0], x[2]$, or $x[0], x[3]$

Case 2.1: Assume that they are $x[0], x[1]$. Then observe that

$$\hat{X} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -j \\ 1 & -1 \\ 1 & j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

Observe that any pair of rows is linearly independent. Therefore, if two entries of $\hat{X}$ is zero, then x must be zero. A contradiction. Therefore, this case cannot occur.

Case 2.2: Assume that they are $\boldsymbol{x[0], x[3]}$. Then observe that

$$\hat{\boldsymbol{X}} = \frac{\mathbf{1}}{\mathbf{2}} \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{j} \\ \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \boldsymbol{-j} \end{bmatrix} \begin{bmatrix} \boldsymbol{x[0]} \\ \boldsymbol{x[1]} \end{bmatrix}$$

Observe, again, that any pair of rows is linearly independent. Therefore, if two entries of $\hat{\boldsymbol{X}}$ is zero, then $\boldsymbol{x}$ must be zero. A contradiction. Therefore, this case cannot occur.

Case 2.3: Assume that they are $\boldsymbol{x[0], x[2]}$. Then observe that

$$\hat{\boldsymbol{X}} = \frac{\mathbf{1}}{\mathbf{2}} \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{x[0]} \\ \boldsymbol{x[1]} \end{bmatrix}$$

Then $\hat{\boldsymbol{X}}[\mathbf{0}] = \hat{\boldsymbol{X}}[\mathbf{2}]$ and $\hat{\boldsymbol{X}}[\mathbf{1}] = \hat{\boldsymbol{X}}[\mathbf{3}]$. Thus, either $\hat{\boldsymbol{X}}[\mathbf{0}] = \hat{\boldsymbol{X}}[\mathbf{2}] = \mathbf{0}$, implying $\boldsymbol{x[2]} = \boldsymbol{-x[0]}$; or $\hat{\boldsymbol{X}}[\mathbf{1}] = \hat{\boldsymbol{X}}[\mathbf{3}] = \mathbf{0}$, implying $\boldsymbol{x[2]} = \boldsymbol{x[0]}$.

Thus, the only vectors in this case are of the form $\begin{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\boldsymbol{e^{j\theta}} & \mathbf{0} & \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\boldsymbol{e^{j\theta}} & \mathbf{0} \end{bmatrix}$ or $\begin{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\boldsymbol{e^{j\theta}} & \mathbf{0} & -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\boldsymbol{e^{j\theta}} & \mathbf{0} \end{bmatrix}$.

Question 5 [0 points]: Question - compulsory for elite students; optional for others.

(a) Let $\boldsymbol{\theta = \frac{\pi}{10}}$. Use $\mathbf{sin(2\theta) = cos(3\theta)}$ to argue that

$$\mathbf{4\sin^2(\theta) + 2\sin(\theta) + 1 = 0}$$

Hence, compute the value of $\mathbf{cos(\frac{2\pi}{5})}$.

(b) Using the above, compute the multiplicities of the eigenvalues $\mathbf{1, -1}, \boldsymbol{j, -j}$ for the DFT matrix when $\boldsymbol{N = 5}$.