The Chinese University of Hong Kong,

MIEG 2051/ESTR 2360: Fourier Analysis with Engineering Applications

Lecture Notes

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Contents

Chapter 1

Signals and systems

1.1 Signals

A *signal* is a function of one or multiple variables that describes some changing quantities. For example, a sound signal is a function that maps time to instantaneous intensity; the temperature field in a room is a function that maps position in the room to temperature; a ".jpg" image is a function that maps a quantized pixel location to a value that represents a particular color.

We can classify signals into *continuous-time* signals and *discrete-time* signals based on the size of the domain. If the function is defined on countably many values, such as $x[n], n \in \mathbb{Z}$, then we call it a *discrete-time* signal; if the variable takes uncountably many values, such as $x(t), t \in \mathbb{R}$, then we call it a *continuous-time* signal.

In the above example, sound and temperature are continuous-time signals as the domain varies continuously over time and space. An image is a discrete-time signal whose domain consists of discrete (quantized) values corresponding to pixel locations.

Notation: If not otherwise specified, we use $x(t), x(s), x(\tau), x(\sigma)$ etc. to denote continuous-time signals and use $x[n], x[m], x[k], x[\ell]$ etc. to denote discrete-time signals.

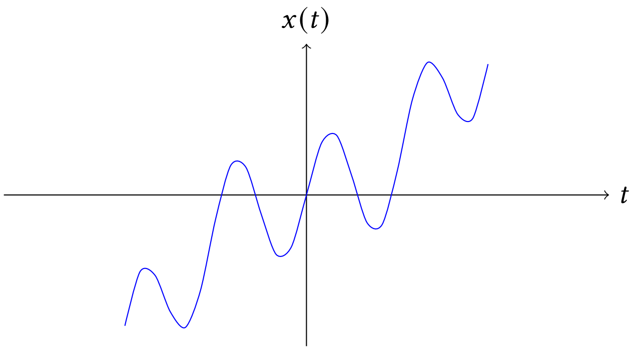


Figure 1.1: A continuous-time signal $x(t)$

Definition 1.1. A *continuous-time* signal is a function over an uncountably infinite space, such as $\mathbb{R}^k$. A *discrete-time* signal is a function over a finite or countably infinite space, such as $\mathbb{Z}^k$. ○

1.2 Systems

A *system* is used to process signals. A system can be imagined as a black box that maps input signals to output signals. Formally, a system is a mapping T from some input signal space $\mathcal{S}_{in}$ to some output signal space $\mathcal{S}_{out}$. If $\mathbf{x} \in \mathcal{S}_{in}$, then the system T induces an output signal $\mathbf{y} = T(\mathbf{x})$, where $\mathbf{y} \in \mathcal{S}_{out}$.

In this course, we do not study the implementation of a system but focus on the relationship between input and output signals, imposed by a system, using mathematical tools. In other words, this course deals with analyzing and understanding system behavior.

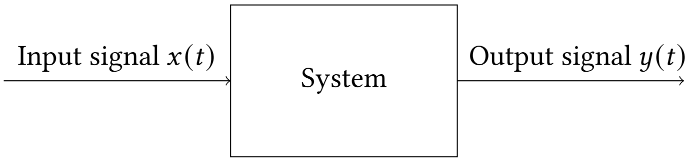


Figure 1.2: A system maps an input signal to an output signal

1.3 Transformation of signals

In this section, we will consider signals that are functions of time (though some of these notions can be applied to other domains as well). Consider a signal depicted in Figure 1.3. Let us explore some basic operations.

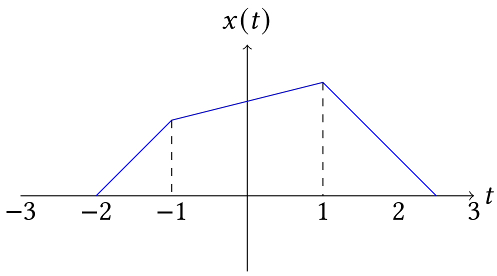


Figure 1.3: A continuous-time signal $x(t)$

1.3.1 Time reversal

The time-reversed signal is given by $y(t) = x(-t)$. The graph of $y(t)$ can be obtained by reflecting the graph of $x(t)$ about the y-axis. See Figure 1.4.

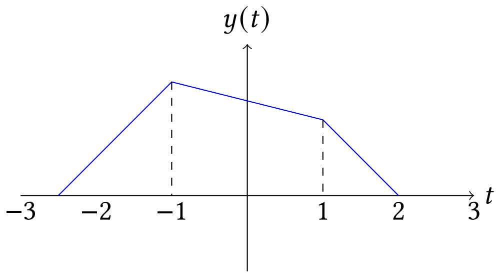


Figure 1.4: $y(t) = x(-t)$

1.3.2 Time scaling

The time-scaled signal is given by $y(t) = x(at)$, where $a > 0$. The graph of $y(t)$ can be obtained by scaling the graph of $x(t)$ along the x-axis, with the point a on the x-axis mapped to the point 1 on the x-axis. Hence it is an expansion when $a < 1$ and it is a compression when $a > 1$. See Figure 1.5 for an example of scaling $x(t)$ by half.

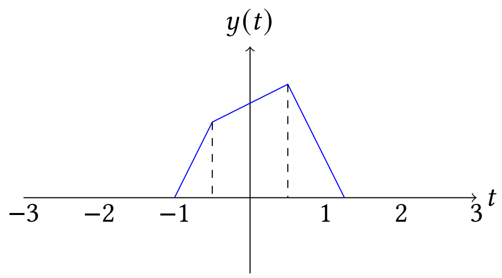


Figure 1.5: $y(t) = x(2t)$

1.3.3 Time shifting

The time-shifted signal is given by $y(t) = x(t - t_0)$, where $t_0 \in \mathbb{R}$. The graph of $y(t)$ can be obtained by shifting the graph of $x(t)$ along the x-axis, with the origin mapped to the point t_0 on the x-axis. Hence it is a shift to the left (time advance) when $t_0 < 0$ and it is a shift to the right (time delay) when $t_0 > 0$. See Figure 1.6 for an example of shifting $x(t)$ to the left by 1 unit.

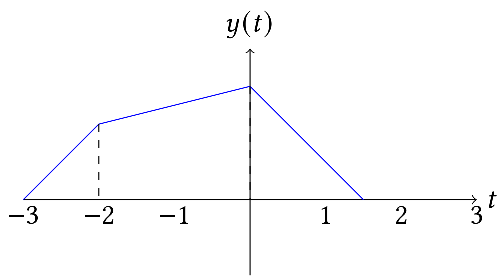


Figure 1.6: $y(t) = x(t + 1)$

1.3.4 Composition of transformations

Suppose we have a series of composition of transformations u, v on $x(t)$ such that the output signal is $y(t) = x(v(u(t)))$. The composition can be understood by applying the transformations one by one, from outer to inner. That is, first find $(x \circ v)(t) = x(v(t))$, then find $(x \circ v \circ u)(t) = (x \circ v)(u(t)) = x(v(u(t)))$.

In the signal $x(t)$ depicted in Figure 1.3, one can view $t = -2, -1, 1, 2$ as *critical points*. A way to understand the composition is to identify the critical points of the new figure. Note that t_0 is a critical point of the new figure if and only if $v(u(t_0)) = t_c$, where t_c is a critical point of the original figure. Thus the new sets of critical points are given by $t_0 = u^{-1}(v^{-1}(t_c))$.

1.4 Properties of signals

1.4.1 Even signals and odd signals

Definition 1.2. A signal $x(t)$ is *even* if $x(-t) = x(t)$ for any t. A signal $x(t)$ is *odd* if $x(-t) = -x(t)$ for any t. ○

Remark 1.1. The *zero signal*, i.e. the signal that equals 0 everywhere, is the only signal that is both even and odd. To see this, if a signal $x(t)$ is both even and odd, then $x(t) = x(-t) = -x(t)$ for all t, which means that $x(t) = 0$ for all t. ○

Proposition 1. Every signal $x(t)$ can be uniquely written as a sum of an even signal $x_{\text{even}}(t)$ and an odd signal $x_{\text{odd}}(t)$, with the decomposition given by

$$x_{\text{even}}(t) = \frac{x(t) + x(-t)}{2},$$
$$x_{\text{odd}}(t) = \frac{x(t) - x(-t)}{2}.$$

○

Proof. It is straightforward to verify that $x(t) = x_{\text{even}}(t) + x_{\text{odd}}(t)$, that $x_{\text{even}}(t)$ is even and that $x_{\text{odd}}(t)$ is odd. It remains to show the uniqueness of such decomposition. Suppose

$$x(t) = \tilde{x}_{\text{even}}(t) + \tilde{x}_{\text{odd}}(t),$$

where $\tilde{x}_{\text{even}}(t)$ is even and $\tilde{x}_{\text{odd}}(t)$ is odd, is another way to write $x(t)$ as a sum of an even signal and an odd signal. Then we also have

$$\begin{aligned} x(-t) &= \tilde{x}_{\text{even}}(-t) + \tilde{x}_{\text{odd}}(-t) \\ &= \tilde{x}_{\text{even}}(t) - \tilde{x}_{\text{odd}}(t). \end{aligned}$$

Combining the above two equations, we have

$$\begin{aligned} \tilde{x}_{\text{even}}(t) &= \frac{(\tilde{x}_{\text{even}}(t) + \tilde{x}_{\text{odd}}(t)) + (\tilde{x}_{\text{even}}(t) - \tilde{x}_{\text{odd}}(t))}{2} \\ &= \frac{x(t) + x(-t)}{2} \\ &= x_{\text{even}}(t), \\ \tilde{x}_{\text{odd}}(t) &= \frac{(\tilde{x}_{\text{even}}(t) + \tilde{x}_{\text{odd}}(t)) - (\tilde{x}_{\text{even}}(t) - \tilde{x}_{\text{odd}}(t))}{2} \\ &= \frac{x(t) - x(-t)}{2} \\ &= x_{\text{odd}}(t). \end{aligned}$$

Hence the decomposition is unique. ■

1.4.2 Periodic signals

Definition 1.3. A signal $x(t)$ is *periodic* if there exists $T \neq 0$ such that $x(t) = x(t + T)$ for all t. Such a T is called a *period* of $x(t)$. ○

Proposition 2. Suppose that T is a period of $x(t)$. Then for any $n \in \mathbb{Z} \setminus \{0\}$, nT is also a period of $x(t)$. ○

Proof. If $n > 0$ then

$$\begin{aligned} x(t + nT) &= x(t + (n-1)T) \\ &= x(t + (n-2)T) \\ &= \dots \\ &= x(t + T) \\ &= x(t) \end{aligned}$$

for all t. Hence nT is a period of $x(t)$. Now, if $n < 0$, since we have just shown that $-nT$ is a period of $x(t)$, we have

$$\begin{aligned} x(t+nT) &= x((t+nT)-nT) \\ &= x(t) \end{aligned}$$

for all t. Hence nT is a period of $x(t)$. ■

Proposition 3. Suppose that T_1 and T_2 are both periods of $x(t)$, where $T_1 + T_2 \neq 0$. Then $T_1 + T_2$ is also a period of $x(t)$. ○

Proof.

$$\begin{aligned} x(t) &= x(t+T_1) \\ &= x(t+T_1+T_2) \end{aligned}$$

for all t. Hence $T_1 + T_2$ is a period of $x(t)$. ■

Remark 1.2. In general, if $T_1, \ldots, T_k$ are periods of $x(t)$ and $n_1, \ldots, n_k \in \mathbb{Z}$, then $\sum_{\ell=1}^{k} n_\ell T_\ell$ is also a period of $x(t)$ if it is nonzero. ○

Definition 1.4. The *fundamental period* of a periodic signal $x(t)$ is defined as

$$\inf \{T > 0 : T \text{ is a period of } x(t)\}.$$

○

Proposition 4. Let $x(t)$ be a periodic signal with fundamental period T_0. Suppose that $T_0 \neq 0$. Then the set of periods of $x(t)$ is $\{nT_0 : n \in \mathbb{Z} \setminus \{0\}\}$. ○

Proof. We will first show that T_0 is a period of $x(t)$. Suppose otherwise that it is not. By the definition of fundamental period, since $2T_0 > T_0$, we can find a period T_1 with $2T_0 > T_1 \geq T_0$. Now $T_1 > T_0$ since T_0 is not a period. By the definition of fundamental period, we can find a period T_2 with $T_1 > T_2 \geq T_0$. Now $T_1 - T_2$ is a period, but $T_1 - T_2 < 2T_0 - T_0 = T_0$, which contradicts with the definition of fundamental period.

We will then show that if T is a period of $x(t)$ then $\frac{T}{T_0}$ is an integer. Note that $T - \left\lfloor \frac{T}{T_0} \right\rfloor T_0$ is a period if it is nonzero. But

$$\begin{aligned} T - \left\lfloor \frac{T}{T_0} \right\rfloor T_0 &= \left(\frac{T}{T_0} - \left\lfloor \frac{T}{T_0} \right\rfloor \right) T_0 \\ &< T_0. \end{aligned}$$

Hence $T - \left\lfloor \frac{T}{T_0} \right\rfloor T_0$ cannot be a period but must equal 0, which means that $\frac{T}{T_0} = \left\lfloor \frac{T}{T_0} \right\rfloor$ and hence is an integer. ■

Remark 1.3. We have shown that if a periodic signal has a nonzero fundamental period, then the integer multiples of the fundamental period is exactly the full set of periods. Now, if a periodic signal has fundamental period 0, must it be a constant? The answer is no. Consider the *Dirichlet function* $x(t)$ defined as the indicator function of the rationals,

$$x(t) = \begin{cases} 1, & t \in \mathbb{Q}, \\ 0, & t \notin \mathbb{Q}. \end{cases}$$

For any rational number $r \in \mathbb{Q}$, we have $r + t \in \mathbb{Q}$ for any $t \in \mathbb{Q}$ and $r + t \notin \mathbb{Q}$ for $t \notin \mathbb{Q}$. Thus, every nonzero rational number is a period of $x(t)$ and hence $x(t)$ has fundamental period 0.

However, if a periodic signal with fundamental period 0 is continuous at any point, then it must be a constant signal. You will prove this fact in homework. ○

Proposition 5. If a and b are two positive real numbers such that $\frac{a}{b} \notin \mathbb{Q}$ (i.e. their ratio is not a rational number), then for any $\epsilon > 0$, one can find integers k_1, k_2 such that $0 < k_1 a + k_2 b < \epsilon$. ○

Proof. We will outline the argument here without giving all the details. Without loss of generality, let $a > b$. Define a sequence of real numbers as follows: $s_1 = a, s_2 = b$ and

$$s_n = s_{n-2} - \left\lfloor \frac{s_{n-2}}{s_{n-1}} \right\rfloor s_{n-1} \qquad (n \geq 3).$$

It is immediate (by induction) that s_n can be expressed as $s_n = k_{1,n} a + k_{2,n} b$ for some integers $k_{1,n}$ and $k_{2,n}$, and further that s_n is a strictly decreasing sequence bounded below by zero. Further, $s_n \leq s_{n-2} - s_{n-1}$ as $\left\lfloor \frac{s_{n-2}}{s_{n-1}} \right\rfloor \geq 1$. Since s_n is a strictly decreasing sequence bounded below by zero, we know that the sequence converges and let $\lim_{n\to\infty} s_n = s_*$ for some $s_* \geq 0$. To complete the proof, it suffices to show that $s_* = 0$. Suppose $s_* > 0$, then we have $s_n > s_*$ (as s_n is a strictly decreasing sequence) and for any $\epsilon > 0$, there exists N_ϵ such that for all $n > N_\epsilon$, we have $s_n - s_* < \epsilon$. As $s_n - s_{n+1} < s_n - s_*$, we have $s_n - s_{n+1} < \epsilon$. Further, as $s_{n+2} \leq s_n - s_{n+1}$, we have $s_{n+2} < \epsilon$ for any $n > N_\epsilon$. Take $\epsilon = \frac{s_*}{2}$ and we see that for N large enough, $s_* < s_n < \frac{s_*}{2}$, a contradiction. (Question: Which step fails when $s_* = 0$?) ■

Remark 1.4. The sum of two periodic signals is not necessarily periodic. Consider $x(t) = \sin t$ and $y(t) = \sin(\pi t)$. Why? We know that $T_{1,k} := 2k\pi$ are the periods of $x(t)$ and $T_{2,k} := 2k$ are the periods of $y(t)$. Suppose T is a period of $x(t)+y(t)$, then we have $x(t+T)+y(t+T) = x(t)+y(t)$, or in other words, $x(t + T) - x(t) = y(t + T) - y(t) =: g(t)$. Observe that 2π and 2 are both periods of $g(t)$. However, since the ratio of the two periods is irrational, there is a decreasing non-negative sequence of periods, $\hat{T}_n$, of $g(t)$ such that $T_n \to 0$. Since $g(t)$ is continuous, we must have that $g(T)$ is a constant. Further, by considering $t = \frac{\pi}{2}$ and $t = \frac{\pi}{2} - T$, we see that the constant must be zero. Thus $g(t) = 0$, implying that $T \in \{T_{1,k}\}_{k=1}^{\infty} \cap \{T_{2,k}\}_{k=1}^{\infty} = \emptyset$. This yields a contradiction. ○

Corollary 1. Let a and b are two positive real numbers such that $\frac{a}{b} \notin \mathbb{Q}$ (i.e. their ratio is not a rational number). Further, assume a and b be two periods of a function $x(t)$. Then, the fundamental period of $x(t) = 0$. ○

Proof. We know that if a and b are periods of $x(t)$, then so is $ma + nb$ for any $m, n \in \mathbb{Z}$. For any $\epsilon > 0$, we know from Proposition 5 that there exists integers m_ϵ, n_ϵ such that $0 < m_\epsilon a + n_\epsilon b < \epsilon$. Therefore $T_0 \leq \epsilon$ for every $\epsilon > 0$. Hence, $T_0 = 0$. ■

1.5 Special signals

1.5.1 Complex exponential signal

The *complex exponential signal* is the continuous-time signal

$$t \mapsto e^{j2\pi f t},$$

where $f \in \mathbb{R}$. The parameter f is usually called the *frequency*. Some authors may alternatively parameterize with $\omega := 2\pi f$ which is usually called the *angular frequency*.

For $f \in \mathbb{R} \setminus \{0\}$, the complex exponential signal $t \mapsto e^{j2\pi ft}$ is periodic with fundamental period $\frac{1}{|f|}$. To see this, note that T is a period of $e^{j2\pi ft}$ if and only if $e^{j2\pi f(t+T)} = e^{j2\pi ft}$ for all $t \in \mathbb{R}$, which is equivalent to that $e^{j2\pi fT} = 1$, or equivalently $fT \in \mathbb{Z}$. Hence, the smallest positive period is $\frac{1}{|f|}$ (note that f can be negative).

In acoustics, $|f|$ is often called the *fundamental frequency* and $k|f|$ ($k = 2, 3, 4, \ldots$) is often called the *harmonics*.

Note that the real part and imaginary part of the complex exponential are the sinusoidal functions,

$$\begin{aligned} \mathrm{Re}(e^{j2\pi ft}) &= \cos 2\pi ft, \\ \mathrm{Im}(e^{j2\pi ft}) &= \sin 2\pi ft, \end{aligned}$$

and that the sinusoidal functions themselves are linear combinations of complex exponentials,

$$\begin{aligned} \sin 2\pi ft &= \frac{e^{j2\pi ft} - e^{-j2\pi ft}}{2j}, \\ \cos 2\pi ft &= \frac{e^{j2\pi ft} + e^{-j2\pi ft}}{2}. \end{aligned}$$

The *discrete-time complex exponential signal* $n \mapsto e^{j2\pi fn}$ is the sampling of the continuous-time complex exponential signal $e^{j2\pi ft}$ at $t \in \mathbb{Z}$. However, it is not periodic unless f is rational.

1.5.2 Delta impulse

We define the *discrete-time delta impulse signal* $\delta[n]$ as

$$\delta[n] := \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

The set of shifted delta impulses $\{n \mapsto \delta[n-m]\}_{m=-\infty}^{+\infty}$ forms a basis for the space of discrete-time signals, since any discrete-time signal $x[n]$ can be written as a linear combination of shifted delta impulses as

$$x[n] = \sum_{m=-\infty}^{+\infty} x[m]\delta[n-m].$$

A defining property of $\delta[n]$ is that it is the only discrete-time signal such that

$$\sum_{n=-\infty}^{+\infty} x[n]\delta[n] = x[0]$$

for any discrete-time signal $x[n]$. We want to mimic the definition of delta impulse in continuous-time, that is, we want to find a continuous-time signal $\delta(t)$ such that

$$\int_{-\infty}^{+\infty} x(t)\delta(t)dt = x(0)$$

for any continuous-time signal $x(t)$.

Definition 1.5. The (continuous-time) *Dirac delta impulse* $\delta(t)$ is defined as a linear map from the set of all complex-valued functions that are continuous at $t = 0$ to their value at $t = 0$. That is,

$$\delta\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\delta(t)dt := x(0)$$

whenever $x(t)$ is continuous at $t = 0$. ○

Remark 1.5. Linearity of integration yields that

$$\begin{aligned} \delta\{ax(t) + by(t)\} &= ax(0) + by(0) \\ &= a \cdot \delta\{x(t)\} + b \cdot \delta\{y(t)\}. \end{aligned}$$

Hence $\delta\{x(t)\}$ is indeed a linear map. ○

Remark 1.6. For convenience, we often simply write

$$\int_{-\infty}^{+\infty} x(t)\delta(t)dt = x(0).$$

○

Remark 1.7. Similarly to the discrete-time setting, a continuous-time signal $x(t)$ can be expressed as a linear combination of shifted delta impulses as

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau.$$

To see this, consider

$$\begin{aligned} \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau &= \int_{-\infty}^{+\infty} x(t - \sigma)\delta(\sigma)d\sigma \qquad \text{(with } \sigma := t - \tau\text{)} \\ &= x(t). \end{aligned}$$

○

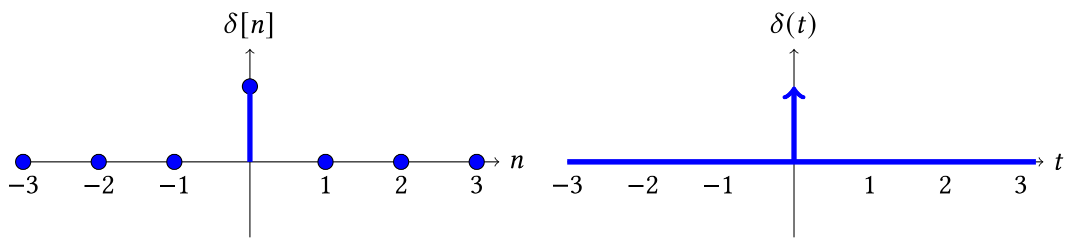


Figure 1.7: $\delta[n]$ and $\delta(t)$

We may approximate the Dirac delta signal by a sequence of signals $\{\delta_n(t)\}_{n=1}^{\infty}$. There are multiple ways to construct the sequence $\delta_n(t)$. For example, we can choose the sequence $\delta_n(t) = n \cdot \text{rect}(nt)$, where the rectangular signal is defined by

$$\text{rect}(t) := \begin{cases} 1, & t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \frac{1}{2}, & t \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 6. Let $\delta_n(t) := n \cdot \text{rect}(nt)$. Suppose that $x(t)$ is continuous at $t = 0$. Then

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} x(t)\delta_n(t)dt = x(0).$$

○

Proof. Let $\epsilon > 0$. Observe that $\int_{-\infty}^{+\infty} \delta_n(t)dt = 1$. Since $x(t)$ is continuous at $t = 0$, there exists $\delta > 0$ such that $|x(t) - x(0)| < \epsilon$ for any $|t - 0| < \delta$. Let N be the smallest positive integer such that $\frac{1}{2N} < \delta$. Then, for any $n \geq N$, we have,

$$\begin{aligned}
\left|\int_{-\infty}^{+\infty} x(t)\delta_n(t)dt - x(0)\right| &= \left|\int_{-\infty}^{+\infty} (x(t) - x(0))\delta_n(t)dt\right| \\
&\leq \int_{-\infty}^{+\infty} |(x(t) - x(0))\delta_n(t)|\, dt \\
&= \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n|x(t) - x(0)|dt \\
&\leq \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\epsilon dt \\
&= \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} x(t)\delta_n(t)dt = x(0).$$

■

Remark 1.8. Although $\delta(t)$ is not a signal in the traditional sense, we will still treat it like a signal and derive the properties of $\delta(t)$. These notions can be made precise, but that is beyond the scope of this class. ○

Proposition 7. $\delta(t)$ is an even signal. ○

Proof. For any test signal $x(t)$ continuous at $t = 0$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} x(t)\delta(-t)dt &= \int_{-\infty}^{+\infty} x(-\tau)\delta(\tau)d\tau && \text{(with } \tau := -t\text{)} \\
&= x(0) \\
&= \int_{-\infty}^{+\infty} x(t)\delta(t)dt.
\end{aligned}$$

Since $x(t)$ is arbitrary, we have that $\delta(t) = \delta(-t)$. ■

Remark 1.9. You may be curious why we need to introduce a test signal $x(t)$ to prove that $\delta(t)$ is even. It is tempting to claim directly the parity of $\delta(t)$ from its definition. However, $\delta(t)$ is never a valid signal that you can manipulate solely. Therefore, if you want to claim any property associated with $\delta(t)$ (e.g., parity, derivative), you have to check whether it holds for any test signal $x(t)$.

The rigorous formulation of $\delta(t)$ is out of the scope of this class. ○

Proposition 8. $\delta(at) = \frac{1}{|a|}\delta(t)$ for any $a \in \mathbb{R} \setminus \{0\}$. ○

Proof. For any test signal $x(t)$ continuous at $t = 0$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} x(t)\delta(at)dt &= \int_{-\infty}^{+\infty} x\left(\frac{\tau}{a}\right)\delta(\tau)\cdot\frac{1}{|a|}d\tau \qquad \text{(with } \tau := at\text{)} \\
&= \frac{1}{|a|}\int_{-\infty}^{+\infty} x\left(\frac{\tau}{a}\right)\delta(\tau)d\tau \\
&= \frac{1}{|a|}x(0) \\
&= \frac{1}{|a|}\int_{-\infty}^{+\infty} x(t)\delta(t)dt \\
&= \int_{-\infty}^{+\infty} x(t)\left(\frac{1}{|a|}\delta(t)\right)dt.
\end{aligned}$$

Since $x(t)$ is arbitrary, we have that $\delta(at) = \frac{1}{|a|}\delta(t)$. ■

Proposition 9. $x(t)\delta(t) = x(0)\delta(t)$ for any signal $x(t)$ continuous at $t = 0$. ○

Proof. For any test signal $y(t)$ continuous at $t = 0$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} y(t)x(t)\delta(t)dt &= y(0)x(0) \\
&= \int_{-\infty}^{+\infty} y(t)x(0)\delta(t)dt.
\end{aligned}$$

Since $y(t)$ is arbitrary, we have that $x(t)\delta(t) = x(0)\delta(t)$. ■

Proposition 10. The derivative of $\delta(t)$ satisfies

$$\int_{-\infty}^{+\infty} x(t)\delta'(t)dt = -x'(0),$$

for any signal $x(t)$ differentible at $t = 0$. ○

Proof. Integration by parts gives that

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \frac{d}{dt}\left(x(t)\delta(t)\right)dt \\
&= \int_{-\infty}^{+\infty} \left(x'(t)\delta(t) + x(t)\delta'(t)\right)dt \\
&= x'(0) + \int_{-\infty}^{+\infty} x(t)\delta'(t)dt.
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{+\infty} x(t)\delta'(t)dt = -x'(0).$$

■

Proposition 11. The n-th derivative of $\delta(t)$ satisfies

$$\int_{-\infty}^{+\infty} x(t)\delta^{(n)}(t)dt = (-1)^n x^{(n)}(0),$$

for any signal $x(t)$ n-times differentible at $t = 0$. ○

Proof. Integration by parts gives that

$$\begin{aligned}0 &= \int_{-\infty}^{+\infty} \frac{d}{dt}\left(x(t)\delta^{(n-1)}(t)\right) dt \\ &= \int_{-\infty}^{+\infty} \left(x'(t)\delta^{(n-1)}(t) + x(t)\delta^{(n)}(t)\right) dt \\ &= \int_{-\infty}^{+\infty} x'(t)\delta^{(n-1)}(t)dt + \int_{-\infty}^{+\infty} x(t)\delta^{(n)}(t)dt.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{-\infty}^{+\infty} x(t)\delta^{(n)}(t)dt &= \int_{-\infty}^{+\infty} -x'(t)\delta^{(n-1)}(t)dt \\ &= \int_{-\infty}^{+\infty} x''(t)\delta^{(n-2)}(t)dt \\ &= \int_{-\infty}^{+\infty} -x'''(t)\delta^{(n-3)}(t)dt \\ &= \dots \\ &= \int_{-\infty}^{+\infty} (-1)^{n-1}x^{(n-1)}(t)\delta'(t)dt \\ &= \int_{-\infty}^{+\infty} (-1)^{n}x^{(n)}(t)\delta(t)dt \\ &= (-1)^n x^{(n)}(0).\end{aligned}$$

■

1.5.3 Unit step signal

The continuous-time *unit step signal* $u(t)$ is defined as

$$u(t) := \begin{cases} 0, & t < 0, \\ \frac{1}{2} & t = 0, \\ 1, & t > 0. \end{cases}$$

In general, we do not care about the values at $t = 0$.

Proposition 12.

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

In other words, $u(t)$ is the indefinite integral of $\delta(t)$. ○

Proof.

$$\begin{aligned}\int_{-\infty}^{t} \delta(\tau)d\tau &= \int_{-\infty}^{+\infty} u(t-\tau)\delta(\tau)d\tau \\ &= u(t).\end{aligned}$$

■

Remark 1.10. Hence $u'(t) = \delta(t)$. A useful application of this is to calculate the derivative of $x(t)u(t)$:

$$\begin{aligned}\frac{d}{dt}(x(t)u(t)) &= x'(t)u(t) + x(t)u'(t) \\ &= x'(t)u(t) + x(t)\delta(t).\end{aligned}$$

○

We define the *discrete-time unit step signal* analogously as

$$u[n] := \begin{cases} 1, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

1.5.4 Rectangular signal

The continuous-time *rectangular signal* rect(t) is defined as

$$\text{rect}(t) := \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2}, \\ \frac{1}{2} & t = -\frac{1}{2} \text{ or } t = \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

In general, we do not care about the values at $t \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

Remark 1.11. We can write rect(t) in terms of $u(t)$ as

$$\text{rect}(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right),$$

and we can write $u(t)$ in terms of rect(t) as

$$u(t) = \sum_{n=0}^{+\infty} \text{rect}\left(t - n - \frac{1}{2}\right).$$

○

Exercise 1. Express rect$(2t)$ as a linear combination of rect(t) and its shifts only. ○

Solution.

$$\text{rect}(2t) = \text{rect}\left(t - \frac{1}{4}\right) + \sum_{n=1}^{+\infty} \left(\text{rect}\left(t - n - \frac{1}{4}\right) - \text{rect}\left(t - n + \frac{1}{4}\right)\right).$$

♣

1.6 Properties of systems

Suppose that a system produces an output signal $y(t)$ from an input signal $x(t)$. The value of the output signal at a certain time t, $y(t)$, is completely determined by the time t and the input signal itself $\{x(\tau)\}_{\tau\in\mathbb{R}}$. The input signal itself can further be divided into its past $\{x(\tau)\}_{\tau<t}$, its present $x(t)$ and its future $\{x(\tau)\}_{\tau>t}$. Considering this gives three properties of interest of a system:

- If the output does not depend on the past input $\{x(\tau)\}_{\tau<t}$ nor the future input $\{x(\tau)\}_{\tau>t}$, i.e. $y(t)$ is completely determined by the current time t and the current input $x(t)$, then the system is said to be *memoryless*.
- If the output does not depend on the future input $\{x(\tau)\}_{\tau>t}$, i.e. $y(t)$ is completely determined by the current time t and the input up until now $\{x(\tau)\}_{\tau\leq t}$, then the system is said to be *causal*.
- If the system is agnostic about the current time t, i.e. $y(t)$ is completely determined by $\{x(\tau)\}_{\tau\in\mathbb{R}}$, then the system is said to be *time-invariant*.

In particular, one may consider memoryless systems as those that can be represented by a family of functions $\{f_t(\cdot)\}_{t\in\mathbb{R}}$ indexed by time, with the relationship between input and output given by $y(t) = f_t(x(t))$. Furthermore, memoryless time-invariant systems are those that can be represented by a function f, with the relationship between input and output given by $y(t) = f(x(t))$.

Other properties of interest of a system are:

- If the output is linear in the input $\{x(\tau)\}_{\tau\in\mathbb{R}}$, then the system is said to be *linear*.
- If the output is bounded whenever the input is bounded, then the system is said to be *stable*.
- If the input can be reconstructed from the output, then the system is said to be *invertible*.

1.6.1 Memorylessness

Definition 1.6. A system is *memoryless* if, for any t, $x_1(t) = x_2(t)$ implies $y_1(t) = y_2(t)$, where $y_i(t)$ is the output of the system when the input is $x_i(t)$ ($i = 1, 2$). In other words, if two inputs agree at a certain time, then their outputs must also agree at that time. ○

Example 1. The system defined by $y(t) = tx(t)^2$ is memoryless.

1.6.2 Causality

Definition 1.7. A system is *causal* if, for any t, ($x_1(\tau) = x_2(\tau)$ for any $\tau \leq t$) implies ($y_1(\tau) = y_2(\tau)$ for any $\tau \leq t$), where $y_i(t)$ is the output of the system when the input is $x_i(t)$ ($i = 1, 2$). In other words, if two inputs agree until a certain time, then their outputs must also agree until that time. ○

Remark 1.12. Memoryless systems are also causal. ○

Example 2. The system defined by $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ is causal.

1.6.3 Time-invariance

Definition 1.8. A system is *time-invariant* if, for any t_0, the time-shifted input $x(t-t_0)$ produces the time-shifted output $y(t - t_0)$, where $y(t)$ is the output of the system when the input is $x(t)$. ○

Example 3. The system defined by $y(t) = \int_{t-2}^{t-1} x(\tau)d\tau$ is time-invariant.

Example 4. The system defined by $y(t) = \int_{0}^{\max\{0,t\}} x(\tau)d\tau$ is not time-invariant.

1.6.4 Linearity

Definition 1.9. A system is *linear* if the input $ax_1(t)+bx_2(t)$ produces the output $ay_1(t)+by_2(t)$ for any a, b, where $y_i(t)$ is the output of the system when the input is $x_i(t)$ $(i = 1, 2)$. In other words, the system is a linear map between signal spaces. ○

Remark 1.13. In this class, we would also impose the condition that a linear system must have the following property: If $\{x_\tau(t)\}_{\tau\in\mathbb{R}}$ is a collection of signals and $a(\tau)$ is a function such that $x(t) = \int_{-\infty}^{+\infty} a(\tau)x_\tau(t)d\tau$ is well defined, then the output of $x(t)$ should be $y(t) = \int_{-\infty}^{+\infty} a(\tau)y_\tau(t)d\tau$, where $y_\tau(t)$ is the output of the system when the input is $x_\tau(t)$. ○

Example 5. The system defined by $y(t) = t^2x(t)$ is linear. Note that a linear system does not mean that the system is linear in t, but that the system is linear in $x(t)$.

1.6.5 Stability

Definition 1.10. A signal is *bounded* if there exists some finite value B such that $|x(t)| \leq B$ for all t. A system is *stable* if bounded input signals produce bounded output signals. ○

Example 6. The system defined by $y(t) = e^{x(t)}$, $y(t) = \int_{-1}^{1} x(t)dt$ is stable.

Example 7. The system defined by $y(t) = tx(t)$ is not stable.

1.6.6 Invertibility

Definition 1.11. A system is *invertible* if $(y_1(t) = y_2(t)$ for any $t)$ implies $(x_1(t) = x_2(t)$ for any $t)$, where $y_i(t)$ is the output of the system when the input is $x_i(t)$ $(i = 1, 2)$. In other words, same output signals must be produced by same input signals. In other words, the system is a injection between signal spaces. ○

Example 8. The system defined by $y(t) = \frac{x(t)}{|x(t)|+1}$ is invertible (with inverse explicitly given by $x(t) = \frac{y(t)}{1-|y(t)|}$).

1.7 Examples of systems

Example 1. Suppose the system is $y(t) = x^2(t + 1)$. Then it is

- Not memoryless: $y(t)$ depends on $x(t + 1)$, a future time.
- Not causal: $y(t)$ depends on $x(t + 1)$, a future time.
- Stable: Let $|x(t)| < M$ be a bounded input. Then $y(t) < M^2$.
- Not invertible: Let $x_1(t) = 1$ and $x_2(t) = -1$. We have $y_1(t) = y_2(t) = 1$, but $x_1(t) \neq x_2(t)$. (Distinct inputs produce the same output.)
- Not linear: The output of $x(t) = ax_1(t) + bx_2(t)$ is $(ax_1(t + 1) + bx_2(t + 1))^2 \neq ax_1(t + 1)^2 + bx_2(t + 1)^2$.
- Time invariant: The output of $x_1(t) = x(t - t_0)$ is $x_1^2(t + 1) = x(t - t_0 + 1)^2 = y(t - t_0)$.

Example 2. Suppose the system is $y(t) = t|x(t - 1)|$. Then it is

- Not memoryless: $y(t)$ depends on $x(t-1)$, a past time.
- Causal: $y(t)$ depends only on $x(t-1)$.
- Not stable: Let $x(t) = 1$. This is a bounded signal. Then observe that $\sup_y |y(t)| = \sup_t |t| = \infty$, hence $y(t)$ is not bounded.
- Not invertible: Let $x_1(t) = 1$ and $x_2(t) = -x_1(t) = -1$. We have $y_1(t) = y_2(t) = t$ for any t, but $x_1(t) \neq x_2(t)$.
- Not Linear: Let $x_2(t) = -x_1(t)$. Then the output of $x_1(t) + x_2(t)$ is 0 for any t and not in general equal to $2t|x_1(t-1)|$, which is $y_1(t) + y_2(t)$.
- Not time invariant: The output of $x_1(t) = x(t-t_0)$ is $t|x_1(t-1)| = t|x_1(t-t_0-1)| \neq (t-t_0)|x(t-t_0-1)| = y(t-t_0)$.

Example 3. Suppose the system is $y(t) = \frac{x(t)}{\ln(1+|x(t)|)}$. Then it is

- Memoryless: Set $f_t(x) = f(x) = \frac{x}{\ln(1+|x|)}$ to be the mapping from the input value at time t to the output value at time t.
- Causal: As the system is memoryless, it is also causal (as memoryless systems are a subset of causal systems).
- Stable: The function $f(x) = \frac{x}{\ln(1+|x|)}$ is strictly increasing on the reals (please show this). Therefore, $|x(t)| \leq B\, \forall t$ implies that $|y(t)| \leq \frac{B}{\ln(1+B)}\ \forall t$. Hence a bounded input produces a bounded output, implying stability.
- Invertible: This again follows from the fact that $f(x) = \frac{x}{\ln(1+|x|)}$ is strictly increasing on the reals. If $x_1(t_0) \neq x_2(t_0)$, then we have $y_1(t_0) = \frac{x_1(t_0)}{\ln(1+|x_1(t_0)|)} \neq \frac{x_2(t_0)}{\ln(1+|x_2(t_0)|)} = y_2(t_0)$. This implies that different inputs produce different outputs.
- Not Linear: Let $x_1(t) = 1,\ \forall t$. Then $y_1(t) = \frac{1}{\ln 2}$. Let $x_2(t) = 2,\ \forall t$. Then $y_1(t) = \frac{2}{\ln 3}$. Now set $x(t) = x_1(t) + x_2(t) = 3 \forall t$. Then we have $y(t) = \frac{3}{\ln 4} \neq \frac{1}{\ln 2} + \frac{2}{\ln 3}$.
- Time invariant: The output of $x_1(t) = x(t-t_0)$ is $\frac{x_1(t)}{\ln(1+|x_1(t)|)} = \frac{x(t-t_0)}{\ln(1+|x(t-t_0)|)} = y(t-t_0)$.

Example 4. Suppose the system is $y(t) = \int_{-\infty}^{t} \frac{|x(\tau)|}{1+\tau^2} d\tau$. Then it is

- Not memoryless: $y(t)$ depends on $x(\tau)$ for $-\infty < \tau < t$.
- Causal: $y(t)$ depends only on $x(\tau)$ for $-\infty < \tau < t$.
- Stable: Suppose $|x(t)| \leq B,\ \forall t$. Then $|y(t)| \leq \int_{-\infty}^{t} \frac{B}{1+\tau^2} d\tau \leq \int_{-\infty}^{+\infty} \frac{B}{1+\tau^2} d\tau = B\pi < \infty$.
- Not Invertible: Omitted.
- Not Linear: Omitted.
- Not Time Invariant: Omitted.

Exercises

1. Find the even and odd decomposition for the following signals and plot the even and odd signals

   (a) $x(t) = \begin{cases} t, & 0 \leq t \leq 7 \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = 4e^{-2t}u(t)$. Here $u(t)$ is the unit-step function defined above.

(c) $x[n] = \frac{1}{\pi n}\sin(\pi n)$. Take $x[0] = 1$.

2. Are the following signals periodic? If yes, determine the fundamental period.

   (a) $x(t) = \begin{cases} \cos(t + \frac{\pi}{4}), & t \leq 0 \\ \sin(t + \frac{\pi}{4}), & t \geq 0 \end{cases}$

   (b) $x[n] = \sin(\frac{6\pi}{7}n + 1)$

   (c) $x[n] = \cos\frac{\pi n}{2} + \cos\frac{\pi n}{4}$

   (d) (i) $x_1[n] = \frac{1}{2}\cos\frac{(\pi+2)n}{2}$

   (ii) $x_2[n] = \cos\frac{(\pi+1)n}{2}\cos\frac{n}{2}$

   (iii) $x[n] = x_1[n] - x_2[n] = \frac{1}{2}\cos\frac{(\pi+2)n}{2} - \cos\frac{(\pi+1)n}{2}\cos\frac{n}{2}$

3. *Frequently used signals*

   (a) Unit impulse function $\delta(t)$.

   Definition: Any function $f(t)$ that is continuous at t_0 obeys

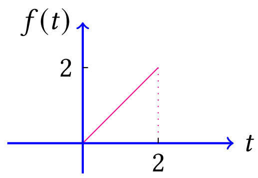
$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\mathrm{d}t = f(t_0).$$

   Compute the following integrals

   (i) $\int_{-\infty}^{\infty} e^{-at}\delta(t-5)\mathrm{d}t$

   (ii) $\int_{-\infty}^{\infty} \cos(2\pi f_0 t)\delta(t + \frac{1}{f_0})\mathrm{d}t$

   (b) Unit step function $u(t)$. Consider $f(t)$ in the following figure



   (i) Use $u(t)$ (and shifted versions) to write a mathematical expression (hint: using integration) for $f(t)$

   (ii) Compute and plot the derivative $\frac{\mathrm{d}f(t)}{\mathrm{d}t}$ (Remark: In this class, differentiation of functions that are not continuous produces δ-functions. For instance $\frac{d}{dt}u(t) = \delta(t)$.)

   (c) Discrete cases: $u[n]$ and $\delta[n]$. Consider the signal $x[n] = a^n u[n]$

   (i) Find and plot signal $g[n] = x[n] - ax[n-1]$

   (ii) Use signal $g[n]$ to express $x[n]$

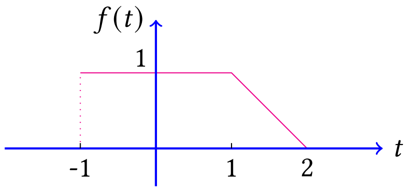
4. *Properties of systems*: Determine if each of the systems is: linear, causal, time-invariant, memoryless, invertible, or stable.

   (a) $y(t) = e^{x(t)}$

   (b) $y[n] = x[n]x[n-1]$

(c) $y[n] = x[2n]$

5. Given $f(t)$ shown below, plot $g(t) = 2f(5-2t)$ by using the following intermediate steps: Plot each of the intermediate figures $f_1(t), f_2(t), f_3(t)$ and finally $g(t)$.



(1) Perform a shift to obtain $f_1(t) = f(t+5)$.

(2) Perform a reflection to obtain $f_2(t) = f_1(-t) = f(-t+5)$.

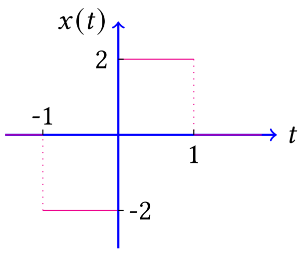
(3) Perform a scaling of time to obtain $f_3(t) = f_2(2t) = f_1(-2t) = f(-2t+5)$.

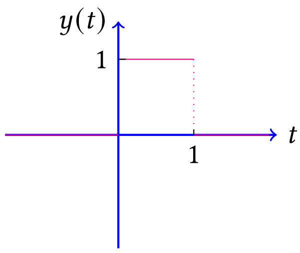
(4) Finally perform the amplification to obtain $g(t) = 2f_3(t) = 2f_2(2t) = 2f_1(-2t) = 2f(5-2t)$.

Questions to think about [no need to include in your solutions]: Can I also do this in a different orders? If so, what other orders? Can one say something more about the various possibilities; e.g. the location of the shifting operation w.r.t to reflection and scaling?

6. Suppose a continuous signal $x(t)$ has periods $T = 1$ and $T = \sqrt{2}$. Then argue that $x(t)$ must be a constant.

7. Write the signal $y(t)$ as a linear combination of the signal $x(t)$ and its time shifts.





8. **(22 final exam)** Consider the following continuous-time system with $x(t)$ and $y(t)$ being the input and output of the system, respectively. Determine whether the system is (1) Memoryless, (2) Time-invariant, (3) Linear, (4) Causal, (5) Stable, and (6) Invertible. Justify your answer for each property. (Marking: 1 mark for each property and 1 mark bonus if all six properties are correct. If you do not justify your answers or if your justification is wrong, you will not get any mark for that property.)

System: Let us assume that all the input signals are continuously differentiable and tend to zero as $|t| \to \infty$.

$$y(t) = \int_{t-2}^{t-1} x(\tau+2)d\tau.$$

For invertibility, you may assume there exists t_0, such that all inputs satisfy $x(t) = 0$ for $t < t_0$, and that they are continuously differentiable signals.

9. **(22 midterm 1 make-up)** Consider the following systems with $x(t)$ and $y(t)$ being the input and output of the system, respectively. Determine whether they are (1) Memoryless,

(2) Time-invariant, (3) Linear, (4) Causal, (5) Stable, and (6) Invertible. Justify your answer for each property. (Marking: 1 mark for each property. If you do not justify your answers or if your justification is wrong, you will not get any mark for that property.)

System 1: $y[n] = \sum_{m=-2|n|}^{n} x[m]e^{-|m|}$.

System 2:

$$y(t) = \frac{1}{|t|+1}\log\left(1 + \left|\int_0^{|t|} x(\tau)d\tau\right|\right).$$

10. **(22 midterm 1)** Consider the following systems with $x(t)$ and $y(t)$ being the input and output of the system, respectively. Determine whether they are (1) Memoryless, (2) Time-invariant, (3) Linear, (4) Causal, (5) Stable, and (6) Invertible. Justify your answer for each property. (Marking: 1 mark for each property. If you do not justify your answers or if your justification is wrong, you will not get any mark for that property.)

    System 1: A continuous-time system

    $$y(t) = \int_{t-3}^{t-2} x(\tau^2)d\tau.$$

    System 2: A discrete-time system

    $$y[n] - ay[n-1] = x[n]$$

    where $d \in \mathbb{N}$ and $0 < |a| < 1$. You can restrict the space of input signals to be $\mathcal{I} := \{\mathbf{x} : x[n] \to 0 \text{ as } n \to -\infty.\}$. For invertibility, you have to determine if any two distinct inputs that are both in $\mathcal{I}$ will lead to distinct outputs.

11. *Properties of systems*: For each statement, state if they are **true** or **false**:

    (1) A memoryless system is definitely a causal system.

    (2) A causal system is definitely a memoryless system.

    (3) A system is invertible if distinct inputs lead to distinct outputs.

    (4) A system is time invariant if a time shift in the input signal produces no change in the output signal.

    (5) Given a zero input to a linear system, it is possible to have a nonzero output.

12. Prove that 2π is the fundamental period of $x(t) = \sin t$.

Chapter 2

LTI systems

The *impulse response* of a system is the output signal when the delta impulse $\delta(t)$ is applied at the input. We will soon see that *linear time-invariant* (LTI) systems are a class of systems that are completely characterized by their impulse response. Consider an LTI system with impulse response $h(t)$. The time-invariant property of the system implies that the output produced by the shifted delta impulse $\delta(t-\tau)$ is $h(t-\tau)$. Recall that any signal $x(t)$ can be written a linear combination of $\delta(t)$ and its shifts as

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau.$$

Hence, the linearity property of the system implies that the output $y(t)$ produced by $x(t)$ is

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau.$$

Analogously, in the discrete-time setting, the *impulse response* of a discrete-time system is the output signal when the discrete-time delta impulse $\delta[n]$ is applied at the input. For a discrete-time LTI system with impulse response $h[n]$, since any discrete-time signal $x[n]$ can be written as a linear combination of $\delta[n]$ and its shifts as

$$x[n] = \sum_{m=-\infty}^{+\infty} x[m]\delta[n-m],$$

the output $y[n]$ produced by the input $x[n]$ is

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]h[n-m].$$

It will be useful to define the *convolution* operation, denoted by $*$. Then an LTI system with impulse response $h(t)$ maps any input signal $x(t)$ to the output signal $x(t) * h(t)$, and a discrete-time LTI system with impulse response $h[n]$ maps any input signal $x[n]$ to the output signal $x[n] * h[n]$.

Definition 2.1. The *convolution* of two signals $x(t), y(t)$ is defined as

$$x(t) * y(t) := \int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau.$$

○

Definition 2.2. The *convolution* of two discrete-time signals $x[n], y[n]$ is defined as

$$x[n] * y[n] := \sum_{m=-\infty}^{+\infty} x[m]y[n-m].$$

○

Proposition 13. A system is LTI if and only if it maps any input signal to the convolution of the input signal and the system's impulse response. ○

2.1 Properties of convolution

Remark 2.1. Most of the following properties of convolution also hold for convolution of discrete-time signals. These properties can be shown, for example, via a similar proof to their continuous-time counterparts, or by considering discrete-time signals as a special kind of continuous-time signals in the form

$$x(t) = \sum_{n=-\infty}^{+\infty} x[n]\delta(t-n),$$

with which one can identify $x(t) * y(t)$ with $x[n] * y[n]$. ○

2.1.1 Commutativity

Proposition 14. For any signals $x(t), y(t)$,

$$x(t) * y(t) = y(t) * x(t).$$

○

Proof.

$$\begin{aligned} x(t) * y(t) &= \int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} x(t-\sigma)y(\sigma)d\sigma && (\text{with } \sigma := t-\tau) \\ &= \int_{-\infty}^{+\infty} y(\sigma)x(t-\sigma)d\sigma \\ &= y(t) * x(t). \end{aligned}$$

■

2.1.2 Associativity

Proposition 15. For any signals $x(t), y(t), z(t)$,

$$x(t) * (y(t) * z(t)) = (x(t) * y(t)) * z(t).$$

○

Proof.

$$\begin{aligned}
x(t) * (y(t) * z(t)) &= \int_{-\infty}^{+\infty} x(\tau)(y * z)(t-\tau)d\tau \\
&= \int_{-\infty}^{+\infty} x(\tau)\left(\int_{-\infty}^{+\infty} y(s)z(t-\tau-s)ds\right)d\tau \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)y(s)z(t-\tau-s)dsd\tau \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)y(\sigma-\tau)z(t-\sigma)d\sigma d\tau && (\text{with } \sigma := \tau + s) \\
&= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} x(\tau)y(\sigma-\tau)d\tau\right)z(t-\sigma)d\sigma \\
&= \int_{-\infty}^{+\infty} (x * y)(\sigma)z(t-\sigma)d\sigma \\
&= (x(t) * y(t)) * z(t).
\end{aligned}$$

■

Remark 2.2. This implies that the impulse response of two cascaded LTI systems is the convolution of their impulse responses. Since convolution is also commutative, the order of cascaded LTI systems can be interchanged without affecting the output signal. ○

Remark 2.3. Since convolution is associative, we can simply write $x(t) * y(t) * z(t)$ without ambiguity. ○

2.1.3 Identity

Proposition 16. For any signal $x(t)$,

$$x(t) = x(t) * \delta(t).$$

○

Proof.

$$\begin{aligned}
x(t) * \delta(t) &= \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \\
&= \int_{-\infty}^{+\infty} x(t-\sigma)\delta(\sigma)d\sigma && (\text{with } \sigma := t - \tau) \\
&= x(t).
\end{aligned}$$

■

Remark 2.4. Since convolution is also associative and commutative, the space of all signals forms an Abelian monoid (i.e. Abelian group without inverses) under convolution, with the delta impulse signal being the identity element. However, convolution is not invertible in general. ○

2.1.4 Linearity

Proposition 17. For any signals $x(t), y(t), h(t)$ and for any $a, b \in \mathbb{C}$,

$$(ax(t) + by(t)) * h(t) = a(x(t) * h(t)) + b(y(t) * h(t)).$$

○

Proof.

$$\begin{aligned}(ax(t) + by(t)) * h(t) &= \int_{-\infty}^{+\infty} (ax(\tau) + by(\tau))h(t-\tau)d\tau \\ &= a\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau + b\int_{-\infty}^{+\infty} y(\tau)h(t-\tau)d\tau \\ &= a(x(t) * h(t)) + b(y(t) * h(t)).\end{aligned}$$

■

2.1.5 Time-invariance

Proposition 18. Suppose that $y(t) = x(t) * h(t)$. Then $y(t - t_0) = x(t - t_0) * h(y)$. ○

Proof.

$$\begin{aligned}x(t-t_0) * h(t) &= \int_{-\infty}^{+\infty} x(\tau - t_0)h(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} x(\sigma)h(t - t_0 - \sigma)d\sigma \qquad \text{(with } \sigma := \tau - t_0\text{)} \\ &= y(t - t_0).\end{aligned}$$

■

Remark 2.5. We have seen that any LTI system must be of the form $x(t) \mapsto x(t) * h(t)$ where $h(t)$ is the impulse response. Now we have shown that for any signal $h(t)$, the system defined by $x(t) \mapsto x(t) * h(t)$ is indeed an LTI system. ○

2.1.6 Derivative

Proposition 19. For any signals $x(t), y(t)$,

$$\begin{aligned}\frac{d}{dt}\left(x(t) * y(t)\right) &= \left(\frac{d}{dt}x(t)\right) * y(t) \\ &= x(t) * \left(\frac{d}{dt}y(t)\right).\end{aligned}$$

○

Proof.

$$\begin{aligned}\frac{d}{dt}(x(t) * y(t)) &= \frac{d}{dt}\int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} x(\tau)\frac{d}{dt}y(t-\tau)d\tau\end{aligned}$$

$$= x(t) * \left(\frac{d}{dt}y(t)\right).$$

By commutativity we also have

$$\begin{aligned}\frac{d}{dt}(x(t) * y(t)) &= \frac{d}{dt}(y(t) * x(t)) \\ &= y(t) * \left(\frac{d}{dt}x(t)\right) \\ &= \left(\frac{d}{dt}x(t)\right) * y(t).\end{aligned}$$

■

Remark 2.6. Therefore, if $x(t)$ is m-times differentiable and $y(t)$ is n-times differentiable, then $x(t) * y(t)$ is $(m + n)$-times differentiable. ○

2.1.7 Indefinite integral

Proposition 20. For any signals $x(t), y(t)$,

$$\begin{aligned}\int_{-\infty}^{t} (x * y)(\tau)d\tau &= \left(\int_{-\infty}^{t} x(\tau)d\tau\right) * y(t) \\ &= x(t) * \left(\int_{-\infty}^{t} y(\tau)d\tau\right).\end{aligned}$$

○

Proof.

$$\begin{aligned}\int_{-\infty}^{t} (x * y)(\tau)d\tau &= \int_{-\infty}^{t}\left(\int_{-\infty}^{+\infty} x(\sigma)y(\tau-\sigma)d\sigma\right)d\tau \\ &= \int_{-\infty}^{t}\int_{-\infty}^{+\infty} x(\sigma)y(\tau-\sigma)d\sigma d\tau \\ &= \int_{-\infty}^{+\infty} x(\sigma)\left(\int_{-\infty}^{t} y(\tau-\sigma)d\tau\right)d\sigma \\ &= \int_{-\infty}^{+\infty} x(\sigma)\left(\int_{-\infty}^{t-\sigma} y(\tilde{\tau})d\tilde{\tau}\right)d\sigma \qquad (\text{with } \tilde{\tau} := \tau - \sigma) \\ &= x(t) * \left(\int_{-\infty}^{t} y(\tau)d\tau\right).\end{aligned}$$

By commutativity we also have

$$\begin{aligned}\int_{-\infty}^{t} (x * y)(\tau)d\tau &= \int_{-\infty}^{t} (y * x)(\tau)d\tau \\ &= y(t) * \left(\int_{-\infty}^{t} x(\tau)d\tau\right) \\ &= \left(\int_{-\infty}^{t} x(\tau)d\tau\right) * y(t).\end{aligned}$$

■

2.1.8 Area

Proposition 21. For any signals $x(t), h(t)$,

$$\int_{-\infty}^{+\infty} (x(t) * h(t))dt = \left(\int_{-\infty}^{+\infty} x(t)dt\right)\left(\int_{-\infty}^{+\infty} h(t)dt\right).$$

○

Proof.

$$\begin{aligned}
\int_{-\infty}^{+\infty} (x(t) * h(t))dt &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right) dt \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau dt \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau)h(\sigma)d\tau d\sigma \qquad \text{(with } \sigma := t-\tau\text{)} \\
&= \left(\int_{-\infty}^{+\infty} x(\tau)d\tau\right)\left(\int_{-\infty}^{+\infty} h(\sigma)d\sigma\right).
\end{aligned}$$

■

2.2 Examples

Exercise 2. Suppose that the impulse response $h[n]$ of an LTI system is

$$h[n] = \begin{cases} -2, & n = 2, \\ 5, & n = 0, \\ -4, & n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the output signal $y[n]$ when the input signal $x[n]$ is

$$x[n] = \begin{cases} 2, & n = 1, \\ 4, & n = 0, \\ -3, & n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

○

Solution. Note that $x[n]$ can be written as a linear combination of the delta signal and its shifts as

$$x[n] = -3\delta[n+1] + 4\delta[n] + 2\delta[n-1].$$

By linearity and time-invariance,

$$y[n] = -3h[n+1] + 4h[n] + 2h[n-1]$$

$$= \begin{cases} -4, & n = 3, \\ -8, & n = 2, \\ 16, & n = 1, \\ 12, & n = 0, \\ -31, & n = -1, \\ 12, & n = -2, \\ 0, & \text{otherwise.} \end{cases}$$

♣

Exercise 3. Suppose that the impulse response $h(t)$ of an LTI system is

$$h(t) = \begin{cases} t+1, & t \in [-1,0), \\ 1, & t \in [0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Determine the output signal $y(t)$ when the input signal is $\mathrm{rect}(t)$. ○

Solution. We first compute $h(t-\tau)$. We have

$$\begin{aligned} h(t-\tau) &= \begin{cases} 1, & t-\tau \in [0,1), \\ t-\tau+1, & t-\tau \in [-1,0), \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \tau \in (t-1,t], \\ t-\tau+1, & \tau \in (t,t+1], \\ 0, & \text{otherwise} \end{cases} \\ &= \mathrm{rect}\left(\tau - t + \frac{1}{2}\right) + (t-\tau+1)\mathrm{rect}\left(\tau - t - \frac{1}{2}\right). \end{aligned}$$

Then

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} \mathrm{rect}(\tau)\left(\mathrm{rect}\left(\tau - t + \frac{1}{2}\right) + (t-\tau+1)\mathrm{rect}\left(\tau - t - \frac{1}{2}\right)\right) d\tau \\ &= \int_{t-1}^{t} \mathrm{rect}(\tau) d\tau + \int_{t}^{t+1} (t-\tau+1)\mathrm{rect}(\tau) d\tau \\ &= \begin{cases} \frac{1}{2}\left(t+\frac{3}{2}\right)^2, & t \in \left(-\frac{3}{2}, -\frac{1}{2}\right], \\ 1 - \frac{1}{2}\left(\frac{1}{2} - t\right)^2, & t \in \left(-\frac{1}{2}, \frac{1}{2}\right], \\ \frac{3}{2} - t, & t \in \left(\frac{1}{2}, \frac{3}{2}\right), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

♣

Example 1. For an LTI system with impulse response $h(t)$, when the input signal is $u(t)$, the output signal is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} u(t-\sigma)h(\sigma)d\sigma \qquad \text{(with } \sigma := t-\tau\text{)} \\ &= \int_{-\infty}^{t} h(\sigma)d\sigma. \end{aligned}$$

Example 2. We compute the convolution of two rectangular signals,

$$\begin{aligned}\mathrm{rect}(t) * \mathrm{rect}(t) &= \int_{-\infty}^{+\infty} \mathrm{rect}(\tau)\mathrm{rect}(t-\tau)d\tau \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{rect}(t-\tau)d\tau.\end{aligned}$$

Note that $\mathrm{rect}(t-\tau)$ is nonzero when $\tau \in \left(t-\frac{1}{2}, t+\frac{1}{2}\right)$. When $t < -1$, $t - \tau < -\frac{1}{2}$, the integral is 0. When $t > 1$, $t - \tau > \frac{1}{2}$, the integral is 0. When $-1 \le t \le 0$,

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{rect}(t-\tau)d\tau &= \int_{-\frac{1}{2}}^{t+\frac{1}{2}} d\tau \\ &= t+1.\end{aligned}$$

When $0 \le t \le 1$,

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{rect}(t-\tau)d\tau &= \int_{t-\frac{1}{2}}^{\frac{1}{2}} d\tau \\ &= 1-t.\end{aligned}$$

Therefore, $\mathrm{rect}(t) * \mathrm{rect}(t) = \mathrm{tri}(t)$, where $\mathrm{tri}(t)$ is the *triangular signal* defined as

$$\mathrm{tri}(t) := \begin{cases} t+1, & -1 \le t < 0, \\ -t+1, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

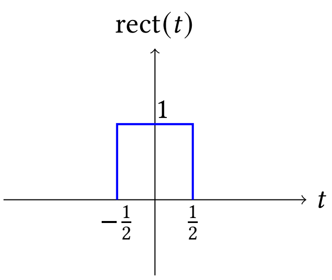


Figure 2.1: The rectangular signal $\mathrm{rect}(t)$

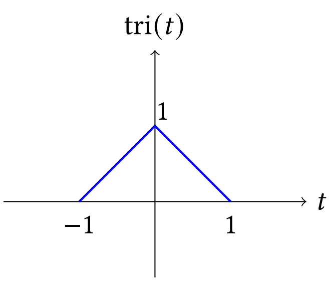


Figure 2.2: The triangular signal $\mathrm{tri}(t)$

Exercise 4. Suppose an LTI system sends $\mathrm{rect}(t)$ to $y(t)$. What is the output of $\mathrm{tri}(t)$? ○

Solution. Let $h(t)$ be the impulse response of the LTI system. Since $\text{tri}(t) = \text{rect}(t) * \text{rect}(t)$ and $\text{rect}(t) * h(t) = y(t)$, the output of $\text{tri}(t)$ is

$$\begin{aligned}\text{tri}(t) * h(t) &= (\text{rect}(t) * \text{rect}(t)) * h(t)\\ &= \text{rect}(t) * (\text{rect}(t) * h(t))\\ &= \text{rect}(t) * y(t).\end{aligned}$$

♣

Exercise 5. Suppose an LTI system sends $\text{tri}(t)$ to $y(t)$. What is the output of $\text{rect}(t)$? ○

Solution. Let $h(t)$ be the impulse response of the LTI system. We have $y(t) = \text{tri}(t) * h(t)$. Now, we want to express $\text{rect}(t)$ in terms of $\text{tri}(t)$. Note that

$$\frac{d}{dt}\text{tri}\left(t - \frac{1}{2}\right) = \text{rect}(t) - \text{rect}(t-1).$$

Hence

$$\begin{aligned}\text{rect}(t) &= \sum_{n=0}^{+\infty} (\text{rect}(t-n) - \text{rect}(t-n-1))\\ &= \sum_{n=0}^{+\infty} \frac{d}{dt}\text{tri}\left(t - n - \frac{1}{2}\right).\end{aligned}$$

Therefore, the output of $\text{rect}(t)$ is

$$\begin{aligned}\text{rect}(t) * h(t) &= \left(\sum_{n=0}^{+\infty} \frac{d}{dt}\text{tri}\left(t - n - \frac{1}{2}\right)\right) * h(t)\\ &= \sum_{n=0}^{+\infty} \frac{d}{dt}\left(\text{tri}\left(t - n - \frac{1}{2}\right) * h(t)\right)\\ &= \sum_{n=0}^{+\infty} \frac{d}{dt}y\left(t - n - \frac{1}{2}\right).\end{aligned}$$

♣

Exercise 6. Suppose that the impulse response of an LTI system is $h(t) = e^{-at}u(t)$. With the input signal $x(t) = (1-t)\text{rect}\left(t - \frac{1}{2}\right)$, compute the output signal $y(t)$. ○

Solution.

$$\begin{aligned}y(t) &= x(t) * h(t)\\ &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\\ &= \int_{-\infty}^{+\infty} (1-\tau)\text{rect}\left(\tau - \frac{1}{2}\right) e^{-a(t-\tau)}u(t-\tau)d\tau\\ &= e^{-at}\int_0^1 (1-\tau)e^{a\tau}u(t-\tau)d\tau.\end{aligned}$$

When $t < 0$, the integrand is always 0 and hence $y(t) = 0$. When $t \geq 0$,

$$\begin{aligned}
y(t) &= e^{-at} \int_0^{\min\{t,1\}} (1-\tau)e^{a\tau} d\tau \\
&= e^{-at} \cdot \frac{1}{a} \left(\left(\frac{1}{a} + (1-\tau) \right) e^{a\tau} \right) \Bigg|_0^{\min\{t,1\}} \\
&= \frac{1}{a} \left(\frac{1}{a} + (1 - \min\{t,1\}) \right) e^{a \cdot \min\{t,1\}} e^{-at} - \frac{1}{a} \left(\frac{1}{a} + 1 \right) e^{-at} \\
&= \begin{cases} -\frac{1}{a} \left(\left(\frac{1}{a} + 1 \right) \left(e^{-at} - 1 \right) + t \right), & 0 \leq t \leq 1, \\ -\frac{1}{a} \left(\frac{1-e^a}{a} + 1 \right) e^{-at}, & t > 1. \end{cases}
\end{aligned}$$

Therefore,

$$y(t) = \begin{cases} 0, & t < 0, \\ -\frac{1}{a} \left(\left(\frac{1}{a} + 1 \right) \left(e^{-at} - 1 \right) + t \right), & 0 \leq t \leq 1, \\ -\frac{1}{a} \left(\frac{1-e^a}{a} + 1 \right) e^{-at}, & t > 1. \end{cases}$$

♣

2.3 Properties of LTI systems

Proposition 22. An LTI system with impulse response $h(t)$ is causal if and only if $h(t) = 0$ for all $t < 0$. ○

Remark 2.7. A similar result for discrete-time LTI systems follows from a similar proof: An discrete-time LTI system with impulse response $h[n]$ is causal if and only if $h[n] = 0$ for all $n < 0$. ○

Proof. We first show the "if" part. Suppose that the input signals $x_1(t), x_2(t)$ produce output signals $y_1(t), y_2(t)$ respectively. If $x_1(t) = x_2(t)$ for any $t \leq t_0$, then for any $t \leq t_0$,

$$\begin{aligned}
y_1(t) &= x_1(t) * h(t) \\
&= \int_{-\infty}^{+\infty} x_1(\tau) h(t-\tau) d\tau \\
&= \int_{-\infty}^{t} x_1(\tau) h(t-\tau) d\tau \\
&= \int_{-\infty}^{t} x_2(\tau) h(t-\tau) d\tau \\
&= \int_{-\infty}^{+\infty} x_2(\tau) h(t-\tau) d\tau \\
&= x_2(t) * h(t) \\
&= y_2(t).
\end{aligned}$$

Therefore, the system is causal.

Now we show the "only if" part. Consider the signals $x_1(t) = \delta(t)$ and $x_2(t) = 0$. Suppose that $x_1(t), x_2(t)$ produce output signals $y_1(t), y_2(t)$ respectively. Observe that for any $t < 0$ we have $x_1(t) = 0 = x_2(t)$. By causality of the system, we have $y_1(t) = y_2(t)$ for all $t < 0$. Note that

$$y_1(t) = x_1(t) * h(t)$$

$$\begin{aligned} &= \delta(t) * h(t) \\ &= h(t), \\ y_2(t) &= x_2(t) * h(t) \\ &= 0 * h(t) \\ &= 0. \end{aligned}$$

Therefore, $h(t) = 0$ for all $t < 0$. ■

Proposition 23. An LTI system with impulse response $h(t)$ is stable if and only if

$$\int_{-\infty}^{+\infty} |h(t)| dt < \infty.$$

○

Remark 2.8. A similar result for discrete-time LTI systems follows from a similar proof: An discrete-time LTI system with impulse response $h[n]$ is stable if and only if

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty.$$

○

Proof. We first prove the "if" part. Suppose $x(t)$ is a bounded signal whose absolute value is bounded by M. We have

$$\begin{aligned} |x(t) * h(t)| &= \left| \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau \right| \\ &\le \int_{-\infty}^{+\infty} |x(\tau) h(t-\tau)| d\tau \\ &\le \int_{-\infty}^{+\infty} M |h(t-\tau)| d\tau \\ &\le M \int_{-\infty}^{+\infty} |h(\sigma)| d\sigma \qquad \text{(with } \sigma := t - \tau) \\ &< \infty. \end{aligned}$$

This shows that the system output is bounded whenever the input is bounded.

Now we show the "only if" part. Suppose that the LTI system is stable. Let $x(t) = \text{sgn}(h(-t))$, where the *sign function* $\text{sgn}(z)$ is defined by

$$\text{sgn}(z) := \begin{cases} \frac{z}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Observe that $x(t)$ is a bounded signal with $|x(t)| \le 1$. Since the system is stable, there exists some $M > 0$ such that $|y(t)| \le M$ for any t, where $y(t) = x(t) * h(t)$ is the output signal produced by input signal $x(t)$. We have

$$\begin{aligned} M &\ge y(0) \\ &= \int_{-\infty}^{+\infty} x(\tau) h(-\tau) d\tau \end{aligned}$$

$$= \int_{-\infty}^{+\infty} \operatorname{sgn}(h(-\tau))h(-\tau)d\tau$$
$$= \int_{-\infty}^{+\infty} |h(-\tau)|d\tau$$
$$= \int_{-\infty}^{+\infty} |h(\sigma)|d\sigma \qquad \text{(with } \sigma := -\tau\text{)}.$$

Thus, $\int_{-\infty}^{+\infty} |h(t)|dt < \infty$. ■

2.4 Eigenfunctions of LTI systems

Similar to the idea of eigenvectors of square matrices, an *eigenfunction* of a system is an input signal $x(t)$ such that its corresponding output signal $y(t)$ satisfies $y(t) = \lambda x(t)$ for some constant $\lambda \in \mathbb{C}$, where such constant λ is also called the *eigenvalue* correspond to the eigenfunction.

For a continuous-time LTI system with impulse response $h(t)$, consider an input signal $t \mapsto e^{st}$ for some constant $s \in \mathbb{C}$. This produces an output signal

$$\begin{aligned} e^{st} * h(t) &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= e^{st}\left(\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right). \end{aligned}$$

This shows that $t \mapsto e^{st}$ is an eigenfunction of every continuous-time LTI system, with eigenvalue $\int_{-\infty}^{+\infty} h(t)e^{-st}dt$.

For a discrete-time LTI system with impulse response $h[n]$, consider an input signal $n \mapsto z^n$ for some constant $z \in \mathbb{C}$. This produces an output signal

$$\begin{aligned} z^n * h[n] &= \sum_{m=-\infty}^{+\infty} h[m]z^{n-m} \\ &= z^n\left(\sum_{m=-\infty}^{+\infty} h[m]z^{-m}\right). \end{aligned}$$

This shows that $n \mapsto z^n$ is an eigenfunction of every discrete-time LTI system, with eigenvalue $\sum_{n=-\infty}^{+\infty} h[n]z^{-n}$.

2.5 Extension to higher dimensions

Convolution can be extended to multivariable functions. For example, image processing deals with two-dimensional signals. Two-dimensional convolution is defined as

$$x(t_1, t_2) * y(t_1, t_2) := \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau_1, \tau_2)y(t_1 - \tau_1, t_2 - \tau_2)d\tau_1 d\tau_2.$$

In general, n-dimensional convolution is defined as

$$x(t_1, \ldots, t_n) * y(t_1, \ldots, t_n) := \int_{\mathbb{R}^n} x(\tau_1, \ldots, \tau_n)y(t_1 - \tau_1, \ldots, t_n - \tau_n)d\tau.$$

The n-dimensional convolution still has many properties like the one-dimensional convolution:

- Commutativity: $x * y = y * x$.
- Associativity: $x * (y * z) = (x * y) * z$.
- Identity: $x = x * \delta$, where $\delta(t_1, \ldots, t_n)$ is the n-dimentional delta impulse:

$$\int_{\mathbb{R}^n} x(t_1, \ldots, t_n)\delta(t_1, \ldots, t_n)dt = x(0, \ldots, 0)$$

  for all $x(t_1, \ldots, t_n)$.
- Linearity: $(ax + by) * h = a(x * h) + b(y * h)$.
- Translation invariance: If $y = x * h$, then $Ty = Tx * h$ for any translation operator T (i.e. $(Tx)(t_1, \ldots, t_n) = x\left(t_1 - \tilde{t}_1, \ldots, t_n - \tilde{t}_n\right)$).
- Directional derivative: $\mathbf{v}^T\nabla(x * y) = \left(\mathbf{v}^T\nabla x\right) * y = x * \left(\mathbf{v}^T\nabla y\right)$ for any $\mathbf{v} \in \mathbb{C}^n$, where ∇ is the gradient operator:

$$\nabla x(t_1, \ldots, t_n) := \begin{pmatrix} \frac{\partial x(t_1,\ldots,t_n)}{\partial t_1} \\ \frac{\partial x(t_1,\ldots,t_n)}{\partial t_2} \\ \vdots \\ \frac{\partial x(t_1,\ldots,t_n)}{\partial t_n} \end{pmatrix}.$$

- Volume:

$$\int_{\mathbb{R}^n} (x * y)(t_1, \ldots, t_n)dt = \left(\int_{\mathbb{R}^n} x(t_1, \ldots, t_n)dt\right)\left(\int_{\mathbb{R}^n} y(t_1, \ldots, t_n)dt\right).$$

2.6 Finite discrete signals: linear cyclic-shift-invariant systems

In this section, we consider systems whose input and output signals are (column) vectors in $\mathbb{C}^N$ (for notational convenience, labeled $\mathbf{x} := \begin{pmatrix} x[0] & x[1] & x[2] & \cdots & x[N-1] \end{pmatrix}^T$).

Observe (justify why?) that a linear system is characterized by an $N \times N$ matrix, A, and that the input signal $\mathbf{x}$ and output signal $\mathbf{y}$ satisfy

$$\mathbf{y} = A\mathbf{x}.$$

How do we determine the matrix A? Let $\mathbf{h}_k := \begin{pmatrix} h_k[0] & h_k[1] & h_k[2] & \cdots & h_k[N-1] \end{pmatrix}^T$ denote the output signal when the input signal is $\mathbf{e}_k := \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}^T$ where the number 1 occurs at the k-th location, $0 \le k \le N-1$. Then the matrix A is defined by

$$A = \begin{pmatrix} h_0[0] & h_1[0] & \cdots & h_{N-1}[0] \\ h_0[1] & h_1[1] & \cdots & h_{N-1}[1] \\ \vdots & \vdots & & \vdots \\ h_0[N-1] & h_1[N-1] & \cdots & h_{N-1}[N-1] \end{pmatrix}.$$

The natural equivalent of time-invariance for such systems turns out to be cyclic-shift invariance. A right cyclic shift of $x[n]$ by n_0 creates a new signal $n \mapsto x[(n-n_0)_N]$. Here, $(\cdot)_N$ denotes the modulo operation defined as the only integer $0 \le n < N$ such that $(n)_N \equiv n \pmod N$. A system is called *cyclic-shift-invariant* if for any n_0, the cyclically shifted input $x[(n-n_0)_N]$ produces the cyclically shifted output $y[(n-n_0)_N]$, where $y[n]$ is the output of the system when the input is $x[n]$. Thus, for a linear cyclic-shift-invariant system, we have $h_k[n] = h_0[(n-k)_N]$.

For convenience, we drop the subscript 0 in $h_0[n]$. Therefore, we have that for a linear cyclic-shift-invariant system, the input signal $\mathbf{x}$ and output signal $\mathbf{y}$ satisfy $\mathbf{y} = A\mathbf{x}$, where the $N \times N$ matrix A must be in the form

$$A = \begin{pmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & \cdots & h[2] \\ \vdots & \vdots & & \vdots \\ h[N-2] & h[N-3] & \cdots & h[N-1] \\ h[N-1] & h[N-2] & \cdots & h[0] \end{pmatrix},$$

for some $h[n] \in \mathbb{C}^N$. Matrices of this form are called *circulant* matrices.

Definition 2.3. The *conjugate transpose* (or *Hermitian transpose*) of an $M \times N$ matrix A is the $N \times M$ matrix, denoted by $A^\dagger$, defined by

$$A^\dagger_{n,m} := \overline{A_{m,n}},$$

for $m = 0, 1, \ldots, M-1$ and $n = 0, 1, \ldots, N-1$. ○

Definition 2.4. A square matrix A is *normal* if $AA^\dagger = A^\dagger A$. ○

Definition 2.5. A square matrix A is *unitary* if $AA^\dagger = A^\dagger A = I$, i.e. $A^{-1} = A^\dagger$. ○

Proposition 24. All normal matrices are diagonalizable, i.e. for any $N \times N$ matrix A, there exist an $N \times N$ diagonal matrix Λ and an $N \times N$ unitary matrix H such that $A = H\Lambda H^\dagger$. ○

Proof. In the proof we can assume A has N distinct eigenvalues.

We shall first prove that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ then $\mathbf{v}$ is also an eigenvector of $A^\dagger$ with eigenvalue $\overline{\lambda}$. Let $\mathbf{v} \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$. Let

$$\begin{aligned} \mathbf{u} &:= A\mathbf{v} - \lambda\mathbf{v}, \\ \mathbf{w} &:= A^\dagger\mathbf{v} - \overline{\lambda}\mathbf{v}. \end{aligned}$$

Then we have

$$\begin{aligned} \|\mathbf{u}\|^2 &= \mathbf{u}^\dagger\mathbf{u} \\ &= (A\mathbf{v} - \lambda\mathbf{v})^\dagger(A\mathbf{v} - \lambda\mathbf{v}) \\ &= (\mathbf{v}^\dagger A^\dagger - \overline{\lambda}\mathbf{v}^\dagger)(A\mathbf{v} - \lambda\mathbf{v}) \\ &= \mathbf{v}^\dagger A^\dagger A\mathbf{v} - \overline{\lambda}\mathbf{v}^\dagger A\mathbf{v} - \lambda\mathbf{v}^\dagger A^\dagger\mathbf{v} + \|\lambda\|^2\mathbf{v}^\dagger\mathbf{v}, \\ \|\mathbf{w}\|^2 &= \mathbf{w}^\dagger\mathbf{w} \\ &= (A^\dagger\mathbf{v} - \overline{\lambda}\mathbf{v})^\dagger(A^\dagger\mathbf{v} - \overline{\lambda}\mathbf{v}) \\ &= (\mathbf{v}^\dagger A - \lambda\mathbf{v}^\dagger)(A^\dagger\mathbf{v} - \overline{\lambda}\mathbf{v}) \\ &= \mathbf{v}^\dagger AA^\dagger\mathbf{v} - \overline{\lambda}\mathbf{v}^\dagger A\mathbf{v} - \lambda\mathbf{v}^\dagger A^\dagger\mathbf{v} + \|\lambda\|^2\mathbf{v}^\dagger\mathbf{v} \\ &= \|\mathbf{u}\|^2 + \mathbf{v}^\dagger(AA^\dagger - A^\dagger A)\mathbf{v}. \end{aligned}$$

As A is normal, we have $\|\mathbf{u}\| = \|\mathbf{w}\|$. Take λ to be one of the eigenvalues of A, and $\mathbf{v}$ to be a corresponding eigenvector, then $\mathbf{u} = 0$. Hence $\mathbf{w} = 0$. Therefore, $A^\dagger\mathbf{v} = \overline{\lambda}\mathbf{v}$, i.e. $\mathbf{v}$ is an eigenvector of $A^\dagger$ with eigenvalue $\overline{\lambda}$.

Assume $\mathbf{h}_k, \mathbf{h}_\ell$ are two eigenvectors of A with distinct eigenvalues λ_k, λ_ℓ respectively. Then

$$\begin{aligned}\lambda_k \mathbf{h}_\ell^\dagger \mathbf{h}_k &= \mathbf{h}_\ell^\dagger (\lambda_k \mathbf{h}_k) \\ &= \mathbf{h}_\ell^\dagger A \mathbf{h}_k \\ &= (A^\dagger \mathbf{h}_\ell)^\dagger \mathbf{h}_k \\ &= (\overline{\lambda}_\ell \mathbf{h}_\ell)^\dagger \mathbf{h}_k \\ &= \lambda_\ell \mathbf{h}_\ell^\dagger \mathbf{h}_k.\end{aligned}$$

As $\lambda_k \neq \lambda_\ell$, we must have $\mathbf{h}_\ell^\dagger \mathbf{h}_k = 0$, i.e. $\mathbf{h}_k$ and $\mathbf{h}_\ell$ are orthogonal.

Let λ_k $(0 \leq k \leq N-1)$ be the distinct eigenvalues of A, and $\mathbf{h}_k$ be an eigenvector of A with eigenvalue λ_k and length 1. Let

$$H := \begin{pmatrix} \mathbf{h}_0 & \mathbf{h}_1 & \cdots & \mathbf{h}_{N-1} \end{pmatrix}.$$

and Λ be a diagonal matrix with diagonal entries $\Lambda_{k,k} = \lambda_k$. It is easy to check that H is unitary since the columns of H are orthonormal. Then

$$\begin{aligned} H^\dagger A H &= \begin{pmatrix} \mathbf{h}_0^\dagger \\ \mathbf{h}_1^\dagger \\ \vdots \\ \mathbf{h}_{N-1}^\dagger \end{pmatrix} \begin{pmatrix} A\mathbf{h}_0 & A\mathbf{h}_1 & \cdots & A\mathbf{h}_{N-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{h}_0^\dagger \\ \mathbf{h}_1^\dagger \\ \vdots \\ \mathbf{h}_{N-1}^\dagger \end{pmatrix} \begin{pmatrix} \lambda_0\mathbf{h}_0 & \lambda_1\mathbf{h}_1 & \cdots & \lambda_{N-1}\mathbf{h}_{N-1} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} \end{pmatrix} \\ &= \Lambda. \end{aligned}$$

As H is unitary, we have $A = H\Lambda H^\dagger$. ■

Proposition 25. All circulant matrices are normal. ○

Proof. Let A be an $N \times N$ circulant matrix. Since A is circulant, we have $A_{n,m} = h[(n-m)_N]$ where

$$\begin{pmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-1] \end{pmatrix}$$

is the first column of A. Then,

$$(AA^\dagger)_{n,m} = \sum_{k=0}^{N-1} A_{n,k} (A^\dagger)_{k,m}$$

$$
\begin{aligned}
&= \sum_{k=0}^{N-1} A_{n,k}\overline{A_{m,k}} \\
&= \sum_{k=0}^{N-1} h[(n-k)_N]\overline{h[(m-k)_N]} \\
&= \sum_{\ell=0}^{N-1} h[(\ell-m)_N]\overline{h[(\ell-n)_N]} && (\text{with } \ell := (n+m-k)_N) \\
&= \sum_{\ell=0}^{N-1} A_{\ell,m}\overline{A_{\ell,n}} \\
&= \sum_{\ell=0}^{N-1} (A^\dagger)_{n,\ell}A_{\ell,m} \\
&= (A^\dagger A)_{n,m}.
\end{aligned}
$$

Hence $AA^\dagger = A^\dagger A$. ■

Proposition 26. Let A be an $N \times N$ circulant matrix. Then the following statements hold:

1. $\mathbf{h}_k := \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & e^{j2\pi\frac{k}{N}} & e^{j2\pi\frac{2k}{N}} & \cdots & e^{j2\pi\frac{(N-1)k}{N}} \end{pmatrix}^T$ $(0 \le k \le N-1)$ are N orthonormal eigenvectors of A.
2. $H := \begin{pmatrix} \mathbf{h}_0 & \mathbf{h}_1 & \cdots & \mathbf{h}_{N-1} \end{pmatrix}$ is a unitary matrix.
3. $A = H\Lambda H^\dagger$, where Λ is the diagonal matrix with diagonal entries $\Lambda_{k,k}$ being the eigenvalue of A corresponding to the eigenvector $\mathbf{h}_k$.

○

Proof. 1. Since A is circulant, we have $A_{n,m} = y[(n-m)_N]$ where

$$
\begin{pmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{pmatrix}
$$

is the first column of A. Consider

$$
\begin{aligned}
(A\mathbf{h}_k)_\ell &= \sum_{n=0}^{N-1} A_{\ell,n}(\mathbf{h}_k)_n \\
&= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} y[(\ell-n)_N]e^{j2\pi\frac{kn}{N}} \\
&= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} y[m]e^{j2\pi\frac{k(\ell-m)}{N}} && (\text{with } m := (\ell-n)_N) \\
&= \frac{1}{\sqrt{N}}e^{j2\pi\frac{k\ell}{N}}\sum_{m=0}^{N-1} y[m]e^{-j2\pi\frac{km}{N}} \\
&= (\mathbf{h}_k)_\ell\lambda_k,
\end{aligned}
$$

where $\lambda_k := \sum_{n=0}^{N-1} y[n]e^{-j2\pi\frac{kn}{N}}$. Thus $A\mathbf{h}_k = \lambda_k\mathbf{h}_k$, i.e. $\mathbf{h}_k$ is an eigenvector of A with eigenvalue λ_k.

Now, for $0 \le k \le N-1$ and $0 \le \ell \le N-1$,

$$\begin{aligned}
\mathbf{h}_\ell^\dagger \mathbf{h}_k &= \sum_{n=0}^{N-1} \overline{(\mathbf{h}_\ell)_n} (\mathbf{h}_k)_n \\
&= \sum_{n=0}^{N-1} \overline{\frac{1}{\sqrt{N}} e^{j2\pi \frac{\ell n}{N}}} \frac{1}{\sqrt{N}} e^{j2\pi \frac{kn}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi \frac{k-\ell}{N} n} \\
&= \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1} 1, & k-\ell \equiv 0 \pmod N, \\ \frac{1}{N} \frac{e^{j2\pi \frac{k-\ell}{N} N} - 1}{e^{j2\pi \frac{k-\ell}{N}} - 1}, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & k-\ell \equiv 0 \pmod N, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & k = \ell, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, $\mathbf{h}_k$ $(0 \le k \le N-1)$ are orthonormal.

2. It follows from orthonormality of $\mathbf{h}_k$ that $H^\dagger H = I$. Now, observe that $H^T = H$. Then

$$\begin{aligned}
HH^\dagger &= (H^T)(H^T)^\dagger \\
&= (\overline{H^\dagger})(\overline{H}) \\
&= \overline{H^\dagger H} \\
&= \overline{I} \\
&= I.
\end{aligned}$$

Therefore, $H^\dagger H = HH^\dagger = I$, i.e. H is unitary.

3. Consider

$$\begin{aligned}
H^\dagger A H &= \begin{pmatrix} \mathbf{h}_0^\dagger \\ \mathbf{h}_1^\dagger \\ \vdots \\ \mathbf{h}_{N-1}^\dagger \end{pmatrix} \begin{pmatrix} A\mathbf{h}_0 & A\mathbf{h}_1 & \cdots & A\mathbf{h}_{N-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{h}_0^\dagger \\ \mathbf{h}_1^\dagger \\ \vdots \\ \mathbf{h}_{N-1}^\dagger \end{pmatrix} \begin{pmatrix} \lambda_0 \mathbf{h}_0 & \lambda_1 \mathbf{h}_1 & \cdots & \lambda_{N-1} \mathbf{h}_{N-1} \end{pmatrix} \\
&= \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} \end{pmatrix} \\
&= \Lambda.
\end{aligned}$$

As H is unitary, we have $A = H \Lambda H^\dagger$.

■

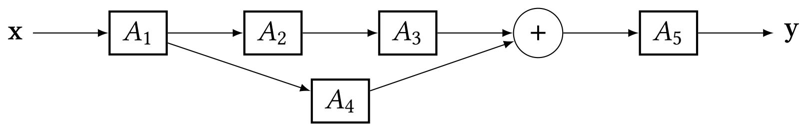
2.6.1 The curious case of common eigenvectors

As we have shown that all linear cyclic-shift-invariant systems share the same set of eigenvectors. Note that the k-th eigenvector has the form

$$\mathbf{h}_k = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & e^{j2\pi\frac{k}{N}} & e^{j2\pi\frac{2k}{N}} & \dots & e^{j2\pi\frac{(N-1)k}{N}} \end{pmatrix}^T.$$

This fact has massive practical implications and will form the basis of our further study of this and LTI systems (that will also demonstrate similar behavior).

To get a sense of the practical implications, consider a large system composed of a series of parallel connections of smaller LTI systems.



In particular, let us say that (see figure above)

$$\mathbf{y} = A_5(A_4 + A_3A_2)A_1\mathbf{x}.$$

Note that these matrix multiplications correspond to convolutions in LTI systems, and the overall complexity would roughly be $O(N^2)$ for each $\mathbf{x}$, or if we want to pre-compute the equivalent LSI system, we would need to compute $A_5(A_4 + A_3A_2)A_1$ which takes the time for matrix multiplication (currently at $O(N^{2.3\cdots})$, but most practical algorithms work at $O(N^3)$). This gets prohibitively time-consuming for large N. However, the fact that they have a common set of eigenvectors (along with a cute property of the eigenvectors) comes to our rescue.

Note that replacing $A_i = H\Lambda_i H^\dagger$ and using $H^\dagger H = I$ we get

$$\mathbf{y} = H\Lambda_5(\Lambda_4 + \Lambda_3\Lambda_2)\Lambda_1 H^\dagger \mathbf{x}.$$

Further, it turns out that we can exploit the structure of H to compute $H^\dagger\mathbf{x}$ in $O(N\log_2 N)$ time efficiently (see *fast Fourier transform* (FFT)). Therefore the output can be computed in $O(N\log_2 N)$.

Looking ahead, instead of working with signals $\mathbf{x}$, we often work with the vector $\hat{\mathbf{X}} = H^\dagger\mathbf{x}$ (called its *Fourier transform*). Under this transformation, we have

$$\hat{\mathbf{Y}} = \Lambda_5(\Lambda_4 + \Lambda_3\Lambda_2)\Lambda_1\hat{\mathbf{X}}$$

or

$$\hat{\mathbf{Y}}[k] = \lambda_5[k](\lambda_4[k] + \lambda_3[k]\lambda_2[k])\lambda_1[k]\hat{\mathbf{Y}}[k], \qquad 0 \le k \le N-1.$$

Thus all operations become just a multiplication of complex scalars.

In effect, if we express any input signal as a linear combination of its "common eigenvectors", all linear cyclic-shift-invariant (or LTI) systems just behave like amplifiers that scale each component with corresponding eigenvalues.

Exercises

1. **(22 Midterm 1)** A **causal** LTI system produces the following input-output relationship shown in Figure 2.3. The signal

$$x(t) = \begin{cases} 4t(1-t), & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad y(t) = \begin{cases} |\sin(\pi t)|, & t > 0 \\ 0, & \text{otherwise} \end{cases}.$$

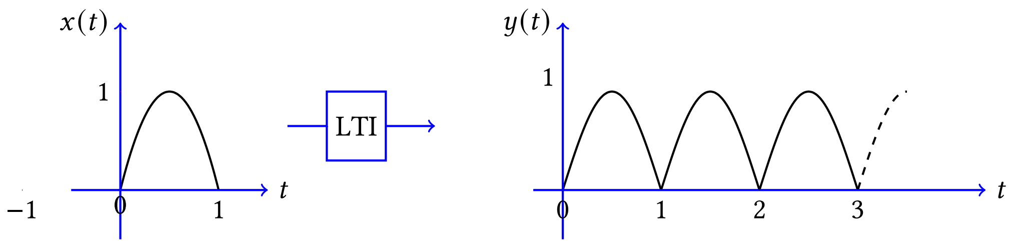


Figure 2.3: An input-output pair

(1) Compute the output of the LTI system when the input is $x_1(t)$ shown below.

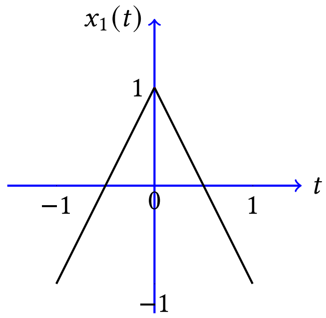


Figure 2.4: An input $x_1(t)$

$$x_1(t) = \begin{cases} 1 + 2t & -1 < t \leq 0 \\ 1 - 2t & 0 \leq t < 1 \\ 0, & \text{o.w.} \end{cases}.$$

(2) Compute an input, $x_2(t)$, to the LTI system that produces the output $y_2(t)$ given below. (See Figure 2.5).

$$y(t) = \begin{cases} \frac{1}{2^k}|\sin(\pi t)|, & k \leq t < k+1, \ \& \ k \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

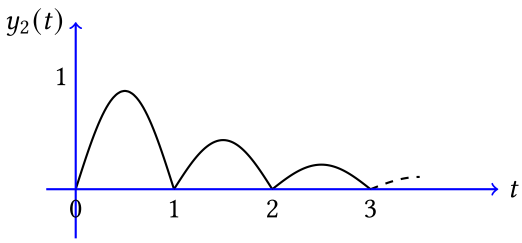


Figure 2.5: Output $y_2(t)$

(3) Is the LTI system stable? Justify

2. **(22 Midterm 1)** A linear system produces the following input-output relationship for every shifted $\delta(t)$. (see Figure 2.6). Here

$$\text{Tri}(t) = \begin{cases} 1 - |t|, & -1 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}.$$

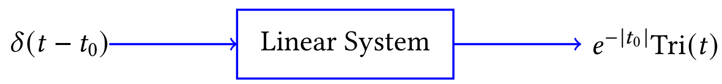


Figure 2.6: An input-output pair

(1) Express the output $y(t)$, of the above system, for a generic input, $x(t)$.

(2) Is this system stable?

(3) Compute the output of the system, when the input is $\mathrm{rect}(t)$, where

$$\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}.$$

(4) Compute the output of the system, when $x(t)$ is an integrable odd function. (An integrable function is one such that $\int_{\mathbb{R}} |x(t)|dt < \infty$.)

3. *Understanding LTI system behavior*: Consider an LTI system that maps an input $x(t)$ to the output $y(t)$ as shown in the Figure 2.7. (If one worries about values of $x(t)$ at $t = -1, 1,$

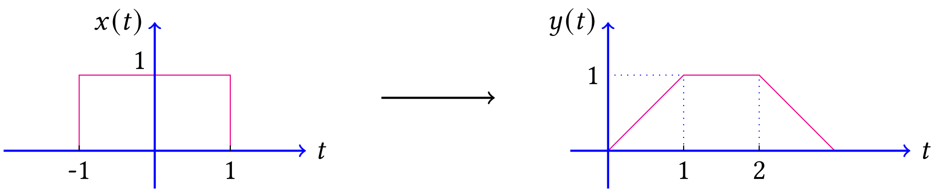


Figure 2.7: Input and output of an LTI system

take them to be $\frac{1}{2}$. In principle, the exact value at these points does not matter.)

Compute the output of the same LTI system for the following three inputs:

(1) Input $x_1(t)$ shown in Figure 2.8. (Hint: try to express $x_1(t)$ as a linear combination of $x(t+3)$ and $x(t+2)$.)

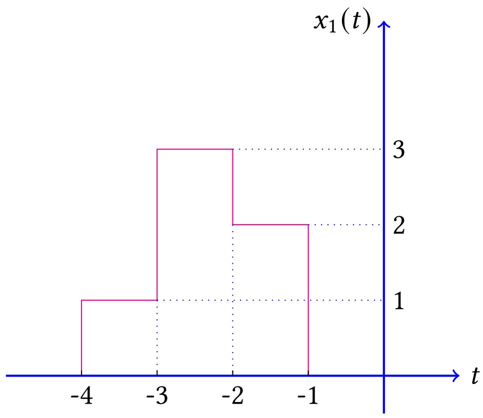


Figure 2.8: Input $x_1(t)$

(2) Input $x_2(t) = u(t)$, where $u(t)$ is the unit step function defined in the notes. (Hint: Consider $\sum_{m=0}^{\infty} x(t-2m-1)$)

(3) Input $x_3(t) = \delta(t)$, where $\delta(t)$ is the delta function. (Hint: use the above part and the differentiation property of convolution.)

4. *An exercise in convolution*: Let

$$\text{Rect}(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ \frac{1}{2} & |t| = \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases} \qquad \text{and } \text{Tri}(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & |t| \geq 1 \end{cases} .$$

Show that: $\frac{1}{a}\text{Rect}(\frac{t}{a}) * \frac{1}{a}\text{Rect}(\frac{t}{a}) = \frac{1}{a}\text{Tri}(\frac{t}{a})$. (Assume $a > 0$)

(Remark: Integrations in the convolution can be done either algebraically or geometrically using pictures.)

5. Let $x(t) = \sum_{l=2}^{\infty} \delta(t - l)$ be the input, and $h(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$ be the impulse response.

Question: What's the output $y(t) = x(t) * h(t)$?

6. *Area and center of gravity theorems*: Let $y(t) = x(t) * h(t)$. Show that

(1)
$$\int_{-\infty}^{\infty} y(t)dt = \left(\int_{-\infty}^{\infty} x(t)dt\right)\left(\int_{-\infty}^{\infty} h(t)dt.\right)$$

(2)
$$\frac{\int_{-\infty}^{\infty} ty(t)dt}{\int_{-\infty}^{\infty} y(t)dt} = \left(\frac{\int_{-\infty}^{\infty} tx(t)dt}{\int_{-\infty}^{\infty} x(t)dt}\right) + \left(\frac{\int_{-\infty}^{\infty} th(t)dt}{\int_{-\infty}^{\infty} h(t)dt}\right).$$

(For the second part, assume that $\int_{-\infty}^{\infty} y(t)dt, \int_{-\infty}^{\infty} x(t)dt, \int_{-\infty}^{\infty} h(t)dt$ are non-zero.)

7. **Linear Systems (Midterm 2012) - Compulsory for Elite students**

Consider a linear system (this system is *not time-invariant*) which has the following output $h_{t_0}(t)$ (shown in Figure 2.9) when the input is $\delta(t - t_0)$.

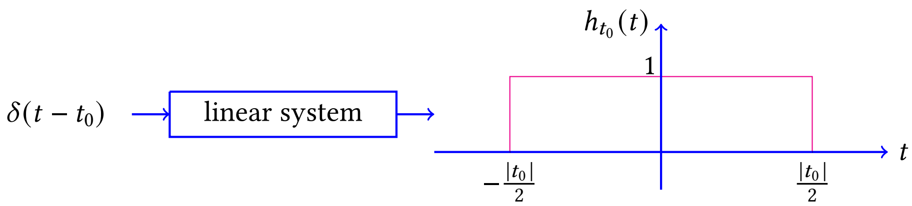


Figure 2.9: The system behavior for shifted δ functions

(1) Write an expression for the output $y(t)$ of the system in terms of the input $x(t)$ to the system. (Here $x(t)$ is a generic input signal). (Hint: Express $x(t)$ as a linear combination of the shifted impulses $\delta(t - t_0)$.)

(2) Compute and plot the output when the input $x(t)$ is as shown below in Figure 2.10.

(3) What is the output of the system when the input $x_1(t) = \frac{dx(t)}{dt}$, where $x(t)$ is the signal shown in Figure 2.10?

(4) Show that the output of $\frac{dx(t)}{dt}$, for any generic $x(t)$, is given by $x(-2|t|) - x(2|t|)$. (Assume that $x(t) \to 0$ as $|t| \to \infty$.)

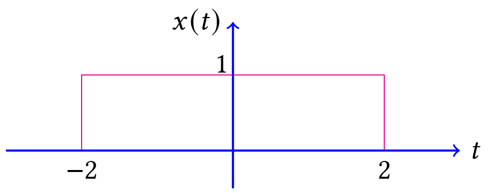


Figure 2.10: Input $x(t)$

8. **(22 Make-up Midterm 1)** An LTI system produces the following input-output relationship shown in Figure 2.11. The signal

$$y(t) = \begin{cases} t+2, & -2 \le t < 1 \\ 1, & -1 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

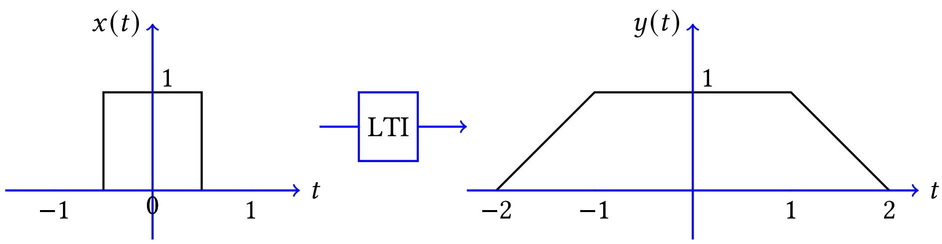


Figure 2.11: An input-output pair

Compute the output of the same LTI system for the input shown in Figure 2.12. The signal

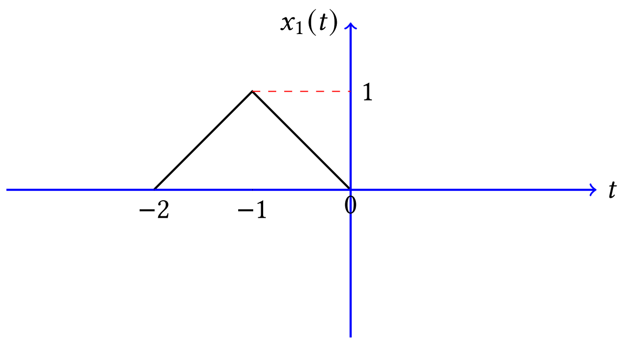


Figure 2.12: An input $x_1(t)$

$$x_1(t) = \begin{cases} t+2, & -2 \le t < 1 \\ -t, & -1 \le t \le 0 \\ 0, & \text{otherwise} \end{cases}.$$

Chapter 3

Fourier series

In this chapter, we consider periodic signals with certain implicitly imposed smoothness conditions. The idea of Fourier series is to write a periodic signal as a linear combination of eigenfunctions of any LTI system. We already know that a set of eigenfunctions of any LTI system is $\{t \mapsto e^{j2\pi ft} : f \in \mathbb{R}\}$. For $e^{j2\pi ft}$ to be periodic with period T, we require $e^{j2\pi f(t+T)} = e^{j2\pi ft}$ for all $t \in \mathbb{R}$, or equivalently $e^{j2\pi fT} = 1$, which means that $f = \frac{k}{T}$ for some $k \in \mathbb{Z}$. Thus, all such eigenfunctions with period T must be of the form $t \mapsto e^{j2\pi \frac{kt}{T}}$.

Definition 3.1. (*Inner product*) An *inner product* $\langle \cdot, \cdot \rangle$ is a complex-valued binary operation having the following properties:

1. $\langle x, x \rangle \geq 0$ for any x, where the equality holds if and only if $x = 0$.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for any x, y.
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for any $a, b \in \mathbb{C}$ and x, y, z.

○

Remark 3.1. The inner product is usually defined such that it is linear in the first argument and *conjugate linear* in the second argument, i.e., $\langle cx, y \rangle = c\langle x, y \rangle$ and $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$. However, some authors may define it the other way around. ○

Proposition 27. Let $T > 0$. The binary operation defined by

$$\langle x(t), y(t) \rangle_T := \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{y(t)} dt$$

is an inner product on the set of periodic signals with period T. ○

Proof. Observe that

$$\begin{aligned} \langle x(t), x(t) \rangle_T &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt \\ &\geq 0, \end{aligned}$$

with equality holds if and only if $x(t) = 0$ almost everywhere.

Secondly, observe that

$$\overline{\langle x(t), y(t) \rangle_T} = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{x(t)\overline{y(t)}} dt$$

$$= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{x(t)} y(t) dt$$
$$= \langle y(t), x(t) \rangle_T.$$

Finally, note that for any $a, b \in \mathbb{C}$,

$$\begin{aligned}
\langle ax(t) + by(t), z(t) \rangle_T &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (ax(t) + by(t)) \overline{z(t)} dt \\
&= a \cdot \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \overline{z(t)} dt + b \cdot \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) \overline{z(t)} dt \\
&= a \langle x(t), z(t) \rangle_T + b \langle x(t), z(t) \rangle_T.
\end{aligned}$$

■

This chapter is based is the following result.

Proposition 28. A periodic signal $x(t)$ with period T (with some additional weak conditions) can be written as

$$x(t) = \sum_{k=-\infty}^{+\infty} \hat{X}[k] e^{j2\pi \frac{kt}{T}},$$

where the discrete-time signal $\hat{X}[k]$, which is called the *Fourier series coefficients* of $x(t)$, is given by

$$\begin{aligned}
\hat{X}[k] &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi \frac{kt}{T}} dt \\
&= \langle x(t), e^{j2\pi \frac{kt}{T}} \rangle_T,
\end{aligned}$$

for $k \in \mathbb{Z}$. ○

Remark 3.2. The convergence of the Fourier series is a well-studied topic in classical harmonic analysis, a branch of pure mathematics. For more details, see https://en.wikipedia.org/wiki/Convergence_of_Fourier_series#Pointwise_convergence. In general, the most common criteria for pointwise convergence of a periodic function $x(t)$ are as follows:

- Given an interval $[a, b] \in \mathbb{R}$, the *total variation* of $x(t)$ in this interval is defined as

  $$V_x([a, b]) = \sup_P \sum_{i=0}^{k-1} |x(t_{i+1}) - x(t_i)|,$$

  where the supremum is taken over all partitions $t_0 = a < t_1 < t_2 < \cdots < t_{k-1} < t_k = b$. Now, $x(t)$ is of bounded variation if $V_x([0, T]) < \infty$.

  If $x(t)$ is of bounded variation, then its Fourier series converges everywhere. In particular, if $x(t)$ has a left limit and a right limit at t_0, then the Fourier series would converge to $\frac{x(t_0^-)+x(t_0^+)}{2}$.

- A function $x(t)$ is said to satisfy a *Hölder condition* of order α if there exists a $c < \infty$ such that $|x(t + h) - x(t)| < ch^\alpha$. If $x(t)$ satisfies a Hölder condition for some $\alpha > 0$, then its Fourier series converges uniformly.

- If $x(t)$ is continuous and its Fourier coefficients are absolutely summable, then the Fourier series converges uniformly.

○

Proposition 29. The set of functions $\left\{t \mapsto e^{j2\pi\frac{kt}{T}}\right\}_{k\in\mathbb{Z}}$ is an orthonormal set under the inner product $\langle\cdot,\cdot\rangle_T$, i.e.

$$\begin{aligned}\langle e^{j2\pi\frac{kt}{T}}, e^{j2\pi\frac{\ell t}{T}}\rangle_T &= \delta[k-\ell] \\ &= \begin{cases}1, & k=\ell,\\ 0, & k\neq\ell.\end{cases}\end{aligned}$$

○

Proof.

$$\begin{aligned}\langle e^{j2\pi\frac{kt}{T}}, e^{j2\pi\frac{\ell t}{T}}\rangle_T &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi\frac{kt}{T}} e^{-j2\pi\frac{\ell t}{T}}\,dt \\ &= \begin{cases}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt, & k=\ell,\\ \frac{1}{T}\left.\frac{e^{j2\pi\frac{(k-\ell)t}{T}}}{j2\pi\frac{k-\ell}{T}}\right|_{-\frac{T}{2}}^{\frac{T}{2}}, & k\neq\ell\end{cases} \\ &= \begin{cases}1, & k=\ell,\\ 0, & k\neq\ell.\end{cases}\end{aligned}$$

■

Remark 3.3. The eigenfunctions $e^{j2\pi\frac{kt}{T}}$ only span a subspace of the set of periodic signals with period T, i.e., not all periodic signals $x(t)$ can be expressed as a linear combination of $e^{j2\pi\frac{kt}{T}}$. This is why, intuitively, one needs to know the conditions under which a periodic function can be written as a linear combination of $e^{j2\pi\frac{kt}{T}}$. ○

Remark 3.4. A sequence of functions $x_n(t)$ converges to $x(t)$ *pointwise*, if for any fixed t, the sequence $x_n(t)$ converges to $x(t)$. For an infinite sum, we can define the sequence of partial sum as follows,

$$S_n(t) = \sum_{k=-n}^{n} \hat{X}[k]e^{j2\pi\frac{kt}{T}}.$$

At points of discontinuity (where left and right limits exist) the functions $S_n(t)$ exhibit an overshoot and undershoot phenomenon called the Gibbs phenomenon. For more details, see https://en.wikipedia.org/wiki/Gibbs_phenomenon. ○

Important: For the rest of the chapter, we will assume that the functions we deal with have a valid Fourier series representation (i.e. a convergent Fourier series representation).

3.1 Properties of Fourier series

3.1.1 Linearity

Proposition 30. Let the Fourier series coefficients of $x(t), y(t)$ be $\hat{X}[k], \hat{Y}[k]$ respectively. Then, for $a, b \in \mathbb{C}$, the Fourier series coefficients of $ax(t) + by(t)$ are $a\hat{X}[k] + b\hat{Y}[k]$. ○

Proof.

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(ax(t)+by(t))e^{-j2\pi\frac{kt}{T}}dt = a\cdot\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)e^{-j2\pi\frac{kt}{T}}dt + b\cdot\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}y(t)e^{-j2\pi\frac{kt}{T}}dt$$
$$= a\hat{X}[k]+b\hat{Y}[k].$$

■

3.1.2 Time shifting

Proposition 31. Let the Fourier series coefficients of $x(t)$ be $\hat{X}[k]$. Then the Fourier series coefficients of $x(t-t_0)$ are $e^{-j2\pi\frac{kt_0}{T}}\hat{X}[k]$. ○

Proof.

$$\begin{aligned}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t-t_0)e^{-j2\pi\frac{kt}{T}}dt &= \frac{1}{T}\int_{-\frac{T}{2}-t_0}^{\frac{T}{2}-t_0}x(\tau)e^{-j2\pi\frac{k(\tau+t_0)}{T}}d\tau && (\text{with } \tau := t-t_0)\\ &= e^{-j2\pi\frac{kt_0}{T}}\cdot\frac{1}{T}\int_{-\frac{T}{2}-t_0}^{\frac{T}{2}-t_0}x(\tau)e^{-j2\pi\frac{k\tau}{T}}d\tau\\ &= e^{-j2\pi\frac{kt_0}{T}}\cdot\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(\tau)e^{-j2\pi\frac{k\tau}{T}}d\tau && (\text{since } x(t) \text{ has period } T)\\ &= e^{-j2\pi\frac{kt_0}{T}}\hat{X}[k].\end{aligned}$$

■

3.1.3 Modulation

Proposition 32. Let the Fourier series coefficients of $x(t)$ be $\hat{X}[k]$. Then the Fourier series coefficients of $\frac{1}{N}\sum_{n=0}^{N-1}x\left(t-\frac{n}{N}T\right)$ are

$$\begin{cases}\hat{X}[k], & k\equiv 0 \pmod N,\\ 0, & \text{otherwise.}\end{cases}$$

○

Proof.

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}x\left(t-\frac{n}{N}T\right) &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=-\infty}^{+\infty}\hat{X}[k]e^{j2\pi\frac{k}{T}\left(t-\frac{n}{N}T\right)}\right)\\ &= \sum_{k=-\infty}^{+\infty}\hat{X}[k]e^{j2\pi\frac{kt}{T}}\left(\frac{1}{N}\sum_{n=0}^{N-1}e^{-j2\pi\frac{kn}{N}}\right)\\ &= \sum_{k=-\infty}^{+\infty}\hat{X}[k]e^{j2\pi\frac{kt}{T}}\left(\begin{cases}1, & k\equiv 0 \pmod N,\\ 0, & \text{otherwise}\end{cases}\right)\\ &= \sum_{k=-\infty}^{+\infty}\left(\begin{cases}\hat{X}[k], & k\equiv 0 \pmod N,\\ 0, & \text{otherwise}\end{cases}\right)e^{j2\pi\frac{kt}{T}}.\end{aligned}$$

■

3.1.4 Time reversal

Proposition 33. Let the Fourier series coefficients of $x(t)$ be $\hat{X}[k]$. Then the Fourier series coefficients of $x(-t)$ are $\hat{X}[-k]$. ○

Proof.

$$\begin{aligned}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(-t)e^{-j2\pi\frac{kt}{T}}dt &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)e^{-j2\pi\frac{k(-\tau)}{T}}d\tau & (\text{with } \tau := -t)\\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)e^{-j2\pi\frac{(-k)\tau}{T}}d\tau \\ &= \hat{X}[-k].\end{aligned}$$

■

3.1.5 Complex conjugation

Proposition 34. Let the Fourier series coefficients of $x(t)$ be $\hat{X}[k]$. Then the Fourier series coefficients of $\overline{x(t)}$ are $\overline{\hat{X}[-k]}$. ○

Proof.

$$\begin{aligned}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{x(t)}e^{-j2\pi\frac{kt}{T}}dt &= \overline{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{j2\pi\frac{kt}{T}}dt} \\ &= \overline{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{(-k)t}{T}}dt} \\ &= \overline{\hat{X}[-k]}.\end{aligned}$$

■

3.1.6 Convolution

Proposition 35. Let the Fourier series coefficients of $x(t), y(t)$ be $\hat{X}[k], \hat{Y}[k]$ respectively. Then the Fourier series coefficients of $x(t) \circledast_T y(t)$ are $\hat{X}[k]\hat{Y}[k]$, where $\circledast_T$ denotes the *periodic convolution* with period T:

$$x(t) \circledast_T y(t) := \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)y(t-\tau)d\tau,$$

and the Fourier series coefficients of $x(t)y(t)$ are $\hat{X}[k] * \hat{Y}[k]$. ○

Proof.

$$\begin{aligned}x(t) \circledast_T y(t) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)y(t-\tau)d\tau \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_{k=-\infty}^{+\infty} \hat{X}[k]e^{j2\pi\frac{k\tau}{T}}\right) y(t-\tau)d\tau\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{+\infty} \hat{X}[k] \left(\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t-\tau) e^{j2\pi \frac{k\tau}{T}} d\tau \right) \\
&= \sum_{k=-\infty}^{+\infty} \hat{X}[k] \left(\frac{1}{T} \int_{-\frac{T}{2}+t}^{\frac{T}{2}+t} y(\sigma) e^{j2\pi \frac{k(t-\sigma)}{T}} d\sigma \right) && \text{(with } \sigma := t-\tau\text{)} \\
&= \sum_{k=-\infty}^{+\infty} \hat{X}[k] e^{j2\pi \frac{kt}{T}} \left(\frac{1}{T} \int_{-\frac{T}{2}+t}^{\frac{T}{2}+t} y(\sigma) e^{-j2\pi \frac{k\sigma}{T}} d\sigma \right) \\
&= \sum_{k=-\infty}^{+\infty} \hat{X}[k] e^{j2\pi \frac{kt}{T}} \left(\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(\sigma) e^{-j2\pi \frac{k\sigma}{T}} d\sigma \right) && \text{(since } y(t) \text{ has period } T\text{)} \\
&= \sum_{k=-\infty}^{+\infty} \hat{X}[k] \hat{Y}[k] e^{j2\pi \frac{kt}{T}}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
x(t)y(t) &= \left(\sum_{k=-\infty}^{+\infty} \hat{X}[k] e^{j2\pi \frac{kt}{T}} \right) \left(\sum_{\ell=-\infty}^{+\infty} \hat{Y}[\ell] e^{j2\pi \frac{\ell t}{T}} \right) \\
&= \sum_{k=-\infty}^{+\infty} \sum_{\ell=-\infty}^{+\infty} \hat{X}[k] \hat{Y}[\ell] e^{j2\pi \frac{(k+\ell)t}{T}} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \hat{X}[k] \hat{Y}[m-k] e^{j2\pi \frac{mt}{T}} && \text{(with } m := k+\ell\text{)} \\
&= \sum_{m=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} \hat{X}[k] \hat{Y}[m-k] \right) e^{j2\pi \frac{mt}{T}} \\
&= \sum_{m=-\infty}^{+\infty} (\hat{X}[m] * \hat{Y}[m]) e^{j2\pi \frac{mt}{T}}.
\end{aligned}
$$

■

3.1.7 Differentiation

Proposition 36. Let the Fourier series coefficients of $x(t)$ be $\hat{X}[k]$. Then the Fourier series coefficients of $\frac{d}{dt}x(t)$ are $j2\pi\frac{k}{T}\hat{X}[k]$. ○

Proof.

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt} \sum_{k=-\infty}^{+\infty} \hat{X}[k] e^{j2\pi \frac{kt}{T}} \\
&= \sum_{k=-\infty}^{+\infty} \hat{X}[k] \frac{d}{dt} e^{j2\pi \frac{kt}{T}} \\
&= \sum_{k=-\infty}^{+\infty} j2\pi \frac{k}{T} \hat{X}[k] e^{j2\pi \frac{kt}{T}}.
\end{aligned}
$$

■

3.1.8 Parseval's theorem

Proposition 37. Let the Fourier series coefficients of $x(t), y(t)$ be $\hat{X}[k], \hat{Y}[k]$ respectively. Then

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{y(t)}dt = \sum_{k=-\infty}^{+\infty} \hat{X}[k]\overline{\hat{Y}[k]}.$$

In particular,

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |\hat{X}[k]|^2.$$

○

Proof.

$$\begin{aligned}
\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{y(t)}dt &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_{k=-\infty}^{+\infty} \hat{X}[k]e^{j2\pi\frac{kt}{T}}\right)\overline{\left(\sum_{\ell=-\infty}^{+\infty} \hat{Y}[\ell]e^{j2\pi\frac{\ell t}{T}}\right)}dt \\
&= \sum_{k=-\infty}^{+\infty}\sum_{\ell=-\infty}^{+\infty} \hat{X}[k]\overline{\hat{Y}[\ell]}\left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi\frac{kt}{T}}e^{-j2\pi\frac{\ell t}{T}}dt\right) \\
&= \sum_{k=-\infty}^{+\infty}\sum_{\ell=-\infty}^{+\infty} \hat{X}[k]\overline{\hat{Y}[\ell]}\left(\begin{cases}1, & k=\ell, \\ 0, & k\neq\ell\end{cases}\right) \\
&= \sum_{k=-\infty}^{+\infty} \hat{X}[k]\overline{\hat{Y}[k]}.
\end{aligned}$$

In particular, putting $y(t) = x(t)$ gives

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |\hat{X}[k]|^2.$$

■

3.2 Examples of Fourier series

Example 1. The *Riemann zeta function* $\zeta(s)$ is defined as

$$\zeta(s) := \sum_{k=1}^{+\infty}\frac{1}{k^s},$$

for $\mathrm{Re}(s) > 1$ (i.e. on which the summation is absolutely convergent). We will show that

$$\zeta(2) = \sum_{k=1}^{+\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

by Parseval's theorem. In fact, it is possible to explicitly compute all $\zeta(2n)$, where $n = 1, 2, 3, \ldots$.

Consider a square wave signal $x(t)$ as in Figure 3.1,

$$x(t) = \begin{cases}1, & t\in\left(nT, nT+\frac{T}{2}\right), n\in\mathbb{Z}, \\ 0, & \text{otherwise.}\end{cases}$$

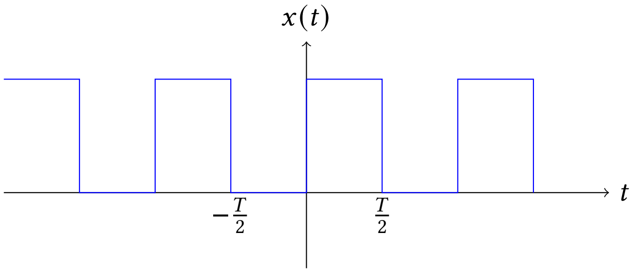


Figure 3.1: Square wave signal $x(t)$

The Fourier series coefficients of $x(t)$ are

$$\begin{aligned}\hat{X}[k] &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{kt}{T}}dt \\ &= \frac{1}{T}\int_{0}^{\frac{T}{2}} e^{-j2\pi\frac{kt}{T}}dt \\ &= \begin{cases}\frac{1}{2}, & k=0, \\ \frac{1}{j\pi k}, & k \text{ is odd}, \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

By Parseval's theorem,

$$\begin{aligned}\frac{1}{2} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt \\ &= \frac{1}{4} + \sum_{k\,\in\,\mathbb{Z},\,k \text{ odd}} \frac{1}{\pi^2 k^2} \\ &= \frac{1}{4} + 2\sum_{k\,\geq\,1,\,k \text{ odd}} \frac{1}{\pi^2 k^2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\pi^2}{8} &= \sum_{k\,\geq\,1,\,k \text{ odd}} \frac{1}{k^2} \\ &= \sum_{k\geq 1}\frac{1}{k^2} - \sum_{k\,\geq\,1,\,k \text{ even}} \frac{1}{k^2} \\ &= \sum_{k\geq 1}\frac{1}{k^2} - \frac{1}{4}\sum_{k\geq 1}\frac{1}{k^2} \\ &= \frac{3}{4}\zeta(2).\end{aligned}$$

Therefore $\zeta(2) = \frac{\pi^2}{6}$.

Exercise 7. Compute the Fourier series coefficients (with period T) of $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$, which is known as the *shah function*. ○

Solution. Note that $x(t) = \delta(t)$ on $\left[-\frac{T}{2}, \frac{T}{2}\right]$. We have

$$\hat{X}[k] = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{kt}{T}}dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-j2\pi\frac{kt}{T}}dt$$
$$= \frac{1}{T}.$$

Thus, $\hat{X}[k] = \frac{1}{T}$ for all $k \in \mathbb{Z}$. ♣

Exercise 8. Compute the Fourier series coefficients (with period $T = 1$) of $x(t) = \sin 6\pi t \cos 2\pi t$. ○

Solution. By Euler's formula, we have

$$\begin{aligned} x(t) &= \sin 6\pi t \cos 2\pi t \\ &= \frac{e^{j6\pi t} - e^{-j6\pi t}}{2j} \cdot \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} \\ &= \frac{1}{4j}(e^{j8\pi t} + e^{j4\pi t} - e^{-j4\pi t} - e^{-j8\pi t}). \end{aligned}$$

Therefore, the Fourier series coefficients are

$$\hat{X}[k] = \begin{cases} -\frac{1}{4j}, & k = -4, -2, \\ \frac{1}{4j}, & k = 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

♣

Exercise 9. Compute the Fourier series coefficients (with period $T = 1$) of $x(t) = \sum_{k=-\infty}^{+\infty} x_0(t) * \delta(t-k)$, where

$$x_0(t) = \begin{cases} \sin \pi t, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

○

Method 1. We first try to solve this by direct computation.

$$\begin{aligned} \hat{X}[k] &= \int_0^1 (\sin \pi t)e^{-j2\pi kt}dt \\ &= \int_0^1 \frac{e^{j\pi t} - e^{-j\pi t}}{2j} e^{-j2\pi kt}dt \\ &= \frac{1}{2j}\left(\frac{1}{-j2\pi k + j\pi}e^{(-j2\pi k + j\pi)t} - \frac{1}{-j2\pi k - j\pi}e^{(-j2\pi k - j\pi)t}\right)\Bigg|_0^1 \\ &= -\frac{2}{(4k^2 - 1)\pi}. \end{aligned}$$

♣

Method 2. Consider

$$x'(t) = \sum_{k=-\infty}^{+\infty} x_0'(t) * \delta(t-k)$$

$$= \sum_{k=-\infty}^{+\infty} \pi x_1(t) * \delta(t-k),$$

where

$$x_1(t) = \begin{cases} \cos \pi t, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also,

$$\begin{aligned} x''(t) &= \sum_{k=-\infty}^{+\infty} \pi x_1'(t) * \delta(t-k) \\ &= \sum_{k=-\infty}^{+\infty} (-\pi^2 x_0(t) + 2\pi\delta(t)) * \delta(t-k) \\ &= -\pi^2 x(t) + 2\pi \sum_{k=-\infty}^{+\infty} \delta(t-k). \end{aligned}$$

Recall that the shah function $\sum_{k=-\infty}^{+\infty} \delta(t-k)$ has Fourier series $\sum_{k=-\infty}^{+\infty} 1 \cdot e^{j2\pi k}$. Then

$$(j2\pi k)^2 \hat{X}[k] = -\pi^2 \hat{X}[k] + 2\pi.$$

It follows that

$$\hat{X}[k] = -\frac{2}{(4k^2-1)\pi}.$$

♣

Exercise 10. Suppose that the Fourier series coefficients of $x(t)$ with period 1 are $\hat{X}[k]$. Find the function whose Fourier series coefficients are

$$\begin{cases} \hat{X}[k], & k \equiv 0 \pmod 5, \\ 0, & \text{otherwise.} \end{cases}$$

○

Solution. The answer is

$$\frac{1}{5}\left(x(t) + x\left(t - \frac{1}{5}\right) + x\left(t - \frac{2}{5}\right) + x\left(t - \frac{3}{5}\right) + x\left(t - \frac{4}{5}\right)\right).$$

To verify, note that the Fourier series coefficients of this function are

$$\frac{1}{5}(1 + \omega^k + \omega^{2k} + \omega^{3k} + \omega^{4k})\hat{X}[k],$$

where $\omega = e^{-j2\pi\frac{1}{5}}$. And observe that

$$1 + \omega^k + \omega^{2k} + \omega^{3k} + \omega^{4k} = \begin{cases} 5, & k \equiv 0 \pmod 5, \\ 0, & \text{otherwise.} \end{cases}$$

♣

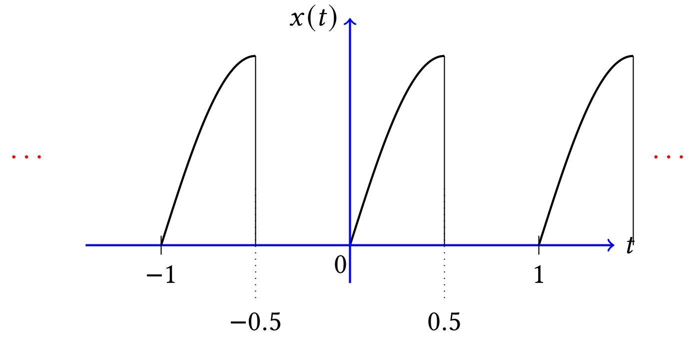


Figure 3.2: The signal $x(t)$

Exercises

1. **(22 Midterm 1)** Let $x(t)$ be the periodic extension (with $T_0 = 1$) of $x_p(t)$ where

$$x_p(t) = \begin{cases} \sin(\pi t) & t \in [0, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}.$$

   (1) Compute, using integration, the (exponential) Fourier Series coefficients of $x(t)$, i.e, compute $\{\hat{X}[k]\}$ where

$$x(t) = \sum_k \hat{X}[k] e^{j2\pi kt}.$$

   (2) Compute $x_1(t)$ and $r(t)$

$$x'(t) = x_1(t) - \sum_{k \in \mathbb{Z}} \delta(t - k - \frac{1}{2})$$
$$x_1'(t) = -\pi^2 x(t) + r(t)$$

   (3) Use the differentiation property of the Fourier Series to re-derive the Fourier Series co-efficients $\{\hat{X}[k]\}$that you obtained in the first part.

   (4) Using the Fourier Series co-efficients computed above, determine the value of (justify your steps)

$$\sum_{k \geq 1} \frac{4k^2 + 1}{(4k^2 - 1)^2}.$$

2. **(22 Make-up Midterm 1)** Let $x(t)$ be the periodic extension (with $T_0 = 1$) of $x_p(t)$ where

$$x_p(t) = \begin{cases} te^{-|t|} & t \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}.$$

   Compute the Fourier Series coefficients of $x(t)$.

3. **(22 Make-up Midterm 1)** *Short answer questions*

   (1) Consider a memoryless time-invariant system such that for any input $x(t)$ with $x(0) = a$, the output value at $t = 0$, $y(0) = 1 + a^2$. Compute and plot the output $y(t)$

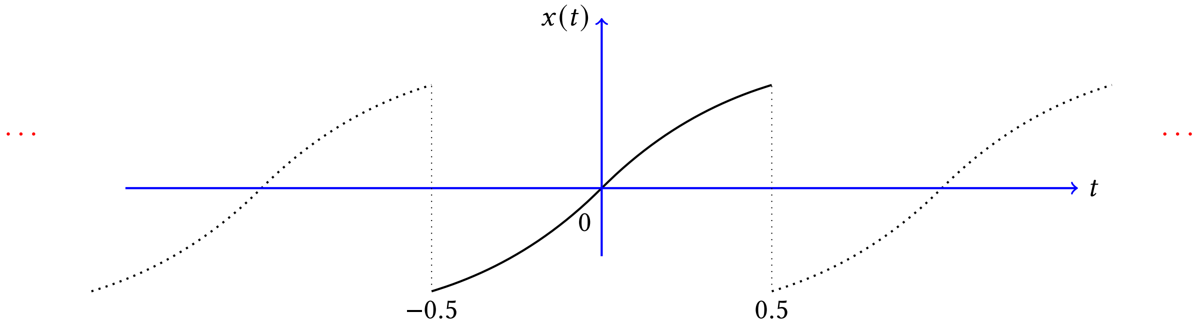


Figure 3.3: The signal $x(t)$

of the system when the input

$$x(t) = \begin{cases} t+1, & -1 \le t < 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

(2) Suppose a periodic function $x(t)$, with period 1, has all its even coefficients in its Fourier Series representation to be zero. Then compute the function given by

$$x_1(t) = x(t) + x\left(t - \frac{2k+1}{2}\right)$$

where $k \in \mathbb{N}$ is some fixed integer.

(3) Let $x(t)$ be periodic with period T. The signal $\hat{x}(t)$ is not necessarily periodic. Is $y(t)$ periodic where

$$y(t) = \int_{-T/2}^{T/2} x(\tau)\hat{x}(t-\tau)d\tau.$$

(4) Is the cascade (series combination) of two stable systems (not necessarily LTI) stable?

4. **(22 Midterm 1)** *ELITE question: compulsory for ELITE students; optional for others.*

We know that for a linear shift invariant system, the eigen-basis depends only on N. Here we are considering $N = 2$. Let $[x[0],\ x[1]]$ be a vector such that $|x[0]|^2 + |x[1]|^2 = 1$. Assume that all scalars are complex numbers.

Define

$$\hat{X}[0] = \frac{1}{\sqrt{2}}(x[0] + x[1])$$
$$\hat{X}[1] = \frac{1}{\sqrt{2}}(x[0] - x[1])$$

(1) Show that $|\hat{X}[0]|^2 + |\hat{X}[1]|^2 = 1$.

(2) Define two probability vectors according to $p_0 = |x[0]|^2, p_1 = |x[1]|^2$ and $q_0 = |\hat{X}[0]|^2, q_1 = |\hat{X}[1]|^2$. Prove that

$$-p_0 \log p_0 - p_1 \log p_1 - q_0 \log q_0 - q_1 \log q_1 \ge \log 2.$$

5. **(22 Final)** Let $x(t)$ be the periodic extension with $T_0 = 1$ of $x_p(t)$ where

$$x_p(t) = \begin{cases} (2+4t) & t \in [-\frac{1}{2}, -\frac{1}{4}) \\ \frac{3}{2} + 2t & t \in [-\frac{1}{4}, 0) \\ \frac{3}{2} - 2t & t \in [0, \frac{1}{4}) \\ (2-4t) & t \in [\frac{1}{4}, \frac{1}{2}) \end{cases}.$$

That is $x(t) = \sum_{n\in\mathbb{Z}} x_p(t-n)$. (as shown in Figure 3.4) Compute the Fourier Series

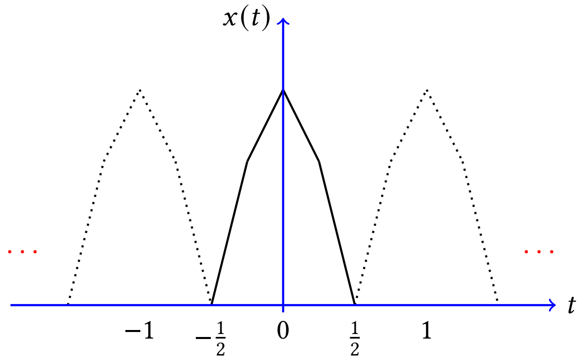


Figure 3.4: The signal $x(t)$

coefficients of $x(t)$, where $x(t) = \sum_k \hat{X}[k]e^{j2\pi kt}$.

(Hint: This question can be done in a very time-efficient manner if you think about it)

6. Calculate the Fourier series with the given period interval (Hint: Use properties of Fourier-Series to deduce the answers from earlier parts easily.)

   a) $x^a(t) = \begin{cases} 0, & t \in [-1, 0] \\ t, & t \in [0, 1] \end{cases}$.

   b) $x^b(t) = t, [-1, 1]$

   c) $x^c(t) = t + 1, [-1, 1]$

   d) $x^d(t) = t, [0, 2]$

   e) $x^e(t) = 2t, [0, 1]$

   f) $x^f(t) = |t|, [-1, 1]$

   g) $x^g(t) = x^a(2t)$

      - Do this part assuming the period remains the same, i.e. $T = 2$
      - Do this part with the new period, i.e. $T = 1$.

7. Let $x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt}$ and let $x_2(t) = \sum_{k=-\infty}^{\infty} b_k e^{j2\pi kt}$. Here assume that $a_k > 0\ \forall k$ and that $b_k = \frac{k}{a_k}$, when $|k| = 1$ and $b_k = 0$ otherwise. Further, define $y(t) = \int_{-1/2}^{1/2} x_1(\tau)x_2(t-\tau)d\tau$.

   Compute and plot $y(t)$. [Hint: Use orthogonality.]

8. Assume $T_0 = 1$, i.e. express $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt}$. Compute a_k when

   a) $x(t) = \cos(50\pi t)$

b) $x(t) = \cos(50\pi t - \pi/4)$

9. **Cauchy-Schwartz inequality and Triangle inequality**: Consider a vector space V over complex numbers (Vector space is a collection of objects closed under addition and scalar multiplication). An *inner product* is a function defined $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$ that satisfies the following properties:

   - $\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle$. The operation is linear in the first variable.
   - $\langle x, y \rangle = \langle y, x \rangle^*$ where a^* denotes the complex conjugation.
   - $\langle x, x \rangle \geq 0$ with equality *if and only if* $x = 0$. This property is called positive definiteness.
   - **Prove** the following inequality for any inner product (also called Cauchy-Schwartz inequality).
   $$|\langle x, y \rangle| \quad \leq \quad \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}.$$
   (Hint: consider $\langle x - ay, x - ay \rangle$ where $a = \frac{\langle x, y \rangle}{\langle y, y \rangle}$.)

   In an inner product space, the length (or norm) of a vector is usually computed as
   $$\|x\| = \langle x, x \rangle^{\frac{1}{2}}.$$

   - Show that the lengths of vectors satisfy *the triangle inequality*, i.e.
   $$\|x + y\| \leq \|x\| + \|y\|.$$

Chapter 4

Fourier transform

The key idea of the Fourier transform is to express signals as a linear combination of a set $\left\{t \mapsto e^{j2\pi f t}\right\}_{f\in\mathbb{R}}$ of eigenfunctions of LTI systems. While the Fourier series deals with periodic functions, the Fourier transform extends this idea to aperiodic signals.

4.1 Introduction

4.1.1 Fourier series with period tending to infinity

One can view the Fourier transform as the Fourier series with period $T \to \infty$. Consider an aperiodic signal $x(t)$. Let $x_T(t)$ denote its truncation to the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$ and let $x_T^{(p)}(t) := x_T(t) * \left(\sum_{k=-\infty}^{+\infty} \delta(t-kT)\right)$ be its periodic extension. Then Fourier series allows us to get

$$x_T^{(p)}(t) = \sum_{k=-\infty}^{+\infty} \hat{X}^{(p)}[k] e^{j2\pi \frac{kt}{T}},$$

where

$$\begin{aligned}\hat{X}^{(p)}[k] &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_T^{(p)}(t) e^{-j2\pi\frac{kt}{T}}\,dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi\frac{kt}{T}}\,dt.\end{aligned}$$

Let us denote $\hat{X}_T\left(\frac{k}{T}\right) := T\hat{X}^{(p)}[k]$ and $f_k := \frac{k}{T}$. Under this notation, we have

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} \frac{1}{T}\hat{X}_T\left(\frac{k}{T}\right) e^{j2\pi\frac{kt}{T}} \\ &= \sum_{k=-\infty}^{+\infty} \frac{1}{T}\hat{X}_T(f_k) e^{j2\pi f_k t}, \qquad \left(\text{for } t \in \left(-\frac{T}{2}, \frac{T}{2}\right)\right) \\ \hat{X}_T(f_k) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi f_k t} dt.\end{aligned}$$

Finally, define $\hat{X}_T(f) := \hat{X}_T(f_{\lfloor fT \rfloor}) = \hat{X}_T\left(\frac{\lfloor fT \rfloor}{T}\right)$ for $f \in \mathbb{R}$. Then we have

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} \frac{1}{T}\hat{X}_T(f_k) e^{j2\pi f_k t} \\ &= \sum_{k=-\infty}^{+\infty} \int_{\frac{k}{T}}^{\frac{k+1}{T}} \hat{X}_T(f_k) e^{j2\pi f_k t} df \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{+\infty} \int_{\frac{k}{T}}^{\frac{k+1}{T}} \hat{X}_T(f) e^{j2\pi \frac{\lfloor fT \rfloor}{T} t} df \\
&= \int_{-\infty}^{+\infty} \hat{X}_T(f) e^{j2\pi \frac{\lfloor fT \rfloor}{T} t} df, \qquad \left(\text{for } t \in \left(-\frac{T}{2}, \frac{T}{2}\right)\right) \\
\hat{X}_T(f) &= \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi \frac{\lfloor fT \rfloor}{T} t} dt.
\end{aligned}
$$

Now, assuming the existence of various limits and smoothness, denote $\hat{X}(f) := \lim_{T\to\infty} \hat{X}_T(f)$ and observe that

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{+\infty} \hat{X}(f) e^{j2\pi f t} df, \\
\hat{X}(f) &= \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt.
\end{aligned}
$$

The two integrals above will be referred to as the *inverse Fourier transform* and *Fourier transform* integrals respectively.

4.1.2 Interpretation as an eigenvalue

Consider an LTI system with impulse response $h(t)$. With the input to the system being $x(t) = e^{j2\pi f t}$, the output of the system will be

$$
\begin{aligned}
x(t) * h(t) &= \int_{-\infty}^{+\infty} h(\tau) e^{j2\pi f (t-\tau)} d\tau \\
&= e^{j2\pi f t} \left(\int_{-\infty}^{+\infty} h(\tau) e^{-j2\pi f \tau} d\tau \right) \\
&=: \hat{H}(f) x(t),
\end{aligned}
$$

where we have put $\hat{H}(f) := \int_{-\infty}^{+\infty} h(\tau) e^{-j2\pi f \tau} d\tau$. Thus, $e^{j2\pi f t}$ is an eigenfunction of all LTI systems with eigenvalue $\hat{H}(f)$.

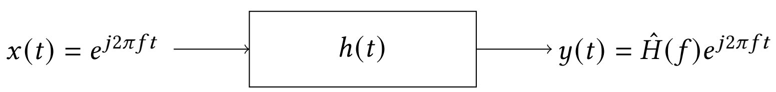


Figure 4.1: Fourier transform as the eigenvalue of $e^{j2\pi f t}$

4.1.3 Definition

Definition 4.1. The *Fourier transform* of a signal $x(t)$ is defined as

$$
\hat{X}(f) = \mathcal{F}\{x(t)\}(f) := \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt.
$$

○

Remark 4.1. It turns out that the Fourier transform can be often inverted (see https://en.wikipedia.org/wiki/Fourier_inversion_theorem). ○

Definition 4.2. The *inverse Fourier transform* of a signal $\hat{X}(f)$ is defined as

$$x(t) = \mathcal{F}^{-1}\{\hat{X}(f)\}(t) := \int_{-\infty}^{+\infty} \hat{X}(f) e^{j2\pi f t} df.$$

○

Remark 4.2. In this class, we are a bit fast and loose in being rigorous for the sake of developing engineering intuition. We say that $x(t) \xrightarrow{\text{FT}} \hat{X}(f)$ forms a *Fourier transform pair* if either $\mathcal{F}\{x(t)\}(f) = \hat{X}(f)$ or $\mathcal{F}^{-1}\{\hat{X}(f)\}(t) = x(t)$. This relaxation allows us to deal with delta functions in a much easier fashion. ○

4.2 Properties of Fourier transform

4.2.1 Linearity

Proposition 38. Let the Fourier transform of $x(t), y(t)$ be $\hat{X}(f), \hat{Y}(f)$ respectively. Then, for any $a, b \in \mathbb{C}$, the Fourier transform of $ax(t) + by(t)$ is $a\hat{X}(f) + b\hat{Y}(f)$. ○

Proof.

$$\begin{aligned}
\mathcal{F}\{ax(t) + by(t)\}(f) &= \int_{-\infty}^{+\infty} (ax(t) + by(t)) e^{-j2\pi f t} dt \\
&= a \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt + b \int_{-\infty}^{+\infty} y(t) e^{-j2\pi f t} dt \\
&= a\hat{X}(f) + b\hat{Y}(f).
\end{aligned}$$

■

4.2.2 Time/frequency shifting

Proposition 39. Let the Fourier transform of $x(t)$ be $\hat{X}(f)$. Then the Fourier transform of $x(t - t_0)$ is $e^{-j2\pi f t_0}\hat{X}(f)$ and the Fourier transform of $e^{j2\pi f_0 t}x(t)$ is $\hat{X}(f - f_0)$. ○

Proof.

$$\begin{aligned}
\mathcal{F}\{x(t - t_0)\}(f) &= \int_{-\infty}^{+\infty} x(t - t_0) e^{-j2\pi f t} dt \\
&= \int_{-\infty}^{+\infty} x(\tau) e^{-j2\pi f(\tau + t_0)} d\tau && \text{(with } \tau := t - t_0\text{)} \\
&= e^{-j2\pi f t_0} \int_{-\infty}^{+\infty} x(\tau) e^{-j2\pi f \tau} d\tau \\
&= e^{-j2\pi f t_0} \hat{X}(f).
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}^{-1}\{\hat{X}(f - f_0)\}(t) &= \int_{-\infty}^{+\infty} \hat{X}(f - f_0) e^{j2\pi f t} df \\
&= \int_{-\infty}^{+\infty} \hat{X}(\nu) e^{j2\pi(\nu + f_0)t} d\nu && \text{(with } \nu := f - f_0\text{)} \\
&= e^{j2\pi f_0 t} \int_{-\infty}^{+\infty} \hat{X}(\nu) e^{j2\pi \nu t} d\nu \\
&= e^{j2\pi f_0 t} x(t).
\end{aligned}$$

■

4.2.3 Scaling

Proposition 40. Let the Fourier transform of $x(t)$ be $\hat{X}(f)$. Then, for any $a \in \mathbb{R} \setminus \{0\}$, the Fourier transform of $x(at)$ is $\frac{1}{|a|}\hat{X}\left(\frac{f}{a}\right)$. ○

Proof.

$$\begin{aligned}
\mathcal{F}\{x(at)\}(f) &= \int_{-\infty}^{+\infty} x(at)e^{-j2\pi f t}dt \\
&= \frac{1}{|a|}\int_{-\infty}^{+\infty} x(\tau)e^{-j2\pi\left(\frac{f}{a}\right)\tau}d\tau \qquad \text{(with } \tau := at\text{)} \\
&= \frac{1}{|a|}\hat{X}\left(\frac{f}{a}\right).
\end{aligned}$$

■

4.2.4 Complex conjugation

Proposition 41. Let the Fourier transform of $x(t)$ be $\hat{X}(f)$. Then the Fourier transform of $\overline{x(t)}$ is $\overline{\hat{X}(-f)}$. ○

Proof.

$$\begin{aligned}
\mathcal{F}\left\{\overline{x(t)}\right\}(f) &= \int_{-\infty}^{+\infty} \overline{x(t)}e^{-j2\pi f t}dt \\
&= \overline{\left(\int_{-\infty}^{+\infty} x(t)e^{-j2\pi(-f)t}dt\right)} \\
&= \overline{\hat{X}(-f)}.
\end{aligned}$$

■

4.2.5 Duality

Proposition 42. Let the Fourier transform of $x(t)$ be $\hat{X}(f)$. Then the Fourier transform of $\hat{X}(f)$ is the reflected signal, $x(-t)$. In other words, $\mathcal{F}\{\mathcal{F}\{x(t)\}(f)\}(t) = x(-t)$. ○

Proof.

$$\begin{aligned}
\mathcal{F}\{\hat{X}(f)\}(t) &= \int_{-\infty}^{+\infty} \hat{X}(f)e^{-j2\pi f t}df \\
&= \int_{-\infty}^{+\infty} \hat{X}(f)e^{j2\pi f(-t)}df \\
&= \mathcal{F}^{-1}\{\hat{X}(f)\}(-t) \\
&= x(-t).
\end{aligned}$$

■

Remark 4.3. The Fourier transform of $\delta(t)$ is 1 by the definition of the delta function. By duality, we see that the Fourier transform of 1 is $\delta(-f) = \delta(f)$. However,

$$\int_{-\infty}^{+\infty} e^{-j2\pi f t}dt$$

is not defined. ○

4.2.6 Convolution

Proposition 43. Let the Fourier transforms of $x(t), y(t)$ be $\hat{X}(f), \hat{Y}(f)$ respectively. Then the Fourier transform of $x(t)*y(t)$ is $\hat{X}(f)\hat{Y}(f)$ and the Fourier transform of $x(t)y(t)$ is $\hat{X}(f)*\hat{Y}(f)$. ○

Proof.

$$\begin{aligned}
\mathcal{F}\{x(t)*y(t)\}(f) &= \int_{-\infty}^{+\infty} (x(t)*y(t))e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau\right) e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)y(t-\tau)e^{-j2\pi f\tau}e^{-j2\pi f(t-\tau)}d\tau dt \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)y(\sigma)e^{-j2\pi f\tau}e^{-j2\pi f\sigma}d\tau d\sigma \qquad (\text{with } \sigma := t-\tau) \\
&= \left(\int_{-\infty}^{+\infty} x(\tau)e^{-j2\pi f\tau}d\tau\right)\left(\int_{-\infty}^{+\infty} y(\sigma)e^{-j2\pi f\sigma}d\sigma\right) \\
&= \hat{X}(f)\hat{Y}(f).
\end{aligned}$$

Now, by duality of the Fourier transform, $x(t), y(t)$ are the Fourier transforms of $\hat{X}(-f), \hat{Y}(-f)$ respectively. Then the Fourier transform of $\hat{X}(-f)*\hat{Y}(-f)$ is $x(t)y(t)$. Again by duality of the Fourier transform, we have

$$\mathcal{F}\{x(t)y(t)\}(f) = \hat{X}(f)*\hat{Y}(f).$$

■

Remark 4.4. Alternatively, this can be understood using the eigenvalue interpretation of the Fourier transform. Cascading two LTI systems with impulse responses $x(t)$ and $y(t)$ gives an LTI system with impulse response $x(t)*y(t)$. Let $e^{j2\pi ft}$ be the input to the system. Considering the system as a whole, we see that the output is $\mathcal{F}\{x(t)*y(t)\}(f)e^{j2\pi ft}$. Considering the system as the combination of its constituents, we see that the output is $\hat{X}(f)\hat{Y}(f)e^{j2\pi ft}$. Thus, $\mathcal{F}\{x(t)*y(t)\}(f) = \hat{X}(f)\hat{Y}(f)$. ○

4.2.7 Differentiation

Proposition 44. Let the Fourier transform of $x(t)$ be $\hat{X}(f)$. Then the Fourier transform of $x'(t)$ is $j2\pi f\hat{X}(f)$. ○

Proof.

$$\begin{aligned}
x'(t) &= \frac{d}{dt}\mathcal{F}^{-1}\{\hat{X}(f)\}(t) \\
&= \frac{d}{dt}\left(\int_{-\infty}^{+\infty} \hat{X}(f)e^{j2\pi ft}df\right) \\
&= \int_{-\infty}^{+\infty} \frac{d}{dt}\left(\hat{X}(f)e^{j2\pi ft}\right)df \\
&= \int_{-\infty}^{+\infty} j2\pi f\hat{X}(f)e^{j2\pi ft}df \\
&= \mathcal{F}^{-1}\{j2\pi f\hat{X}(f)\}(t).
\end{aligned}$$

■

4.2.8 Integration

Proposition 45. Let the Fourier transform of $x(t)$ be $\hat{X}(f)$. Then the Fourier transform of $\int_{-\infty}^{t} x(\tau)d\tau$ is $\frac{\hat{X}(f)}{j2\pi f} + \frac{1}{2}\hat{X}(0)\delta(f)$. ○

Proof. Recall that the unit step signal is given by

$$u(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Note that $\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$. By the convolution property, we only need to show that the Fourier transform of $u(t)$ is $\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$. This is done in Example 10. ■

4.2.9 Parseval's theorem

Proposition 46. Let the Fourier transform of $x(t), y(t)$ be $\hat{X}(f), \hat{Y}(f)$ respectively. Then

$$\int_{-\infty}^{+\infty} x(t)\overline{y(t)}dt = \int_{-\infty}^{+\infty} \hat{X}(f)\overline{\hat{Y}(f)}df.$$

In particular,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |\hat{X}(f)|^2 df.$$

○

Proof.

$$\begin{aligned}
\int_{-\infty}^{+\infty} x(t)\overline{y(t)}dt &= \int_{-\infty}^{+\infty} \mathcal{F}^{-1}\{\hat{X}(f)\}(t)\overline{y(t)}dt \\
&= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \hat{X}(f)e^{j2\pi ft}df\right)\overline{y(t)}dt \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{X}(f)\overline{y(t)}e^{j2\pi ft}dfdt \\
&= \int_{-\infty}^{+\infty} \hat{X}(f)\overline{\left(\int_{-\infty}^{+\infty} y(t)e^{-j2\pi ft}dt\right)}df \\
&= \int_{-\infty}^{+\infty} \hat{X}(f)\overline{\hat{Y}(f)}df.
\end{aligned}$$

In particular, putting $y(t) = x(t)$ gives

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |\hat{X}(f)|^2 df.$$

■

Remark 4.5. We have assumed that both $x(t)$ and $y(t)$ are square-integrable functions to justify the various interchanges of the integral. ○

4.2.10 Poisson summation formula

Proposition 47. Let the Fourier transform of $x(t)$ be $\hat{X}(f)$. Then

$$\sum_{t\in\mathbb{Z}} x(t) = \sum_{f\in\mathbb{Z}} \hat{X}(f).$$

○

Proof. One can show that the Fourier transform of the shah function $\sum_{k\in\mathbb{Z}} \delta(t-k)$ is $\sum_{k\in\mathbb{Z}} \delta(f-k)$. This is done in Example 3. Now consider

$$\begin{aligned}\mathcal{F}\left\{x(t)\sum_{k\in\mathbb{Z}}\delta(t-k)\right\}(f) &= \hat{X}(f) * \sum_{k\in\mathbb{Z}}\delta(f-k)\\ &= \sum_{k\in\mathbb{Z}}\hat{X}(f) * \delta(f-k)\\ &= \sum_{k\in\mathbb{Z}}\hat{X}(f-k).\end{aligned}$$

But

$$\begin{aligned}\mathcal{F}\left\{x(t)\sum_{k\in\mathbb{Z}}\delta(t-k)\right\}(f) &= \mathcal{F}\left\{\sum_{k\in\mathbb{Z}}x(t)\delta(t-k)\right\}(f)\\ &= \mathcal{F}\left\{\sum_{k\in\mathbb{Z}}x(k)\delta(t-k)\right\}(f)\\ &= \sum_{k\in\mathbb{Z}}x(k)\mathcal{F}\left\{\delta(t-k)\right\}(f)\\ &= \sum_{k\in\mathbb{Z}}x(k)\int_{-\infty}^{+\infty}\delta(t-k)e^{-j2\pi ft}dt\\ &= \sum_{k\in\mathbb{Z}}x(k)e^{-j2\pi fk}.\end{aligned}$$

Therefore,

$$\sum_{k\in\mathbb{Z}}x(k)e^{-j2\pi fk} = \sum_{k\in\mathbb{Z}}\hat{X}(f-k).$$

Setting $f = 0$ gives the result. ■

4.3 Examples of Fourier transform

Example 1. $\delta(t) \xrightarrow{\text{FT}} 1$ and $1 \xrightarrow{\text{FT}} \delta(f)$.

Proof.

$$\begin{aligned}\mathcal{F}\{\delta(t)\}(f) &= \int_{-\infty}^{+\infty}\delta(t)e^{-j2\pi ft}dt = 1,\\ \mathcal{F}^{-1}\{\delta(f)\}(t) &= \int_{-\infty}^{+\infty}\delta(f)e^{j2\pi ft}df = 1.\end{aligned}$$

■

Remark 4.6. It is tempting to compute the Fourier transform of $x(t) = 1$ by integration. However, the integral (or even the Cauchy principal value) is not well-defined.

When $f \neq 0$, to compute the Cauchy principal value, consider

$$\begin{aligned} D_T &= \int_{-T}^{T} e^{-j2\pi f t} dt \\ &= \frac{\sin(2\pi f T)}{\pi f}. \end{aligned}$$

As $T \to \infty$, D_T follows a sinusoidal path with limit superior $\frac{1}{\pi f}$ and limit inferior $-\frac{1}{\pi f}$. Therefore, the limit of the integral as $T \to \infty$ does not exist.

However, we will use the duality of the Fourier transform to deduce that, since the inverse Fourier transform of $\delta(f)$ is 1, we take the Fourier transform to be $\delta(f)$.

In essence, if we apply the duality property blindly, we get $\mathcal{F}\{\mathcal{F}\{\delta(t)\}(f)\}(t) = \delta(-t) = \delta(t)$, where the last equality follows from that δ is an even function. Since $\mathcal{F}\{\delta(t)\}(f) = 1$, we have $\mathcal{F}\{1\}(f) = \delta(f)$. ○

Example 2. $\delta(t - t_0) \xrightarrow{\text{FT}} e^{-j2\pi t_0 f}$ and $e^{j2\pi f_0 t} \xrightarrow{\text{FT}} \delta(f - f_0)$.

Proof. We already know this when $t_0 = 0$ and $f_0 = 0$. Apply the time/frequency shifting property of the Fourier transform. ■

Example 3. $\sum_{k\in\mathbb{Z}} \delta(t - k) \xrightarrow{\text{FT}} \sum_{k\in\mathbb{Z}} \delta(f - k)$.

Proof. We have the Fourier series representation of the shah function

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \delta(t - k) &= \sum_{k\in\mathbb{Z}} e^{-j2\pi k t} \\ &= \sum_{k\in\mathbb{Z}} e^{j2\pi k t}. \end{aligned}$$

Taking Fourier transform, we have

$$\begin{aligned} \mathcal{F}\left\{\sum_{k\in\mathbb{Z}} \delta(t - k)\right\}(f) &= \mathcal{F}\left\{\sum_{k\in\mathbb{Z}} e^{j2\pi k t}\right\}(f) \\ &= \sum_{k\in\mathbb{Z}} \mathcal{F}\{e^{j2\pi k t}\}(f) \\ &= \sum_{k\in\mathbb{Z}} \delta(f - k). \end{aligned}$$

■

Example 4. $e^{-at}u(t) \xrightarrow{\text{FT}} \frac{1}{a+j2\pi f}$ and $e^{at}u(-t) \xrightarrow{\text{FT}} \frac{1}{a-j2\pi f}$, for any $a > 0$.

Proof.

$$\begin{aligned} \mathcal{F}\{e^{-at}u(t)\}(f) &= \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j2\pi f t} dt \\ &= \int_{0}^{+\infty} e^{-(a+j2\pi f)t} dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a + j2\pi f}, \\
\mathcal{F}\{e^{at}u(-t)\}(f) &= \int_{-\infty}^{+\infty} e^{at}u(-t)e^{-j2\pi f t}dt \\
&= \int_{-\infty}^{0} e^{(a-j2\pi f)t}dt \\
&= \frac{1}{a - j2\pi f}.
\end{aligned}
$$

■

Example 5. $\frac{1}{a+j2\pi t} \xrightarrow{\text{FT}} e^{af}u(-f)$ and $\frac{1}{a-j2\pi t} \xrightarrow{\text{FT}} e^{-af}u(f)$, for any $a > 0$.

Proof. This follows from duality. ■

Example 6. $\frac{1}{1+t^2} \xrightarrow{\text{FT}} \pi e^{-2\pi|f|}$.

Proof. Recall that for $a > 0$,

$$
\begin{aligned}
\frac{1}{a + j2\pi t} &\xrightarrow{\text{FT}} e^{af}u(-f), \\
\frac{1}{a - j2\pi t} &\xrightarrow{\text{FT}} e^{-af}u(f).
\end{aligned}
$$

Note that

$$
\frac{1}{a + j2\pi t} + \frac{1}{a - j2\pi t} = \frac{2a}{a^2 + 4\pi^2 t^2}.
$$

When $a = 2\pi$, we have

$$
\frac{1}{2\pi + j2\pi t} + \frac{1}{2\pi - j2\pi t} = \frac{1}{\pi(1 + t^2)} \xrightarrow{\text{FT}} e^{-2\pi f}u(f) + e^{2\pi f}u(-f) = e^{-2\pi|f|}.
$$

■

Example 7. $\text{rect}(t) \xrightarrow{\text{FT}} \text{sinc}(f)$ and $\text{sinc}(t) \xrightarrow{\text{FT}} \text{rect}(f)$, where the *sinc function* $\text{sinc}(f)$ is defined as

$$
\text{sinc}(f) := \frac{\sin \pi f}{\pi f}.
$$

Proof.

$$
\begin{aligned}
\mathcal{F}\{\text{rect}(t)\}(f) &= \int_{-\infty}^{+\infty} \text{rect}(t)e^{-j2\pi f t}dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t}dt \\
&= \frac{1}{-j2\pi f}(e^{-j\pi f} - e^{j\pi f}) \\
&= \frac{-j2 \sin \pi f}{-j2\pi f} \\
&= \frac{\sin \pi f}{\pi f} \\
&= \text{sinc}(f).
\end{aligned}
$$

By duality, we have $\text{sinc}(t) \xrightarrow{\text{FT}} \text{rect}(-f) = \text{rect}(f)$. ■

Exercise 11. Show by contour integration that the Cauchy principal value of $\mathcal{F}\{\text{sinc}(t)\}(f)$ is given by

$$\lim_{M\to\infty}\int_{-M}^{M}\text{sinc}(t)e^{-j2\pi ft}dt = \text{rect}(f).$$

○

Example 8. $\text{tri}(t) \xrightarrow{\text{FT}} \text{sinc}^2(f)$.

Proof. Observe that $\text{tri}(t) = \text{rect}(t) * \text{rect}(t)$. Recall that $\text{rect}(t) \xrightarrow{\text{FT}} \text{sinc}(f)$. Apply the convolution property of the Fourier transform. ■

Example 9. $\text{sinc}(t) * \text{sinc}(t) \xrightarrow{\text{FT}} \text{rect}(f)$.

Proof. Recall that $\text{sinc}(t) \xrightarrow{\text{FT}} \text{rect}(f)$. By the convolution property of the Fourier transform, we have $\mathcal{F}\{\text{sinc}(t) * \text{sinc}(t)\}(f) = \text{rect}^2(f) = \text{rect}(f)$. ■

Example 10. $u(t) \xrightarrow{\text{FT}} \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$.

Proof. Observe that $u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$. Since $\frac{1}{2} \xrightarrow{\text{FT}} \frac{1}{2}\delta(f)$, it suffices to show that $\text{sgn}(t) \xrightarrow{\text{FT}} \frac{1}{j\pi f}$.

Note that $\frac{d}{dt}\text{sgn}(t) = 2\delta(t)$. When $f \neq 0$, we have,

$$(j2\pi f)\mathcal{F}\{\text{sgn}(t)\}(f) = \mathcal{F}\left\{\frac{d}{dt}\text{sgn}(t)\right\}(f) = \mathcal{F}\{2\delta(t)\}(f) = 2.$$

Hence $\mathcal{F}\{\text{sgn}(t)\}(f) = \frac{1}{j\pi f}$.

When $f = 0$, the Cauchy principal value of $\mathcal{F}\{\text{sgn}(t)\}(0)$ is

$$\mathcal{F}\{\text{sgn}(t)\}(0) = \lim_{M\to\infty}\int_{-M}^{M}\text{sgn}(t)dt = \lim_{M\to\infty}(M - M) = 0.$$

Thus, we have $\mathcal{F}\{u(t)\}(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$. ■

Example 11. $-\frac{1}{j2\pi t} \xrightarrow{\text{FT}} \frac{1}{2}\text{sgn}(f)$.

Proof. This can be obtained by duality (or by complex integration). ■

Example 12. $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}} \xrightarrow{\text{FT}} e^{-2\sigma^2\pi^2 f^2}$, for any $\sigma \in \mathbb{R} \setminus \{0\}$.

Proof. We know $\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}dt = 1$.

By definition, we have,

$$\mathcal{F}\left\{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}\right\}(f) = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{t^2}{2\sigma^2}}e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t+j2\sigma^2\pi f)^2}{2\sigma^2}} e^{-2\sigma^2\pi^2 f^2} dt$$
$$= e^{-2\sigma^2\pi^2 f^2} \left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t+j2\sigma^2\pi f)^2}{2\sigma^2}} dt \right).$$

We will show by contour integration that the integral in the last line is equal to 1.

Define the contour $L = L_1 \cup L_2 \cup L_3 \cup L_4$ as follows.

$$\begin{aligned}
&L_1 : z(t) = t, && t \in [-N, M],\\
&L_2 : z(t) = M + jt, && t \in [0, 2\sigma^2\pi^2 f^2],\\
&L_3 : z(t) = -t, && t \in [-M, N],\\
&L_4 : z(t) = -N - jt, && t \in [-2\sigma^2\pi^2 f^2, 0].
\end{aligned}$$

Let $f(z) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}$. Note that L is a closed contour and $f(z)$ does not have any pole. By Cauchy's integral theorem, we have

$$\int_L f(z)dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{L_3} f(z)dz + \int_{L_4} f(z)dz = 0.$$

Note that

$$\int_{L_1} f(z)dz = \int_{-N}^{M} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} dt \overset{M,N\to\infty}{\longrightarrow} 1,$$
$$\int_{L_3} f(z)dz = -\int_{-N}^{M} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t+j2\sigma^2\pi^2 f^2)^2}{2\sigma^2}} dt.$$

It remains to prove that $\int_{L_2} f(z)dz, \int_{L_4} f(z)dz \to 0$ as $M, N \to \infty$. We have,

$$\left|\int_{L_2} f(z)dz\right| = \left|\int_0^{2\sigma^2\pi^2 f^2} e^{-\frac{(M+jt)^2}{2\sigma^2}} dt\right|$$
$$\leq \int_0^{2\sigma^2\pi^2 f^2} \left|e^{-\frac{(M+jt)^2}{2\sigma^2}}\right| dt$$
$$= \int_0^{2\sigma^2\pi^2 f^2} \left|e^{-\frac{M^2+j2Mt-t^2}{2\sigma^2}}\right| dt$$
$$= \int_0^{2\sigma^2\pi^2 f^2} e^{-\frac{M^2}{2\sigma^2}} e^{\frac{t^2}{2\sigma^2}} dt$$
$$= e^{-\frac{M^2}{2\sigma^2}} \int_0^{2\sigma^2\pi^2 f^2} e^{\frac{t^2}{2\sigma^2}} dt$$
$$\leq e^{-\frac{M^2}{2\sigma^2}} \left(2\sigma^2\pi^2 f^2 e^{2\sigma^2\pi^4 f^4}\right) \to 0.$$

Thus, we have $\int_{L_2} f(z)dz \to 0$. Following a similar argument, we have $\int_{L_4} f(z)dz \to 0$. ■

Exercise 12. Compute the Fourier transform of the signal $x(t)$ defined by

$$x(t) = \begin{cases} t+2, & -2 \leq t \leq -1, \\ 1, & -1 \leq t \leq 1, \\ -t+2, & 1 \leq t \leq 2. \end{cases}$$

○

Solution. Observe that $x(t) = \mathrm{tri}(t+1) + \mathrm{tri}(t) + \mathrm{tri}(t-1)$. Then

$$\begin{aligned}\hat{X}(f) &= \mathrm{sinc}^2(f)(e^{j2\pi f} + 1 + e^{-j2\pi f}) \\ &= \mathrm{sinc}^2(f)(1 + 2\cos 2\pi f).\end{aligned}$$

Alternatively, consider $x(t) = 2\mathrm{tri}(\frac{t}{2}) - \mathrm{tri}(t)$.

Alternatively, consider $x'(t) = \mathrm{rect}(t + \frac{3}{2}) - \mathrm{rect}(t - \frac{3}{2})$. ♣

Exercise 13. Compute the Fourier transform of the signal $x(t) = |\cos \pi t|$. ○

Solution. Observe that

$$x''(t) = -\pi^2 x(t) + 2\pi \sum_{k=-\infty}^{+\infty} \delta\left(t - k - \frac{1}{2}\right).$$

Taking Fourier transform gives

$$-4\pi^2 f^2 \hat{X}(f) = -\pi^2 \hat{X}(f) + 2\pi \sum_{k=-\infty}^{+\infty} e^{j2\pi f \cdot \frac{1}{2}} \delta(f - k).$$

Therefore,

$$\begin{aligned}\hat{X}(f) &= \sum_{k=-\infty}^{+\infty} \frac{2\pi e^{j\pi f}}{\pi^2(1 - 4f^2)} \delta(f - k) \\ &= \sum_{k=-\infty}^{+\infty} \frac{2(-1)^k}{\pi(1 - 4k^2)} \delta(f - k).\end{aligned}$$

♣

4.4 Relationship between Fourier series and Fourier transform

Suppose $x(t)$ is a periodic signal with period T. Since $x(t)$ is periodic, it admits a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{+\infty} \hat{X}[k] e^{j2\pi \frac{kt}{T}},$$

where $\hat{X}[k]$ are the Fourier series coefficients. Note that the Fourier transform of $e^{j2\pi \frac{kt}{T}}$ is $\delta\left(f - \frac{k}{T}\right)$. Taking Fourier transform on both sides, we have

$$\hat{X}(f) = \sum_{k=-\infty}^{+\infty} \hat{X}[k] \delta\left(f - \frac{k}{T}\right).$$

This gives the Fourier transform of any periodic signal in terms of its Fourier series.

In particular, the *shah function* $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$ is periodic with period T. Its Fourier series coefficients are given by

$$\hat{X}[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi \frac{kt}{T}} \, dt$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-j2\pi\frac{kt}{T}}\,dt$$
$$= \frac{1}{T}.$$

Hence its Fourier transform is

$$\hat{X}(f) = \sum_{k=-\infty}^{+\infty} \frac{1}{T}\delta\left(f - \frac{k}{T}\right).$$

4.5 Sampling theorem

Definition 4.3. A signal $x(t)$ is called *band-limited* with *bandwidth* $B > 0$ if its Fourier transform $\hat{X}(f) = 0$ for all $|f| > B$. ○

Theorem 4.1. (*Nyquist—Shannon sampling theorem*) Let $s[k]$ be a discrete-time signal, and let $T_s > 0$. Then there is a continuous-time signal $x(t)$, band-limited with bandwidth $\frac{1}{2T_s}$, such that $x(kT_s) = s[k]$ for all $k \in \mathbb{Z}$. ○

Remark 4.7. This theorem means that one needs a bandwidth of $\frac{1}{2T_s}$ to reconstruct a signal sampled with sampling period T_s. This theorem is understood in this way: Any band-limited signal $x(t)$ with bandwidth B can be reconstructed from its samples $\{x(kT_s)\}_{k\in\mathbb{Z}}$ with sampling period T_s as long as $T_s < \frac{1}{2B}$. ○

Proof. Consider the sampled signal $x_s(t)$ defined by

$$x_s(t) := \sum_{k=-\infty}^{+\infty} s[k]\delta(t - kT_s)$$

and its Fourier transform

$$\hat{X}_s(f) = \mathcal{F}\left\{\sum_{k=-\infty}^{+\infty} s[k]\delta(t - kT_s)\right\}(f)$$
$$= \sum_{k=-\infty}^{+\infty} s[k]\mathcal{F}\{\delta(t - kT_s)\}(f)$$
$$= \sum_{k=-\infty}^{+\infty} s[k]e^{-j2\pi f kT_s}.$$

Note that $\hat{X}_s(f)$ is periodic with period $\frac{1}{T_s}$. Hence $\hat{X}_s(f)$ is the periodic extension (with period T_s) of itself truncated to $\left[-\frac{1}{2T_s}, \frac{1}{2T_s}\right]$, i.e.

$$\hat{X}_s(f) = \hat{X}_s(f)\mathrm{rect}(fT_s) * \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_s}\right)$$
$$= \left(\hat{X}_s(f) \cdot T_s\mathrm{rect}(fT_s)\right) * \left(\frac{1}{T_s}\sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_s}\right)\right).$$

Now we let $x(t)$ be the inverse Fourier transform of $\hat{X}(f) := \hat{X}_s(f) \cdot T_s \text{rect}(fT_s)$. We will soon see that $x(t)$ is the reconstructed band-limited signal we want. Indeed $x(t)$ is band-limited with bandwidth $\frac{1}{2T_s}$ since $\hat{X}(f) = 0$ for all $|f| > \frac{1}{2T_s}$. Also, we have

$$\hat{X}_s(f) = \hat{X}(f) * \left(\frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_s}\right) \right).$$

Taking inverse Fourier transform gives

$$\begin{aligned} x_s(t) &= x(t) \sum_{k=-\infty}^{+\infty} \delta(t - kT_s) \\ &= \sum_{k=-\infty}^{+\infty} x(kT_s)\delta(t - kT_s). \end{aligned}$$

This is equivalent to that $s[k] = x(kT_s)$ for all $k \in \mathbb{Z}$, which can be seen by, for example, integrating over a small interval around kT_s. ■

Remark 4.8. We have seen in the proof that the band-limited signal is reconstructed by applying an ideal low-pass filter to the sampled signal. In reality, we cannot create an ideal low-pass filter, that is, we cannot create an LTI system whose frequency response is $\text{rect}\left(\frac{f}{2B}\right)$. Real low-pass filters have a finite roll-off, i.e. a finite slope in the frequency response after the cut-off frequency. You can see `https://www.ti.com/lit/an/sloa049b/sloa049b.pdf` for real filter design ideas. ○

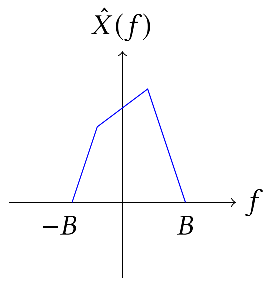


Figure 4.2: The spectrum $\hat{X}(f)$ of a band-limited signal $x(t)$

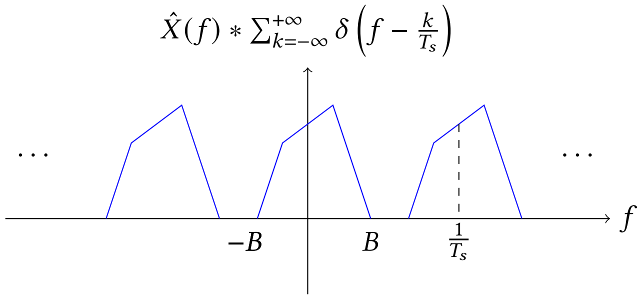


Figure 4.3: Convolution of $\hat{X}(f)$ with the shah function

Exercises

1.

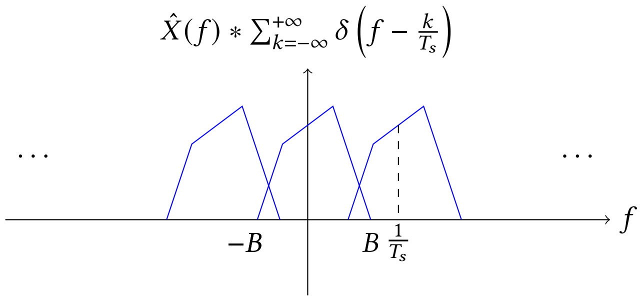


Figure 4.4: Under-sampling when $T_s \geq \frac{1}{2B}$

Chapter 5

Discrete-time Fourier transform

A discrete-time system is called *shift-invariant* if, whenever an input signal $x[n]$ produces an output signal $y[n]$, the shifted input $x[n-n_0]$ produces the shift output $y[n-n_0]$, for any $n_0 \in \mathbb{Z}$. *Linear shift-invariant* (LSI) systems are the discrete-time analog of LTI systems in the continuous-time setting. Similarly to LTI systems, an LSI system can be characterized by its impulse response, which is the output produced by the input $\delta[n]$.

Consider an LSI system with impulse response $h[n]$. The output produced by the input $x[n] = e^{j2\pi fn}$ is given by

$$\begin{aligned} x[n] * h[n] &= \sum_{k=-\infty}^{+\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{j2\pi f(n-k)}h[k] \\ &= e^{j2\pi fn}\left(\sum_{k=-\infty}^{+\infty} h[k]e^{-j2\pi fk}\right) \\ &=: \hat{H}(f)x[n], \end{aligned}$$

where we have put $\hat{H}(f) := \sum_{k=-\infty}^{+\infty} h[k]e^{-j2\pi fk}$. Thus, $e^{j2\pi fn}$ is an eigenfunction of all LSI systems with eigenvalue $\hat{H}(f)$. This gives the definition of the *discrete-time Fourier transform*, which is the analog of the Fourier transform for discrete-time signals.

Definition 5.1. The *discrete-time Fourier transform* (DTFT) of a discrete-time signal $x[n]$ is defined as

$$\hat{X}(f) := \sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi fn}.$$

○

Recall that $\{f \mapsto e^{j2\pi fn}\}_{n\in\mathbb{Z}}$ forms an orthonormal set (under the inner product $\langle x(f), y(f)\rangle := \int_{-\frac{1}{2}}^{\frac{1}{2}} x(f)\overline{y(f)}df$), we have

$$\begin{aligned} \langle \hat{X}(f), e^{-j2\pi fn}\rangle &= \left\langle \sum_{k=-\infty}^{+\infty} x[k]e^{-j2\pi fk}, e^{-j2\pi fn}\right\rangle \\ &= \sum_{k=-\infty}^{+\infty} x[k]\langle e^{-j2\pi fk}, e^{-j2\pi fn}\rangle \end{aligned}$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$
$$= x[n].$$

Thus we have the inverse of DTFT as follows.

Definition 5.2. The *inverse discrete-time Fourier transform* (IDTFT) of a signal $\hat{X}(f)$ is defined as

$$x[n] := \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)e^{j2\pi fn}df.$$

○

5.1 Properties of DTFT

5.1.1 Linearity

Proposition 48. Let the DTFT of $x[n], y[n]$ be $\hat{X}(f), \hat{Y}(f)$ respectively. Then, for any $a, b \in \mathbb{C}$, the DTFT of $ax[n] + by[n]$ is $a\hat{X}(f) + b\hat{Y}(f)$. ○

Proof. The DTFT of $ax[n] + by[n]$ is

$$\sum_{n=-\infty}^{+\infty} (ax[n] + by[n])e^{-j2\pi fn} = a\sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi fn} + b\sum_{n=-\infty}^{+\infty} y[n]e^{-j2\pi fn}$$
$$= a\hat{X}(f) + b\hat{Y}(f).$$

■

5.1.2 Time/frequency shifting

Proposition 49. Let the DTFT of $x[n]$ be $\hat{X}(f)$. Then the DTFT of $x[n-n_0]$ is $e^{-j2\pi fn_0}\hat{X}(f)$ and the DTFT of $e^{j2\pi f_0 n}x[n]$ is $\hat{X}(f-f_0)$. ○

Proof. Consider

$$x[n-n_0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)e^{j2\pi f(n-n_0)}df$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} (e^{-j2\pi fn_0}\hat{X}(f))e^{j2\pi fn}df.$$

Hence the DTFT of $x[n-n_0]$ is $e^{-j2\pi fn_0}\hat{X}(f)$. Also consider

$$\hat{X}(f-f_0) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi(f-f_0)n}$$
$$= \sum_{n=-\infty}^{+\infty} (e^{j2\pi f_0 n}x[n])e^{-j2\pi fn}.$$

Hence the DTFT of $e^{j2\pi f_0 n}x[n]$ is $\hat{X}(f-f_0)$. ■

5.1.3 Zero-padding

Proposition 50. Let the DTFT of $x[n]$ be $\hat{X}(f)$. Let k be a positive integer. Then the DTFT of the zero-padded signal

$$n \mapsto \begin{cases} x[m], & n = km \text{ for some } m \in \mathbb{Z},, \\ 0, & \text{otherwise,} \end{cases}$$

is $\hat{X}(kf)$. ○

Proof. The DTFT of the zero-padded signal is

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} x[m]e^{-j2\pi f(km)} &= \sum_{m=-\infty}^{+\infty} x[m]e^{-j2\pi(kf)m} \\ &= \hat{X}(kf). \end{aligned}$$

■

5.1.4 Compression

Proposition 51. Let the DTFT of $x[n]$ be $\hat{X}(f)$. Let k be a positive integer. Then the DTFT of $x[kn]$ is $\frac{1}{k}\sum_{\ell=0}^{k-1}\hat{X}\left(\frac{f-\ell}{k}\right)$. ○

Proof. Consider

$$\begin{aligned} \frac{1}{k}\sum_{\ell=0}^{k-1}\hat{X}\left(\frac{f-\ell}{k}\right) &= \frac{1}{k}\sum_{\ell=0}^{k-1}\sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi\frac{f-\ell}{k}n} \\ &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi f\frac{n}{k}}\left(\frac{1}{k}\sum_{\ell=0}^{k-1} e^{j2\pi\frac{n}{k}\ell}\right) \\ &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi f\frac{n}{k}}\left(\begin{cases} 1, & n \equiv 0 \pmod k, \\ 0, & \text{otherwise} \end{cases}\right) \\ &= \sum_{m=-\infty}^{+\infty} x[km]e^{-j2\pi fm} && \text{(with } m := \frac{n}{k}\text{),} \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \sum_{\ell=0}^{k-1} e^{j2\pi\frac{n}{k}\ell} &= \begin{cases} \sum_{\ell=0}^{k-1} 1, & n \equiv 0 \pmod k, \\ \frac{e^{j2\pi\frac{n}{k}k}-1}{e^{j2\pi\frac{n}{k}}-1}, & \text{otherwise} \end{cases} \\ &= \begin{cases} k, & n \equiv 0 \pmod k, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the DTFT of $x[kn]$ is $\frac{1}{k}\sum_{\ell=0}^{k-1}\hat{X}\left(\frac{f-\ell}{k}\right)$. ■

5.1.5 Complex conjugation

Proposition 52. Let the DTFT of $x[n]$ be $\hat{X}(f)$. Then the DTFT of $\overline{x[n]}$ is $\overline{\hat{X}(-f)}$. ○

Proof. The DTFT of $\overline{x[n]}$ is

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \overline{x[n]} e^{-j2\pi f n} &= \overline{\sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi(-f)n}} \\ &= \overline{\hat{X}(-f)}.\end{aligned}$$

■

5.1.6 Convolution

Proposition 53. Let the DTFT of $x[n], y[n]$ be $\hat{X}(f), \hat{Y}(f)$ respectively. Then the DTFT of $x[n] * y[n]$ is $\hat{X}(f)\hat{Y}(f)$ and the DTFT of $x[n]y[n]$ is $\hat{X}(f) \circledast \hat{Y}(f)$, where $\circledast$ denotes the *periodic convolution* with period 1:

$$\hat{X}(f) \circledast \hat{Y}(f) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(v)\hat{Y}(f-v)dv.$$

○

Proof. Consider

$$\begin{aligned}\hat{X}(f)\hat{Y}(f) &= \left(\sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi f n}\right)\left(\sum_{m=-\infty}^{+\infty} y[m]e^{-j2\pi f m}\right) \\ &= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} x[n]y[m]e^{-j2\pi f(n+m)} \\ &= \sum_{k=-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty} x[n]y[k-n]\right)e^{-j2\pi f k} && \text{(with } k := n+m) \\ &= \sum_{k=-\infty}^{+\infty} (x[k] * y[k])e^{-j2\pi f k}.\end{aligned}$$

Hence the DTFT of $x[n] * y[n]$ is $\hat{X}(f)\hat{Y}(f)$. Also consider

$$\begin{aligned}x[n]y[n] &= \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(\mu)e^{j2\pi\mu n}d\mu\right)\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{Y}(v)e^{j2\pi v n}dv\right) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(\mu)\hat{Y}(v)e^{j2\pi(\mu+v)n}dvd\mu \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-\frac{1}{2}-\mu}^{\frac{1}{2}-\mu} \hat{X}(\mu)\hat{Y}(v)e^{j2\pi(\mu+v)n}dv\right)d\mu && (v \mapsto \hat{X}(\mu)\hat{Y}(v)e^{j2\pi(\mu+v)n} \text{ has period 1}) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(\mu)\hat{Y}(f-\mu)e^{j2\pi f n}df\right)d\mu && \text{(with } f := \mu+v) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(\mu)\hat{Y}(f-\mu)e^{j2\pi f n}d\mu\right)df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (\hat{X}(f) \circledast \hat{Y}(f))e^{j2\pi f n}df.\end{aligned}$$

Hence the DTFT of $x[n]y[n]$ is $\hat{X}(f) \circledast \hat{Y}(f)$. ■

Remark 5.1. Alternatively, this can be understood using the eigenvalue interpretation of the DTFT. As for the Fourier transform, one can get the convolution property by cascading two LTI systems, one can get the convolution property of the DTFT by cascading two LSI systems instead. ○

5.1.7 Differentiation in frequency

Proposition 54. Let the DTFT of $x[n]$ be $\hat{X}(f)$. Then the DTFT of $-j2\pi nx[n]$ is $\frac{d}{df}\hat{X}(f)$. ○

Proof. Consider

$$\begin{aligned}\frac{d}{df}\hat{X}(f) &= \frac{d}{df}\sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi fn} \\ &= \sum_{n=-\infty}^{+\infty} \frac{d}{df}\left(x[n]e^{-j2\pi fn}\right) \\ &= \sum_{n=-\infty}^{+\infty} (-j2\pi nx[n])e^{-j2\pi fn}.\end{aligned}$$

Hence the DTFT of $-j2\pi nx[n]$ is $\frac{d}{df}\hat{X}(f)$. ■

5.1.8 Parseval's theorem

Proposition 55. Let the DTFT of $x[n], y[n]$ be $\hat{X}(f), \hat{Y}(f)$ respectively. Then

$$\sum_{n=-\infty}^{+\infty} x[n]\overline{y[n]} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)\overline{\hat{Y}(f)}df.$$

In particular,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{X}(f)|^2 df.$$

○

Proof.

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} x[n]\overline{y[n]} &= \sum_{n=-\infty}^{+\infty}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)e^{j2\pi fn}df\right)\overline{y[n]} \\ &= \sum_{n=-\infty}^{+\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)\overline{y[n]}e^{j2\pi fn}df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}}\sum_{n=-\infty}^{+\infty} \hat{X}(f)\overline{y[n]}e^{j2\pi fn}df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)\overline{\left(\sum_{n=-\infty}^{+\infty} y[n]e^{-j2\pi fn}\right)}df \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)\overline{\hat{Y}(f)}df.\end{aligned}$$

In particular, putting $y[n] = x[n]$ gives

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{X}(f)|^2 df.$$

■

5.1.9 Sampling of continuous-time signal

Proposition 56. Let $x(t)$ be a continuous-time signal with Fourier transform $\hat{X}(f)$. Then the DTFT of the discrete-time signal $x[n] := x(n)$ is $\sum_{k=-\infty}^{+\infty} \hat{X}(f-k)$. ○

Proof. Consider

$$\begin{aligned} x(t) \sum_{n=-\infty}^{+\infty} \delta(t-n) &= \sum_{n=-\infty}^{+\infty} x(t)\delta(t-n) \\ &= \sum_{n=-\infty}^{+\infty} x(n)\delta(t-n) \\ &= \sum_{n=-\infty}^{+\infty} x[n]\delta(t-n). \end{aligned}$$

Taking Fourier transform gives

$$\hat{X}(f) * \sum_{n=-\infty}^{+\infty} \delta(f-n) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi fn},$$

i.e. the DTFT of $x[n]$ is

$$\begin{aligned} \hat{X}(f) * \sum_{k=-\infty}^{+\infty} \delta(f-k) &= \sum_{k=-\infty}^{+\infty} \hat{X}(f) * \delta(f-k) \\ &= \sum_{k=-\infty}^{+\infty} \hat{X}(f-k). \end{aligned}$$

■

Remark 5.2. Evaluating the DTFT of $x[n]$ at $f = 0$ gives the Poisson summation formula:

$$\sum_{k=-\infty}^{+\infty} x(k) = \sum_{k=-\infty}^{+\infty} \hat{X}(k).$$

○

5.2 Examples of DTFT

Example 1. $\delta[n] \xrightarrow{\text{DTFT}} 1$ and $1 \xrightarrow{\text{DTFT}} \sum_{k=-\infty}^{+\infty} \delta(f-k)$.

Proof.

$$\delta[n] \xrightarrow{\text{DTFT}} \sum_{n=-\infty}^{+\infty} \delta[n]e^{-j2\pi fn}$$

$$
\begin{aligned}
&= 1, \\
\sum_{k=-\infty}^{+\infty} \delta(f-k) \overset{\text{IDTFT}}{\longrightarrow} &\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k=-\infty}^{+\infty} \delta(f-k) \right) e^{j2\pi f n} df \\
&= \sum_{k=-\infty}^{+\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(f-k) e^{j2\pi f n} df \\
&= \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(f-k) e^{j2\pi f n} \text{rect}(f) df \\
&= \sum_{k=-\infty}^{+\infty} e^{j2\pi k n} \text{rect}(k) \\
&= \sum_{k=-\infty}^{+\infty} \text{rect}(k) \\
&= 1.
\end{aligned}
$$

■

Example 2. $\delta[n-n_0] \overset{\text{DTFT}}{\longrightarrow} e^{-j2\pi f n_0}$ and $e^{j2\pi f_0 n} \overset{\text{DTFT}}{\longrightarrow} \sum_{k=-\infty}^{+\infty} \delta(f-f_0-k)$.

Proof. We already know this when $n_0 = 0$ and $f_0 = 0$. Apply the time/frequency shifting property of the DTFT. ■

Example 3. $a^n u[n] \overset{\text{DTFT}}{\longrightarrow} \frac{1}{1-ae^{-j2\pi f}}$ and $a^{-n} u[-n-1] \overset{\text{DTFT}}{\longrightarrow} -\frac{1}{1-a^{-1}e^{-j2\pi f}}$, for any $0 < |a| < 1$.

Proof.

$$
\begin{aligned}
a^n u[n] \overset{\text{DTFT}}{\longrightarrow} &\sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j2\pi f n} \\
&= \sum_{n=0}^{+\infty} a^n e^{-j2\pi f n} \\
&= \sum_{n=0}^{+\infty} (ae^{-j2\pi f})^n \\
&= \frac{1}{1-ae^{-j2\pi f}}, \\
a^{-n} u[-n-1] \overset{\text{DTFT}}{\longrightarrow} &\sum_{n=-\infty}^{+\infty} a^{-n} u[-n-1] e^{-j2\pi f n} \\
&= \sum_{n=-\infty}^{+\infty} a^n u[n-1] e^{j2\pi f n} \\
&= \sum_{n=1}^{+\infty} a^n e^{j2\pi f n} \\
&= \sum_{n=1}^{+\infty} (ae^{j2\pi f})^n \\
&= \frac{ae^{j2\pi f}}{1-ae^{j2\pi f}}
\end{aligned}
$$

$$= -\frac{1}{1 - a^{-1}e^{-j2\pi f}}.$$

■

Example 4. $u[n] \overset{\text{DTFT}}{\longrightarrow} \frac{1}{1-e^{-j2\pi f}} + \frac{1}{2}\sum_{k=-\infty}^{+\infty} \delta(f-k)$.

Proof. Observe that for all $0 < |a| < 1$ we have

$$\begin{aligned} a^{|n|}(2u[n]-1) &= a^n u[n] - a^{-n}u[-n-1] \\ &\overset{\text{DTFT}}{\longrightarrow} \frac{1}{1-ae^{-j2\pi f}} + \frac{1}{1-a^{-1}e^{-j2\pi f}}. \end{aligned}$$

Taking the limit $a \to 1^-$ gives

$$2u[n] - 1 \overset{\text{DTFT}}{\longrightarrow} \frac{2}{1-e^{-j2\pi f}}.$$

Recall that $1 \overset{\text{DTFT}}{\longrightarrow} \sum_{k=-\infty}^{+\infty} \delta(f-k)$. We can thus conclude with

$$u[n] \overset{\text{DTFT}}{\longrightarrow} \frac{1}{1-e^{-j2\pi f}} + \frac{1}{2}\sum_{k=-\infty}^{+\infty} \delta(f-k).$$

■

Exercise 14. Compute the DTFT of the signal $x[n] = \frac{1}{1+n^2}$. ○

Solution. Consider the continuous-time signal $x(t) := \frac{1}{1+t^2}$. We already know that the Fourier transform of $x(t)$ is given by $\hat{X}(f) = \pi e^{-2\pi|f|}$. Then we can evaluate the DTFT of the discrete-time signal $x[n] = x(n)$ as follows.

$$\begin{aligned} x[n] &\overset{\text{DTFT}}{\longrightarrow} \sum_{k=-\infty}^{+\infty} \hat{X}(f-k) \\ &= \sum_{k=-\infty}^{+\infty} \pi e^{-2\pi|f-k|} \\ &= \sum_{k=-\infty}^{\lfloor f\rfloor} \pi e^{-2\pi|f-k|} + \sum_{k=\lfloor f\rfloor+1}^{+\infty} \pi e^{-2\pi|f-k|} \\ &= \sum_{k=-\infty}^{\lfloor f\rfloor} \pi e^{-2\pi(f-k)} + \sum_{k=\lfloor f\rfloor+1}^{+\infty} \pi e^{-2\pi(k-f)} \\ &= \sum_{k=0}^{+\infty} \pi e^{-2\pi(f-(\lfloor f\rfloor-k))} + \sum_{k=0}^{+\infty} \pi e^{-2\pi((k+\lfloor f\rfloor+1)-f)} \\ &= \pi e^{-2\pi(f-\lfloor f\rfloor)} \sum_{k=0}^{+\infty} e^{-2\pi k} + \pi e^{-2\pi(\lfloor f\rfloor+1-f)} \sum_{k=0}^{+\infty} e^{-2\pi k} \\ &= \frac{\pi e^{-2\pi(f-\lfloor f\rfloor)}}{1-e^{-2\pi}} + \frac{\pi e^{-2\pi(\lfloor f\rfloor+1-f)}}{1-e^{-2\pi}} \\ &= \frac{\pi\left(e^{-2\pi(f-\lfloor f\rfloor)} + e^{2\pi(f-\lfloor f\rfloor-1)}\right)}{1-e^{-2\pi}} \\ &= \frac{\pi\cosh\left(2\pi(f-\lfloor f\rfloor)-\pi\right)}{\sinh\pi}. \end{aligned}$$

♣

Exercises

1.

Chapter 6

Discrete Fourier transform

A system with finite discrete signals (in $\mathbb{C}^N$) as input and output is called *cyclic-shift-invariant* if, whenever an input signal $x[n]$ produces an output signal $y[n]$, the cyclically shifted input $x[(n-n_0)_N]$ produces the cyclically shifted output $y[(n-n_0)_N]$, for any $n_0 \in \mathbb{Z}$. Here, $(\cdot)_N$ denotes the modulo operation defined as the only integer $0 \leq n < N$ such that $(n)_N \equiv n \pmod{N}$.

Recall that an LTI system for continuous-time signals can be characterized by its impulse response, which is the output produced by the input $\delta(t)$, and that an LSI system for aperiodic discrete-time signals can be characterized by its impulse response, which is the output produced by the input $\delta[n]$. Similarly, for finite discrete signals, a *linear cyclic-shift-invariant* system can be characterized by the output produced by the input $(1, 0, \dots, 0) \in \mathbb{C}^N$.

Consider a linear cyclic-shift-invariant system that maps $(1, 0, \dots, 0) \in \mathbb{C}^N$ to $h[n] \in \mathbb{C}^N$. The output produced by the input $x[n] = e^{j2\pi\frac{kn}{N}}$ is given by

$$\begin{aligned}
\sum_{m=0}^{N-1} x[m]h[(n-m)_N] &= \sum_{m=0}^{N-1} x[(n-m)_N]h[m] \\
&= \sum_{m=0}^{N-1} e^{j2\pi\frac{k(n-m)}{N}} h[m] \\
&= e^{j2\pi\frac{kn}{N}} \left(\sum_{m=0}^{N-1} h[m] e^{-j2\pi\frac{km}{N}} \right) \\
&=: \hat{H}[k]x[n],
\end{aligned}$$

where we have put $\hat{H}[k] := \sum_{m=0}^{N-1} h[m]e^{-j2\pi\frac{km}{N}}$. Thus, $x[n] = e^{j2\pi\frac{kn}{N}}$ is an eigenfunction of all linear cyclic-shift-invariant systems with eigenvalue $\hat{H}[k]$. This gives the definition of the *discrete Fourier transform* for finite discrete signals.

Definition 6.1. The *discrete Fourier transform* (DFT) of a finite discrete signal $x[n] \in \mathbb{C}^N$ is defined as

$$\hat{X}[k] := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi\frac{kn}{N}}.$$

○

Remark 6.1. The normalization factor $\frac{1}{\sqrt{N}}$ is introduced so that the DFT becomes a unitary operator, i.e., the DFT preserves the inner product: $\langle x[n], y[n] \rangle = \langle \hat{X}[k], \hat{Y}[k] \rangle$, or in other words, we have Parseval's theorem for the DFT without an extra factor of $\frac{1}{N}$. Some authors may define the DFT and the inverse DFT with normalization factors 1 and $\frac{1}{\sqrt{N}}$, respectively. ○

Note that $\left\{\left(\frac{1}{\sqrt{N}}e^{j2\pi\frac{kn}{N}}\right)_{n=0}^{N-1}\right\}_{k=0}^{N-1}$ forms an orthonormal set under the standard inner product on $\mathbb{C}^N$, $\langle x[n], y[n]\rangle := \sum_{n=0}^{N-1} x[n]\overline{y[n]}$. This can be verified as follows.

$$\begin{aligned}
\sum_{n=0}^{N-1}\left(\frac{1}{\sqrt{N}}e^{j2\pi\frac{kn}{N}}\right)\overline{\left(\frac{1}{\sqrt{N}}e^{j2\pi\frac{\ell n}{N}}\right)} &= \frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi\frac{kn}{N}}e^{-j2\pi\frac{\ell n}{N}} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi\frac{k-\ell}{N}n} \\
&= \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1} 1, & k-\ell \equiv 0 \pmod N, \\ \frac{1}{N}\frac{e^{j2\pi\frac{k-\ell}{N}N}-1}{e^{j2\pi\frac{k-\ell}{N}}-1}, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & k-\ell \equiv 0 \pmod N, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & k=\ell, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Then

$$\begin{aligned}
\left\langle \hat{X}[k], \frac{1}{\sqrt{N}}e^{-j2\pi\frac{kn}{N}}\right\rangle &= \left\langle \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} x[m]e^{-j2\pi\frac{km}{N}}, \frac{1}{\sqrt{N}}e^{-j2\pi\frac{kn}{N}}\right\rangle \\
&= \sum_{m=0}^{N-1} x[m]\left\langle \frac{1}{\sqrt{N}}e^{-j2\pi\frac{km}{N}}, \frac{1}{\sqrt{N}}e^{-j2\pi\frac{kn}{N}}\right\rangle \\
&= \sum_{m=0}^{N-1} x[m]\left(\begin{cases} 1, & m=n, \\ 0, & \text{otherwise} \end{cases}\right) \\
&= x[n].
\end{aligned}$$

Thus we have the inverse of DFT as follows.

Definition 6.2. The *inverse discrete Fourier transform* (IDFT) of a finite discrete signal $\hat{X}[k] \in \mathbb{C}^N$ is defined as

$$x[n] := \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \hat{X}[k]e^{j2\pi\frac{kn}{N}}.$$

○

6.1 Properties of DFT

6.1.1 Linearity

Proposition 57. Let the DFT of $x[n], y[n] \in \mathbb{C}^N$ be $\hat{X}[k], \hat{Y}[k]$ respectively. Then, for any $a, b \in \mathbb{C}$, the DFT of $ax[n] + by[n]$ is $a\hat{X}[k] + b\hat{Y}[k]$. ○

Proof. The DFT of $ax[n] + by[n]$ is

$$\begin{aligned}
\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}(ax[n]+by[n])e^{-j2\pi\frac{kn}{N}} &= a\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}\right) + b\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} y[n]e^{-j2\pi\frac{kn}{N}}\right) \\
&= a\hat{X}[k] + b\hat{Y}[k].
\end{aligned}$$

■

6.1.2 Time/frequency shifting

Proposition 58. Let the DFT of $x[n] \in \mathbb{C}^N$ be $\hat{X}[k]$. Then the DFT of $x[(n-n_0)_N]$ is $e^{-j2\pi\frac{kn_0}{N}}\hat{X}[k]$ and the DFT of $e^{j2\pi\frac{k_0 n}{N}}x[n]$ is $\hat{X}[(k-k_0)_N]$. ○

Proof. The DFT of $x[(n-n_0)_N]$ is

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[(n-n_0)_N]e^{-j2\pi\frac{kn}{N}} &= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} x[m]e^{-j2\pi\frac{k(m+n_0)}{N}} \qquad \text{(with } m := (n-n_0)_N\text{)}\\ &= e^{-j2\pi\frac{kn_0}{N}}\left(\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} x[m]e^{-j2\pi\frac{km}{N}}\right)\\ &= e^{-j2\pi\frac{kn_0}{N}}\hat{X}[k].\end{aligned}$$

The DFT of $e^{j2\pi\frac{k_0 n}{N}}x[n]$ is

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\left(e^{j2\pi\frac{k_0 n}{N}}x[n]\right)e^{-j2\pi\frac{kn}{N}} &= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{(k-k_0)n}{N}}\\ &= \hat{X}[(k-k_0)_N].\end{aligned}$$

■

6.1.3 Time reversal

Proposition 59. Let the DFT of $x[n] \in \mathbb{C}^N$ be $\hat{X}[k]$. Then the DFT of $x[(-n)_N]$ is $\hat{X}[(-k)_N]$. ○

Proof. The DFT of $x[(-n)_N]$ is

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[(-n)_N]e^{-j2\pi\frac{kn}{N}} &= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} x[m]e^{-j2\pi\frac{k(-m)}{N}} \qquad \text{(with } m := (-n)_N\text{)}\\ &= \frac{1}{\sqrt{N}}\sum_{m=0}^{N-1} x[m]e^{-j2\pi\frac{(-k)m}{N}}\\ &= \hat{X}[(-k)_N].\end{aligned}$$

■

6.1.4 Complex conjugation

Proposition 60. Let the DFT of $x[n] \in \mathbb{C}^N$ be $\hat{X}[k]$. Then the DFT of $\overline{x[n]}$ is $\overline{\hat{X}[(-k)_N]}$. ○

Proof. The DFT of $\overline{x[n]}$ is

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} \overline{x[n]}e^{-j2\pi\frac{kn}{N}} &= \overline{\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]e^{j2\pi\frac{kn}{N}}}\\ &= \overline{\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{(-k)n}{N}}}\\ &= \overline{\hat{X}[(-k)_N]}.\end{aligned}$$

■

6.1.5 Duality

Proposition 61. Let the DFT of $x[n] \in \mathbb{C}^N$ be $\hat{X}[k]$. Then the DFT of $\hat{X}[k]$ is

$$x[(-n)_N] = (x[0], x[N-1], x[N-2], \dots, x[1]).$$

○

Proof. The DFT of $\hat{X}[k]$ is

$$\begin{aligned}
\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] e^{-j2\pi \frac{kn}{N}} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} x[m] e^{-j2\pi \frac{km}{N}} \right) e^{-j2\pi \frac{kn}{N}} \\
&= \sum_{m=0}^{N-1} x[m] \left(\frac{1}{N} \sum_{k=0}^{N-1} e^{-j2\pi \frac{k(m+n)}{N}} \right) \\
&= \sum_{m=0}^{N-1} x[m] \left(\begin{cases} 1, & m+n \equiv 0 \pmod{N}, \\ 0, & \text{otherwise} \end{cases} \right) \\
&= x[(-n)_N].
\end{aligned}$$

■

6.1.6 Periodic convolution

Proposition 62. Let the DFT of $x[n], y[n] \in \mathbb{C}^N$ be $\hat{X}[k], \hat{Y}[k]$ respectively. Then the DFT of $\frac{1}{\sqrt{N}} (x[n] \circledast_N y[n])$ is $\hat{X}[k]\hat{Y}[k]$ and the DFT of $x[n]y[n]$ is $\frac{1}{\sqrt{N}} \left(\hat{X}[k] \circledast_N \hat{Y}[k]\right)$, where $\circledast_N$ denotes the *periodic convolution* on $\mathbb{C}^N$:

$$x[n] \circledast_N y[n] = \sum_{m=0}^{N-1} x[m] y[(n-m)_N].$$

○

Proof. Consider

$$\begin{aligned}
\hat{X}[k]\hat{Y}[k] &= \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}} \right) \left(\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} y[m] e^{-j2\pi \frac{km}{N}} \right) \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[n] y[m] e^{-j2\pi \frac{k(n+m)}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} x[n] y[(p-n)_N] e^{-j2\pi \frac{kp}{N}} \qquad \text{(with } p := (n+m)_N\text{)} \\
&= \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] y[(p-n)_N] \right) e^{-j2\pi \frac{kp}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \left(\frac{1}{\sqrt{N}} (x[p] \circledast_N y[p]) \right) e^{-j2\pi \frac{kp}{N}}.
\end{aligned}$$

Hence the DFT of $\frac{1}{\sqrt{N}} (x[n] \circledast_N y[n])$ is $\hat{X}[k]\hat{Y}[k]$.

Now, by duality of the DFT, $x[n], y[n]$ are the Fourier transforms of $\hat{X}[(-k)_N], \hat{Y}[(-k)_N]$ respectively. Then the DFT of $\frac{1}{\sqrt{N}} \left(\hat{X}[(-k)_N] \circledast_N \hat{Y}[(-k)_N]\right)$ is $x[n]y[n]$. Again by duality of the DFT, we have that the DFT of $x[n]y[n]$ is $\frac{1}{\sqrt{N}} \left(\hat{X}[k] \circledast_N \hat{Y}[k]\right)$. ■

6.1.7 Parseval's theorem

Proposition 63. Let the DFT of $x[n], y[n] \in \mathbb{C}^N$ be $\hat{X}[k], \hat{Y}[k]$ respectively. Then

$$\sum_{n=0}^{N-1} x[n]\overline{y[n]} = \sum_{k=0}^{N-1} \hat{X}[k]\overline{\hat{Y}[k]}.$$

In particular,

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |\hat{X}[k]|^2.$$

○

Proof.

$$\begin{aligned}
\sum_{n=0}^{N-1} x[n]\overline{y[n]} &= \sum_{n=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] e^{j2\pi \frac{kn}{N}} \right) \overline{y[n]} \\
&= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} \hat{X}[k] \overline{y[n]} e^{j2\pi \frac{kn}{N}} \\
&= \sum_{k=0}^{N-1} \hat{X}[k] \overline{\left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y[n] e^{-j2\pi \frac{kn}{N}} \right)} \\
&= \sum_{k=0}^{N-1} \hat{X}[k]\overline{\hat{Y}[k]}.
\end{aligned}$$

In particular, putting $y[n] = x[n]$ gives

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |\hat{X}[k]|^2.$$

■

6.1.8 Uncertainty principle

Proposition 64. Let the DFT of $x[n] \in \mathbb{C}^N$ be $\hat{X}[k]$. Then

$$|\text{supp}\,(x[n])| \cdot \left|\text{supp}\left(\hat{X}[k]\right)\right| \geq N,$$

where $|\text{supp}\,(x[n])|$ is the size of the *support* of $x[n]$, i.e. the number of non-zero entries in $x[n]$ for $0 \leq n \leq N-1$. ○

Remark 6.2. The inequality is tight when one observes the following example.

Suppose that $T > 0$ divides N. Let

$$x[n] := \begin{cases} 1, & n \equiv 0 \pmod{T}, \\ 0, & \text{otherwise} \end{cases}$$

for $0 \leq n \leq N-1$. The DFT of $x[n]$ is

$$\begin{aligned}
\hat{X}[k] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{\substack{n=0 \\ n\equiv 0 \pmod T}}^{N-1} e^{-j2\pi \frac{kn}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{\frac{N}{T}-1} e^{-j2\pi \frac{k(mT)}{N}} && (\text{with } m := \frac{n}{T}) \\
&= \begin{cases} \frac{1}{\sqrt{N}} \frac{N}{T}, & e^{-j2\pi \frac{kT}{N}} = 1, \\ \frac{1}{\sqrt{N}} \frac{1-\left(e^{-j2\pi \frac{kT}{N}}\right)^{\frac{N}{T}}}{1-e^{-j2\pi \frac{kT}{N}}}, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{\sqrt{N}}{T}, & k \equiv 0 \pmod{\frac{N}{T}}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, $|\operatorname{supp}(x[n])| = \frac{N}{T}$ and $\left|\operatorname{supp}\left(\hat{X}[k]\right)\right| = T$. Then $|\operatorname{supp}(x[n])| \cdot \left|\operatorname{supp}\left(\hat{X}[k]\right)\right| = N$. ○

Remark 6.3. A stronger form of the uncertainty principle is the following. If $x[n]$ is normalized such that $\sum_{n=0}^{N-1} |x[n]|^2 = 1$ (hence also $\sum_{k=0}^{N-1} \left|\hat{X}[k]\right|^2 = 1$), then

$$\log|\operatorname{supp}(x[n])| + \log\left|\operatorname{supp}\left(\hat{X}[k]\right)\right| \geq H\left(|x[n]|^2\right) + H\left(\left|\hat{X}[k]\right|^2\right) \geq \log N,$$

where

$$\begin{aligned}
H\left(|x[n]|^2\right) &= -\sum_{n=0}^{N-1} |x[n]|^2 \log|x[n]|^2, \\
H\left(\left|\hat{X}[k]\right|^2\right) &= -\sum_{k=0}^{N-1} \left|\hat{X}[k]\right|^2 \log\left|\hat{X}[k]\right|^2
\end{aligned}$$

are the *entropy* of $x[n], \hat{X}[k]$ respectively.

When $x[n]$ has length $N = 2$, consider an inequality of Pinsker's,

$$2\sqrt{p(1-p)} \geq H_2(p) \geq 4p(1-p),$$

where $H_2(p) := -p \log_2 p - (1-p)\log_2(1-p)$, $0 \leq p \leq 1$, is the *binary entropy.* ○

6.2 Examples of DFT

Example 1. Let $a \in \mathbb{C}$. Suppose that $x[n] = a^n$ for $0 \leq n \leq N-1$. Then its DFT is

$$\hat{X}[k] = \begin{cases} \sqrt{N}, & a = e^{j2\pi \frac{k}{N}}, \\ \frac{1}{\sqrt{N}} \frac{1-a^N}{1-ae^{-j2\pi \frac{k}{N}}}, & \text{otherwise.} \end{cases}$$

Proof.

$$\hat{X}[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} a^n e^{-j2\pi \frac{kn}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(ae^{-j2\pi \frac{k}{N}} \right)^n \\
&= \begin{cases} \frac{1}{\sqrt{N}} N, & a = e^{j2\pi \frac{k}{N}}, \\ \frac{1}{\sqrt{N}} \frac{1-\left(ae^{-j2\pi \frac{k}{N}}\right)^N}{1-ae^{-j2\pi \frac{k}{N}}}, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sqrt{N}, & a = e^{j2\pi \frac{k}{N}}, \\ \frac{1}{\sqrt{N}} \frac{1-a^N}{1-ae^{-j2\pi \frac{k}{N}}}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

■

Example 2. Let $a \in \mathbb{C}$. Suppose that $x[n] = \binom{N-1}{n} a^n$ for $0 \le n \le N-1$. Then its DFT is

$$
\hat{X}[k] = \frac{1}{\sqrt{N}} \left(1 + ae^{-j2\pi \frac{k}{N}} \right)^{N-1}.
$$

Proof.

$$
\begin{aligned}
\hat{X}[k] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \binom{N-1}{n} a^n e^{-j2\pi \frac{kn}{N}} \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \binom{N-1}{n} \left(ae^{-j2\pi \frac{k}{N}} \right)^n \\
&= \frac{1}{\sqrt{N}} \left(1 + ae^{-j2\pi \frac{k}{N}} \right)^{N-1}.
\end{aligned}
$$

■

6.3 Matrix representation of the DFT

We adapt the convention that any finite discrete signal $x[n] \in \mathbb{C}^N$ is regarded as a column vector $\mathbf{x} := \begin{pmatrix} x[0] & x[1] & x[2] & \cdots & x[N-1] \end{pmatrix}^T$. Now, the DFT is a linear operator in $\mathbb{C}^N$ and hence can be represented by an $N \times N$ matrix F_N. Then any $\mathbf{x} \in \mathbb{C}^N$ and its DFT $\hat{\mathbf{X}} \in \mathbb{C}^N$ is related by

$$
\hat{\mathbf{X}} = F_N \mathbf{x}.
$$

The DFT matrix F_N is given by the following definition.

Definition 6.3. Let N be a positive integer. The *N-point DFT matrix* $F_N \in \mathbb{C}^{N \times N}$ is defined as

$$
(F_N)_{k,n} := \omega_N^{kn}, \qquad k, n \in \{0, 1, 2, \ldots, N-1\},
$$

where $\omega_N := e^{-j2\pi\frac{1}{N}}$. That is,

$$F_N := \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \omega_N^3 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \omega_N^6 & \cdots & \omega_N^{2(N-1)} \\ 1 & \omega_N^3 & \omega_N^6 & \omega_N^9 & \cdots & \omega_N^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \omega_N^{3(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{pmatrix}.$$

○

Proposition 65. F_N is a unitary matrix, i.e. $F_N^{-1} = F_N^\dagger$. ○

Proof. Consider

$$\begin{aligned}
(F_N^\dagger F_N)_{k,n} &= \sum_{\ell=0}^{N-1} (F_N^\dagger)_{k,\ell}(F_N)_{\ell,n} \\
&= \sum_{\ell=0}^{N-1} \overline{(F_N)_{\ell,k}}(F_N)_{\ell,n} \\
&= \sum_{\ell=0}^{N-1} \overline{\frac{1}{\sqrt{N}}\omega_N^{\ell k}}\frac{1}{\sqrt{N}}\omega_N^{\ell n} \\
&= \frac{1}{N}\sum_{\ell=0}^{N-1} \omega_N^{\ell(n-k)} \\
&= \begin{cases} 1, & n-k \equiv 0 \pmod N, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & k = n, \\ 0, & \text{otherwise,} \end{cases} \\
(F_N F_N^\dagger)_{k,n} &= \sum_{m=0}^{N-1} (F_N)_{k,m}(F_N^\dagger)_{m,n} \\
&= \sum_{m=0}^{N-1} (F_N)_{k,m}\overline{(F_N)_{n,m}} \\
&= \sum_{m=0}^{N-1} \frac{1}{\sqrt{N}}\omega_N^{km}\overline{\frac{1}{\sqrt{N}}\omega_N^{nm}} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} \omega_N^{(k-n)m} \\
&= \begin{cases} 1, & k-n \equiv 0 \pmod N, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & k = n, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, $F_N^\dagger F_N = F_N F_N^\dagger = I$, i.e. $F_N^{-1} = F_N^\dagger$. ■

Remark 6.4. This gives the matrix representation of the inverse DFT, as well as Parseval's theorem which says that $\mathbf{y}^\dagger\mathbf{x} = (F_N\mathbf{y})^\dagger(F_N\mathbf{x})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$. ○

Proposition 66.

$$(F_N^2)_{k,n} = \begin{cases} 1, & n = k = 0 \text{ or } n + k = N, \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$F_N^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

○

Proof.

$$\begin{aligned} (F_N^2)_{k,n} &= \sum_{\ell=0}^{N-1} (F_N)_{k,\ell}(F_N)_{\ell,n} \\ &= \sum_{\ell=0}^{N-1} \frac{1}{\sqrt{N}}\omega_N^{k\ell}\frac{1}{\sqrt{N}}\omega_N^{\ell n} \\ &= \frac{1}{N}\sum_{\ell=0}^{N-1}\omega_N^{(n+k)\ell} \\ &= \begin{cases} 1, & n + k \equiv 0 \pmod N, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & n = k = 0 \text{ or } n + k = N, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

■

Remark 6.5. This is the duality of the DFT. ○

Proposition 67. $F_N^4 = I$. ○

Proof.

$$\begin{aligned} (F_N^4)_{k,n} &= \sum_{\ell=0}^{N-1} (F_N^2)_{k,\ell}(F_N^2)_{\ell,n} \\ &= \sum_{\ell=0}^{N-1} \left(\begin{cases} 1, & k + \ell \equiv 0 \pmod N, \\ 0, & \text{otherwise} \end{cases}\right)\left(\begin{cases} 1, & \ell + n \equiv 0 \pmod N, \\ 0, & \text{otherwise} \end{cases}\right) \\ &= \sum_{\ell=0}^{N-1} \left(\begin{cases} 1, & \ell \equiv -k \equiv -n \pmod N, \\ 0, & \text{otherwise} \end{cases}\right) \\ &= \begin{cases} 1, & k = n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $F_N^4 = I$. ■

Proposition 68. The only possible eigenvalues of F_N are $1, -1, j, -j$. ○

Proof. Suppose $\mathbf{x}$ is an eigenvector of F_N with eigenvalue λ. Since $F_N^4 = I$ we have $\lambda^4\mathbf{x} = F_N^4\mathbf{x} = \mathbf{x}$. Hence $\lambda^4 = 1$. Hence $\lambda \in \{1, -1, j, -j\}$. ■

6.4 Relationship between DTFT and DFT

Consider a finite discrete signal $x[n] \in \mathbb{C}^N$ and its DFT $\hat{X}[k] \in \mathbb{C}^N$. We can naturally extend $x[n]$ to a periodic discrete-time signal by taking $x[n] := x[(n)_N]$ for $n \in \mathbb{Z}$. Then for any $n \in \mathbb{Z}$ we have

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] e^{j2\pi \frac{kn}{N}}.$$

We can write this as

$$\begin{aligned}
x[n] &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] e^{j2\pi \frac{kn}{N}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{\ell=-\infty}^{+\infty} \delta\left(f - \frac{k}{N} - \ell\right) df \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] \sum_{\ell=-\infty}^{+\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta\left(f - \frac{k}{N} - \ell\right) e^{j2\pi \frac{kn}{N}} df \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] \sum_{\ell=-\infty}^{+\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta\left(f - \frac{k}{N} - \ell\right) e^{j2\pi (\frac{k}{N}+\ell) n} df \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] \sum_{\ell=-\infty}^{+\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta\left(f - \frac{k}{N} - \ell\right) e^{j2\pi f n} df \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] \sum_{\ell=-\infty}^{+\infty} \delta\left(f - \frac{k}{N} - \ell\right)\right) e^{j2\pi f n} df.
\end{aligned}$$

Thus, the DTFT $\hat{X}(f)$ of $x[n]$ is given by

$$\begin{aligned}
\hat{X}(f) &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{X}[k] \sum_{\ell=-\infty}^{+\infty} \delta\left(f - \frac{k}{N} - \ell\right) \\
&= \sum_{k=0}^{N-1} \sum_{\ell=-\infty}^{+\infty} \frac{1}{\sqrt{N}} \hat{X}[(k + N\ell)_N] \delta\left(f - \frac{k + N\ell}{N}\right) \\
&= \sum_{k=-\infty}^{+\infty} \frac{1}{\sqrt{N}} \hat{X}[(k)_N] \delta\left(f - \frac{k}{N}\right).
\end{aligned}$$

This gives the DTFT in terms of the DFT. One can further expend this as

$$\hat{X}(f) = \sum_{k=-\infty}^{+\infty} \left(\sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}} \right) \frac{1}{N} \delta\left(f - \frac{k}{N}\right).$$

This gives the DTFT of any periodic discrete-time signal in general.

6.5 Fast Fourier transform

In general, computing $\mathbf{y} = A\mathbf{x}$ where A is an $N \times N$ matrix and $\mathbf{x}$ is an N-dimensional column vector requires $O(N^2)$ scalar multiplications. This could have been the case for computing the N-point DFT $\hat{\mathbf{X}} = F_N\mathbf{x}$. However, by exploiting the properties of DFT, we can do it with $O(N \log N)$ scalar multiplications. The algorithm that makes this speedup possible is the *fast Fourier transform* (FFT).

Let M, N be positive integers. We can compute the DFT $\hat{X}[q]$ of any finite discrete signal $x[p]$ of length MN as follows. For any non-negative integers $k < N$ and $\ell < M$,

$$\begin{aligned}
\hat{X}[k + N\ell] &= \frac{1}{\sqrt{MN}} \sum_{p=0}^{MN-1} x[p] e^{-j2\pi \frac{(k+N\ell)p}{MN}} \\
&= \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x[m + Mn] e^{-j2\pi \frac{(k+N\ell)(m+Mn)}{MN}} \\
&= \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x[m + Mn] e^{-j2\pi \frac{kn}{N}} e^{-j2\pi \frac{\ell m}{M}} e^{-j2\pi \frac{km}{MN}} \\
&= \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \left(e^{-j2\pi \frac{km}{MN}} \left(\frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[m + Mn] e^{-j2\pi \frac{kn}{N}} \right) \right) e^{-j2\pi \frac{\ell m}{M}}.
\end{aligned}$$

This suggests that a DFT of length MN can be done with the following steps:

1. Compute M DFTs of length N: For $m = 0, 1, \ldots, M - 1$, compute

   $$(y_m[0], y_m[1], \ldots, y_m[N-1]) \overset{N\text{-point DFT}}{\longleftarrow} (x[m], x[m+M], \ldots, x[m+(N-1)M]).$$

   Hence the inner bracket is $y_m[k]$.

2. Multiply MN scalar factors: For $m = 0, 1, \ldots, M - 1$ and $k = 0, 1, \ldots, N - 1$, compute

   $$z_k[m] = e^{-j2\pi \frac{km}{MN}} y_m[k].$$

   Hence the outer bracket is $z_k[m]$.

3. Compute N DFTs of length M: For $k = 0, 1, \ldots, N - 1$, compute

   $$(w_k[0], w_k[1], \ldots, w_k[M-1]) \overset{M\text{-point DFT}}{\longleftarrow} (z_k[0], z_k[1], \ldots, z_k[M-1]).$$

   Hence $\hat{X}[k + N\ell] = w_k[\ell]$.

The DFTs of length N and M can be further broken down into DFTs of shorter lengths recursively until the lengths become prime numbers. This recursive algorithm for computing the DFT is called the *fast Fourier transform* (FFT).

6.5.1 Complexity analysis

Let $T(N)$ be the number of scalar multiplications necessary for the N-point FFT. If we break the N-point FFT according to the factorization $N = N_1N_2$, then the overall computation will require $N_2T(N_1) + N_1T(N_2) + N_1N_2$ scalar multiplications. Note that there are multiple ways to break down the FFT. Hence we have

$$T(N) = \min_{\substack{N_1, N_2 \\ N = N_1N_2}} \left(N_2T(N_1) + N_1T(N_2) + N_1N_2 \right).$$

Dividing both sides by N gives

$$\frac{T(N)}{N} = \min_{\substack{N_1,N_2\\ N=N_1N_2}} \left(\frac{T(N_1)}{N_1} + \frac{T(N_2)}{N_2} + 1 \right).$$

From this one can show (for example, by induction) that, if the prime factorization of N is given by $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1, p_2, \dots, p_k$ are distinct primes, then

$$\frac{T(N)}{N} = \sum_{\ell=1}^{k} \alpha_\ell \frac{T(p_\ell)}{p_\ell} + \sum_{\ell=1}^{k} \alpha_\ell - 1.$$

Note that this also implies that the overall complexity does not depend on in what order the FFT is broken down.

In particular, the most common case of computing the DFT is when N is a power of 2. In such case we have

$$\begin{aligned} T(N) &= N\left(\frac{T(2)}{2} \log_2 N + \log_2 N - 1 \right) \\ &= O(N \log N). \end{aligned}$$

6.6 Summary of Fourier transform

Operation	Input	Output
FS	continuous-time T-periodic signal $x(t)$	discrete-time signal $\hat{X}[k]$
FT	continuous-time signal $x(t)$	continuous-time signal $\hat{X}(f)$
DTFT	discrete-time signal $x[n]$	continuous-time 1-periodic signal $\hat{X}(f)$
DFT	finite discrete signal $x[n] \in \mathbb{C}^N$	finite discrete signal $\hat{X}[k] \in \mathbb{C}^N$

Exercises

1.

Chapter 7

Laplace transform

Recall that $t \mapsto e^{j2\pi ft}$, where $f \in \mathbb{R}$, is an eigenfunction of any LTI system. However, these eigenfunctions do not span all signals. For instance, the Fourier transform of $t \mapsto e^t u(t)$ cannot be defined, since

$$\begin{aligned}\lim_{T\to+\infty}\int_{-T}^{T} e^t u(t)e^{-j2\pi ft}dt &= \lim_{T\to+\infty}\int_0^T e^t e^{-j2\pi ft}dt\\ &= \lim_{T\to+\infty}\left.\frac{e^{(1-j2\pi f)t}}{1-j2\pi f}\right|_0^T\\ &= \lim_{T\to+\infty}\frac{e^{(1-j2\pi f)T}-1}{1-j2\pi f}\end{aligned}$$

is not well-defined for $f \neq 0$. Therefore, to deal with signals such as $e^t u(t)$, we must consider a natural generalization of the basis $\{t \mapsto e^{j2\pi ft}\}_{f\in\mathbb{R}}$.

Consider the signal $t \mapsto e^{st}$ where $s \in \mathbb{C}$. With this being the input, the output of the LTI system with impulse response $h(t)$ is

$$\begin{aligned}e^{st} * h(t) &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau\\ &= e^{st}\left(\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right)\\ &=: \hat{H}(s)e^{st},\end{aligned}$$

where we have put $\hat{H}(s) := \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$, assuming that this integral is well-defined. Thus, as long as e^{st} is a valid input to an LTI system, it is also an eigenfunction with eigenvalue $\hat{H}(s)$. This gives the definition of the *Laplace transform*, which generalizes the Fourier transform.

Definition 7.1. The *Laplace transform* of a signal $x(t)$ is defined as

$$\hat{X}(s) = \mathcal{L}\{x(t)\}(s) := \int_{-\infty}^{+\infty} x(t)e^{-st}dt.$$

○

7.1 Region of convergence

In general, the Laplace transform

$$\mathcal{L}\{x(t)\}(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

may not converge for some $s \in \mathbb{C}$. We want to know its *region of convergence*, i.e. the region of $s \in \mathbb{C}$ in which the Laplace transform converges absolutely.

Definition 7.2. The *region of convergence* (ROC) of the Laplace transform of a signal $x(t)$ is the set of all $s \in \mathbb{C}$ such that

$$\int_{-\infty}^{+\infty} |x(t)e^{-st}|dt < \infty.$$

○

Remark 7.1. There exist two different signals with the same Laplace transform. It turns out that they can always be distinguished by the respective ROC of their Laplace transforms.

Consider the two signals $x_1(t) = e^{-at}u(t)$ and $x_2(t) = -e^{-at}u(-t)$, where $a \in \mathbb{R}$. The Laplace transform of $x_1(t)$ is

$$\begin{aligned} \mathcal{L}\{x_1(t)\}(s) &= \int_{-\infty}^{+\infty} x_1(t)e^{-st}dt \\ &= \int_{0}^{+\infty} e^{-at}e^{-st}dt \\ &= \int_{0}^{+\infty} e^{-(s+a)t}dt \\ &= \frac{1}{s+a}, \end{aligned}$$

with ROC $\{s : \mathrm{Re}(s) > -a\}$, since

$$\begin{aligned} \int_{-\infty}^{+\infty} |x_1(t)e^{-st}|dt &= \int_{0}^{+\infty} |e^{-at}e^{-st}|dt \\ &= \int_{0}^{+\infty} e^{-(\mathrm{Re}(s)+a)t}dt \\ &= \begin{cases} \frac{1}{\mathrm{Re}(s)+a}, & \mathrm{Re}(s) > -a, \\ \infty, & \text{otherwise} \end{cases} \end{aligned}$$

is finite if and only if $\mathrm{Re}(s) > -a$. The Laplace transform of $x_2(t)$ is

$$\begin{aligned} \mathcal{L}\{x_2(t)\}(s) &= \int_{-\infty}^{+\infty} x_2(t)e^{-st}dt \\ &= \int_{-\infty}^{0} -e^{-at}e^{-st}dt \\ &= \int_{-\infty}^{0} -e^{-(s+a)t}dt \\ &= \frac{1}{s+a}, \end{aligned}$$

with ROC $\{s : \mathrm{Re}(s) < -a\}$, since

$$\begin{aligned} \int_{-\infty}^{+\infty} |x_2(t)e^{-st}|dt &= \int_{-\infty}^{0} |-e^{-at}e^{-st}|dt \\ &= \int_{-\infty}^{0} e^{-(\mathrm{Re}(s)+a)t}dt \\ &= \begin{cases} -\frac{1}{\mathrm{Re}(s)+a}, & \mathrm{Re}(s) < -a, \\ \infty, & \text{otherwise} \end{cases} \end{aligned}$$

is finite if and only if $\mathrm{Re}(s) < -a$. These two signals have the same Laplace transform but with different ROC. This means that to reconstruct a signal from its Laplace transform, we also need to know the ROC. ○

Proposition 69. The ROC of any Laplace transform is of the form $\{\sigma + j\omega : \sigma \in I, \omega \in \mathbb{R}\}$ where I is some interval in $\mathbb{R}$. In other words, the ROC is always a strip parallel to the imaginary axis. ○

Proof. Suppose that the Laplace transform of $x(t)$ has ROC $\mathcal{R}$. It is enough to show the following two, where the first claim means that $s \in \mathcal{R}$ if and only if the real part of s is in $\mathcal{R}$, and the second claim means that $\mathcal{R} \cap \mathbb{R}$ is an interval.

1. If $s \in \mathcal{R}$, then $s + j\omega \in \mathcal{R}$ for any real number ω.
2. If s_1, s_2 are two real numbers in $\mathcal{R}$, then any $s \in (s_1, s_2)$ is also in $\mathcal{R}$.

To show the first claim, if $s \in \mathcal{R}$ then

$$\begin{aligned}\int_{-\infty}^{+\infty} |x(t)e^{-(s+j\omega)t}|dt &= \int_{-\infty}^{+\infty} |x(t)e^{-st}e^{-j\omega t}|dt \\ &= \int_{-\infty}^{+\infty} |x(t)e^{-st}| \cdot |e^{-j\omega t}|dt \\ &= \int_{-\infty}^{+\infty} |x(t)e^{-st}|dt \\ &< \infty.\end{aligned}$$

Therefore $s + j\omega \in \mathcal{R}$.

To show the second claim, if $s_1, s_2 \in \mathcal{R}$ are real and $s \in (s_1, s_2)$ then

$$\begin{aligned}e^{-st} &\le \max\left\{e^{-s_1 t}, e^{-s_2 t}\right\} \\ &\le e^{-s_1 t} + e^{-s_2 t}\end{aligned}$$

for any $t \in \mathbb{R}$. Hence,

$$\begin{aligned}\int_{-\infty}^{+\infty} |x(t)e^{-st}|dt &= \int_{-\infty}^{+\infty} |x(t)|e^{-st}dt \\ &\le \int_{-\infty}^{+\infty} |x(t)|(e^{-s_1 t} + e^{-s_2 t})dt \\ &= \int_{-\infty}^{+\infty} |x(t)e^{-s_1 t}|dt + \int_{-\infty}^{+\infty} |x(t)e^{-s_2 t}|dt \\ &< \infty.\end{aligned}$$

Therefore $s \in \mathcal{R}$. ■

Proposition 70. Let $\mathcal{R}$ be the ROC of the Laplace transform of a signal $x(t)$.

1. Suppose that $x(t)$ is *right-sided*, i.e. there exists $t_0 \in \mathbb{R}$ such that $x(t) = 0$ for any $t \le t_0$. Then $\mathcal{R}$ is a right half-plane, i.e. if $s_0 \in \mathcal{R}$ then $\{s : \mathrm{Re}(s) > \mathrm{Re}(s_0)\} \subseteq \mathcal{R}$.
2. Suppose that $x(t)$ is *left-sided*, i.e. there exists $t_0 \in \mathbb{R}$ such that $x(t) = 0$ for any $t \ge t_0$. Then $\mathcal{R}$ is a left half-plane, i.e. if $s_0 \in \mathcal{R}$ then $\{s : \mathrm{Re}(s) < \mathrm{Re}(s_0)\} \subseteq \mathcal{R}$.

○

Proof. We only show the result for right-sided signals as the result for left-sided signals follows a similar proof.

If s_0 is in the ROC of the Laplace transform of a right-sided signal $x(t)$ (with $x(t) = 0$ for any $t \leq t_0$), then for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \mathrm{Re}(s_0)$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} |x(t)e^{-st}|dt &= \int_{t_0}^{+\infty} |x(t)e^{-st}|dt \\
&= \int_{t_0}^{+\infty} |x(t)e^{-s_0 t}| \cdot |e^{-(s-s_0)t}|dt \\
&= \int_{t_0}^{+\infty} |x(t)e^{-s_0 t}| \cdot e^{-(\mathrm{Re}(s)-\mathrm{Re}(s_0))t}dt \\
&\leq \int_{t_0}^{+\infty} |x(t)e^{-s_0 t}| \cdot e^{-(\mathrm{Re}(s)-\mathrm{Re}(s_0))t_0}dt \\
&= e^{-(\mathrm{Re}(s)-\mathrm{Re}(s_0))t_0} \int_{-\infty}^{+\infty} |x(t)e^{-s_0 t}|dt \\
&< \infty.
\end{aligned}$$

Hence s is in the ROC. ■

Proposition 71. An LTI system with impulse response $h(t)$ is stable if and only if 0 belongs to the ROC of the Laplace transform of $h(t)$. ○

Proof. Recall that an LTI system is stable if and only if its impulse response $h(t)$ is absolutely integrable, i.e. $\int_{-\infty}^{+\infty} |h(t)|dt < \infty$, which is equivalent to that 0 belongs to the ROC of the Laplace transform of $h(t)$. ■

7.2 Inverse Laplace transform

The Laplace transform $\hat{X}(s)$ of a signal $x(t)$ at $s = \sigma + j2\pi f$ ($\sigma, f \in \mathbb{R}$), where σ is in the ROC, is evaluted as

$$\begin{aligned}
\hat{X}(\sigma + j2\pi f) &= \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j2\pi f t}dt \\
&= \mathcal{F}\{x(t)e^{-\sigma t}\}(f).
\end{aligned}$$

That is, $f \mapsto \hat{X}(\sigma + j2\pi f)$ is the Fourier transform of $t \mapsto x(t)e^{-\sigma t}$. Inverting the Fourier transform gives

$$\begin{aligned}
x(t)e^{-\sigma t} &= \mathcal{F}^{-1}\{\hat{X}(\sigma + j2\pi f)\}(t) \\
&= \int_{-\infty}^{+\infty} \hat{X}(\sigma + j2\pi f)e^{j2\pi f t}df.
\end{aligned}$$

Hence

$$\begin{aligned}
x(t) &= e^{\sigma t} \int_{-\infty}^{+\infty} \hat{X}(\sigma + j2\pi f)e^{j2\pi f t}df \\
&= \int_{-\infty}^{+\infty} \hat{X}(\sigma + j2\pi f)e^{(\sigma + j2\pi f)t}df \\
&= \frac{1}{j2\pi} \int_{\Gamma} \hat{X}(s)e^{st}ds \qquad \text{(with } s := \sigma + j2\pi f\text{)},
\end{aligned}$$

where Γ is the line $\{s : \mathrm{Re}(s) = \sigma\}$ parallel to the imaginary axis, directed upwards. This yields the formula for the inverse Laplace transform.

Definition 7.3. The *inverse Laplace transform* of a function $\hat{X}(s)$ on some subset of $\mathbb{C}$ is defined as the path integral

$$x(t) = \mathcal{L}^{-1}\{\hat{X}(s)\}(t) := \frac{1}{j2\pi}\int_{\Gamma} \hat{X}(s)e^{st}ds,$$

where Γ is any line of the form $\{s : \mathrm{Re}(s) = \sigma\}$, directed upwards, in the domain of $\hat{X}(s)$. ○

Remark 7.2. In general, the value of the path integral does not depend on the choice of the line Γ, and hence the inverse Laplace transfrom is well-defined. This is based on the following observations.

1. The ROC of any Laplace transform is always a strip parallel to the imaginary axis.
2. In general, the Laplace transform is holomorphic in the ROC.
3. The integral along a closed contour of a function that is holomorphic in the enclosed area is zero. This is also known as Cauchy's integral theorem.

○

7.3 Properties of Laplace transform

7.3.1 Linearity

Proposition 72. Let the Laplace transform of $x(t), y(t)$ be $\hat{X}(s), \hat{Y}(s)$ with ROC $\mathcal{R}_1, \mathcal{R}_2$, respectively. Then, for any $a, b \in \mathbb{C}$, the Laplace transform of $ax(t) + by(t)$ is $a\hat{X}(s) + b\hat{Y}(s)$ with ROC $\mathcal{R} \supseteq \mathcal{R}_1 \cap \mathcal{R}_2$. ○

Proof. To show that $s \in \mathcal{R}_1 \cap \mathcal{R}_2$ belongs to the ROC of the Laplace transform of $ax(t) + by(t)$, note that

$$\begin{aligned}\int_{-\infty}^{+\infty} |(ax(t) + by(t))e^{-st}|dt &\le |a|\int_{-\infty}^{+\infty} |x(t)e^{-st}|dt + |b|\int_{-\infty}^{+\infty} |y(t)e^{-st}|dt \\ &< \infty,\end{aligned}$$

since both integrals are finite. Now,

$$\begin{aligned}\mathcal{L}\{ax(t) + by(t)\}(s) &= \int_{-\infty}^{+\infty} (ax(t) + by(t))e^{-st}dt \\ &= a\int_{-\infty}^{+\infty} x(t)e^{-st}dt + b\int_{-\infty}^{+\infty} y(t)e^{-st}dt \\ &= a\hat{X}(s) + b\hat{Y}(s).\end{aligned}$$

■

7.3.2 Time/frequency shifting

Proposition 73. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$ with ROC $\mathcal{R}$. Then the Laplace transform of $x(t - t_0)$ is $e^{-st_0}\hat{X}(s)$ with ROC $\mathcal{R}$, and the Laplace transform of $e^{s_0 t}x(t)$ is $\hat{X}(s - s_0)$ with ROC $\mathcal{R} + s_0 = \{s + s_0 : s \in \mathcal{R}\}$. ○

Proof.

$$\begin{aligned}
\mathcal{L}\{x(t-t_0)\}(s) &= \int_{-\infty}^{+\infty} x(t-t_0)e^{-st}dt \\
&= \int_{-\infty}^{+\infty} x(\tau)e^{-s(\tau+t_0)}d\tau && \text{(with } \tau := t-t_0\text{)} \\
&= e^{-st_0}\int_{-\infty}^{+\infty} x(\tau)e^{-s\tau}d\tau \\
&= e^{-st_0}\hat{X}(s),
\end{aligned}$$

with ROC

$$\begin{aligned}
\left\{s : \int_{-\infty}^{+\infty} |x(t-t_0)e^{-st}|dt < \infty\right\} &= \left\{s : \int_{-\infty}^{+\infty} |x(\tau)e^{-s(\tau+t_0)}|d\tau < \infty\right\} && \text{(with } \tau := t-t_0\text{)} \\
&= \left\{s : |e^{-st_0}|\int_{-\infty}^{+\infty} |x(\tau)e^{-s\tau}|d\tau < \infty\right\} \\
&= \left\{s : \int_{-\infty}^{+\infty} |x(\tau)e^{-s\tau}|d\tau < \infty\right\} \\
&= \mathcal{R}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathcal{L}\{e^{s_0t}x(t)\}(s) &= \int_{-\infty}^{+\infty} e^{s_0t}x(t)e^{-st}dt \\
&= \int_{-\infty}^{+\infty} x(t)e^{-(s-s_0)t}dt \\
&= \hat{X}(s-s_0),
\end{aligned}$$

with ROC

$$\begin{aligned}
\left\{s : \int_{-\infty}^{+\infty} |e^{s_0t}x(t)e^{-st}|dt < \infty\right\} &= \left\{s : \int_{-\infty}^{+\infty} |x(t)e^{-(s-s_0)t}|dt < \infty\right\} \\
&= \{s : s-s_0 \in \mathcal{R}\} \\
&= \{\tilde{s}+s_0 : \tilde{s} \in \mathcal{R}\} \\
&= \mathcal{R}+s_0.
\end{aligned}$$

■

7.3.3 Scaling

Proposition 74. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$ with ROC $\mathcal{R}$. Then, for any $a \in \mathbb{R} \setminus \{0\}$, the Laplace transform of $x(at)$ is $\frac{1}{|a|}\hat{X}\left(\frac{s}{a}\right)$ with ROC $a\mathcal{R} = \{as : s \in \mathcal{R}\}$. ○

Proof.

$$\begin{aligned}
\mathcal{L}\{x(at)\}(s) &= \int_{-\infty}^{+\infty} x(at)e^{-st}dt \\
&= \frac{1}{|a|}\int_{-\infty}^{+\infty} x(\tau)e^{-\left(\frac{s}{a}\right)\tau}d\tau && \text{(with } \tau := at\text{)} \\
&= \frac{1}{|a|}\hat{X}\left(\frac{s}{a}\right),
\end{aligned}$$

with ROC

$$\begin{aligned}\left\{s : \int_{-\infty}^{+\infty} |x(at)e^{-st}|dt < \infty\right\} &= \left\{s : \frac{1}{|a|}\int_{-\infty}^{+\infty} |x(\tau)e^{-\left(\frac{s}{a}\right)\tau}|d\tau < \infty\right\} \qquad \text{(with } \tau := at\text{)} \\ &= \left\{s : \int_{-\infty}^{+\infty} |x(\tau)e^{-\left(\frac{s}{a}\right)\tau}|d\tau < \infty\right\} \\ &= \left\{s : \frac{s}{a} \in \mathcal{R}\right\} \\ &= \{a\tilde{s} : \tilde{s} \in \mathcal{R}\} \\ &= a\mathcal{R}.\end{aligned}$$

■

7.3.4 Complex conjugation

Proposition 75. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$ with ROC $\mathcal{R}$. Then the Laplace transform of $\overline{x(t)}$ is $\overline{\hat{X}\left(\overline{s}\right)}$ with ROC $\mathcal{R}$. ○

Proof.

$$\begin{aligned}\mathcal{L}\left\{\overline{x(t)}\right\}(s) &= \int_{-\infty}^{+\infty} \overline{x(t)}e^{-st}dt \\ &= \overline{\left(\int_{-\infty}^{+\infty} x(t)e^{-\overline{s}t}dt\right)} \\ &= \overline{\hat{X}\left(\overline{s}\right)},\end{aligned}$$

with ROC

$$\begin{aligned}\left\{s : \int_{-\infty}^{+\infty} |\overline{x(t)}e^{-st}|dt < \infty\right\} &= \left\{s : \int_{-\infty}^{+\infty} |x(t)e^{-st}|dt < \infty\right\} \\ &= \mathcal{R}.\end{aligned}$$

■

7.3.5 Convolution

Proposition 76. Let the Laplace transform of $x(t), y(t)$ be $\hat{X}(s), \hat{Y}(s)$ with ROC $\mathcal{R}_1, \mathcal{R}_2$, respectively. Then the Laplace transform of $x(t) * y(t)$ is $\hat{X}(s)\hat{Y}(s)$ with ROC $\mathcal{R} \supseteq \mathcal{R}_1 \cap \mathcal{R}_2$. ○

Proof. To show that $s \in \mathcal{R}_1 \cap \mathcal{R}_2$ belongs to the ROC of the Laplace transform of $x(t) * y(t)$, note that

$$\begin{aligned}\int_{-\infty}^{+\infty} \left|(x(t) * y(t))e^{-st}\right| dt &= \int_{-\infty}^{+\infty} \left|\int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau\right| \left|e^{-st}\right| dt \\ &\le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x(\tau)|\,|y(t-\tau)|\left|e^{-st}\right| d\tau dt \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x(\tau)|\,|y(\sigma)|\left|e^{-s(\tau+\sigma)}\right| d\tau d\sigma \qquad \text{(with } \sigma := t-\tau\text{)} \\ &= \left(\int_{-\infty}^{+\infty} |x(\tau)e^{-s\tau}|\, d\tau\right)\left(\int_{-\infty}^{+\infty} |y(\sigma)e^{-s\sigma}|\, d\sigma\right)\end{aligned}$$

$$< \infty,$$

since both integrals are finite. Now,

$$\begin{aligned}
\mathcal{L}\{x(t) * y(t)\}(s) &= \int_{-\infty}^{+\infty} (x(t) * y(t)) e^{-st} dt \\
&= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(\tau) y(t-\tau) d\tau \right) e^{-st} dt \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau) y(t-\tau) e^{-s\tau} e^{-s(t-\tau)} d\tau dt \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau) y(\sigma) e^{-s\tau} e^{-s\sigma} d\tau d\sigma \qquad \text{(with } \sigma := t-\tau\text{)} \\
&= \left(\int_{-\infty}^{+\infty} x(\tau) e^{-s\tau} d\tau \right) \left(\int_{-\infty}^{+\infty} y(\sigma) e^{-s\sigma} d\sigma \right) \\
&= \hat{X}(s) \hat{Y}(s).
\end{aligned}$$

■

7.3.6 Differentiation in the time-domain

Proposition 77. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$ with ROC $\mathcal{R}$. Then the Laplace transform of $\frac{d}{dt}x(t)$ is $s\hat{X}(s)$ with ROC containing $\mathcal{R}$. ○

Proof. Let $\epsilon \neq 0$. By the time shifting property, the Laplace transform of $x(t-\epsilon)$ is $e^{-\epsilon s}\hat{X}(s)$ with ROC $\mathcal{R}$. By linearity of the Laplace transform, we have

$$\mathcal{L}\left\{ \frac{x(t) - x(t-\epsilon)}{\epsilon} \right\}(s) = \frac{1 - e^{-\epsilon s}}{\epsilon} \hat{X}(s)$$

with ROC at least as large as $\mathcal{R}$. Letting $\epsilon \longrightarrow 0$ yields the desired result. ■

7.3.7 Differentiation in the frequency-domain

Proposition 78. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$ with ROC $\mathcal{R}$. Then the Laplace transform of $-tx(t)$ is $\frac{d}{ds}\hat{X}(s)$ with ROC containing the interior of $\mathcal{R}$. ○

Proof. Let $\epsilon \neq 0$. By the frequency shifting property, the Laplace transform of $e^{\epsilon t}x(t)$ is $\hat{X}(s-\epsilon)$ with ROC $\mathcal{R} + \epsilon$. By linearity of the Laplace transform, we have

$$\mathcal{L}\left\{ \frac{1 - e^{\epsilon t}}{\epsilon} x(t) \right\}(s) = \frac{\hat{X}(s) - \hat{X}(s-\epsilon)}{\epsilon}$$

with ROC at least as large as $\mathcal{R} \cap (\mathcal{R} + \epsilon)$. Letting $\epsilon \longrightarrow 0$ yields the desired result. ■

7.3.8 Integration in the time-domain

Proposition 79. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$ with ROC $\mathcal{R}$. Then the Laplace transform of $\int_{-\infty}^{t} x(\tau)d\tau$ is $\frac{1}{s}\hat{X}(s)$ with ROC containing $\mathcal{R} \cap \{s : \mathrm{Re}(s) > 0\}$. ○

Proof. Note that $\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$ and that $\mathcal{L}\{u(t)\}(s) = \frac{1}{s}$ with ROC $\{s : \mathrm{Re}(s) > 0\}$. The result follows from the convolution property of the Laplace transform. ■

7.3.9 Initial value theorem

Proposition 80. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$. Suppose that $x(t) = 0$ for all $t < 0$. Then

$$\lim_{t\to 0^+} x(t) = \lim_{s\to +\infty} s\hat{X}(s).$$

○

Proof.

$$\begin{aligned}
\lim_{s\to +\infty} s\hat{X}(s) &= \lim_{s\to +\infty} s\int_0^{+\infty} x(t)e^{-st}dt \\
&= \lim_{s\to +\infty} \int_0^{+\infty} x\left(\frac{\tau}{s}\right) e^{-\tau}d\tau && \text{(with } \tau := st\text{)} \\
&= \int_0^{+\infty} \left(\lim_{s\to +\infty} x\left(\frac{\tau}{s}\right)\right) e^{-\tau}d\tau \\
&= \int_0^{+\infty} \left(\lim_{t\to 0^+} x(t)\right) e^{-\tau}d\tau \\
&= \left(\lim_{t\to 0^+} x(t)\right) \int_0^{+\infty} e^{-\tau}d\tau \\
&= \lim_{t\to 0^+} x(t).
\end{aligned}$$

■

7.3.10 Final value theorem

Proposition 81. Let the Laplace transform of $x(t)$ be $\hat{X}(s)$. Suppose that $x(t) = 0$ for all $t < 0$. Then

$$\lim_{t\to +\infty} x(t) = \lim_{s\to 0^+} s\hat{X}(s).$$

○

Proof.

$$\begin{aligned}
\lim_{s\to 0^+} s\hat{X}(s) &= \lim_{s\to 0^+} s\int_0^{+\infty} x(t)e^{-st}dt \\
&= \lim_{s\to 0^+} \int_0^{+\infty} x\left(\frac{\tau}{s}\right) e^{-\tau}d\tau && \text{(with } \tau := st\text{)} \\
&= \int_0^{+\infty} \left(\lim_{s\to 0^+} x\left(\frac{\tau}{s}\right)\right) e^{-\tau}d\tau \\
&= \int_0^{+\infty} \left(\lim_{t\to +\infty} x(t)\right) e^{-\tau}d\tau \\
&= \left(\lim_{t\to +\infty} x(t)\right) \int_0^{+\infty} e^{-\tau}d\tau \\
&= \lim_{t\to +\infty} x(t).
\end{aligned}$$

■

7.4 Examples of Laplace transform

Example 1. $\delta(t) \xrightarrow{\text{LT}} 1$ with ROC $\mathbb{C}$.

Proof.

$$\begin{aligned}\mathcal{L}\{\delta(t)\}(s) &= \int_{-\infty}^{+\infty} \delta(t)e^{-st}dt \\ &= 1,\end{aligned}$$

with ROC

$$\begin{aligned}\left\{s : \int_{-\infty}^{+\infty} |\delta(t)e^{-st}|dt < \infty\right\} &= \{s : 1 < \infty\} \\ &= \mathbb{C}.\end{aligned}$$

■

Example 2. $\delta(t - t_0) \xrightarrow{\text{LT}} e^{-st_0}$ with ROC $\mathbb{C}$.

Proof. We already know this when $t_0 = 0$. Apply the time shifting property of the Laplace transform. ■

Example 3. $u(t) \xrightarrow{\text{LT}} \frac{1}{s}$ with ROC $\{s : \text{Re}(s) > 0\}$, and $-u(-t) \xrightarrow{\text{LT}} \frac{1}{s}$ with ROC $\{s : \text{Re}(s) < 0\}$.

Proof. The ROC of $\mathcal{L}\{u(t)\}(s)$ is

$$\begin{aligned}\left\{s : \int_{-\infty}^{+\infty} |u(t)e^{-st}|dt < \infty\right\} &= \left\{s : \int_{0}^{+\infty} e^{-\text{Re}(s)t}dt < \infty\right\} \\ &= \{s : \text{Re}(s) > 0\}.\end{aligned}$$

Hence, for s in the ROC,

$$\begin{aligned}\mathcal{L}\{u(t)\}(s) &= \int_{-\infty}^{+\infty} u(t)e^{-st}dt \\ &= \int_{0}^{+\infty} e^{-st}dt \\ &= \left.\frac{e^{-st}}{-s}\right|_{0}^{+\infty} \\ &= \frac{1}{s}.\end{aligned}$$

The ROC of $\mathcal{L}\{-u(-t)\}(s)$ is

$$\begin{aligned}\left\{s : \int_{-\infty}^{+\infty} |-u(-t)e^{-st}|dt < \infty\right\} &= \left\{s : \int_{-\infty}^{0} e^{-\text{Re}(s)t}dt < \infty\right\} \\ &= \{s : \text{Re}(s) < 0\}.\end{aligned}$$

Hence, for s in the ROC,

$$\mathcal{L}\{-u(-t)\}(s) = \int_{-\infty}^{+\infty} -u(-t)e^{-st}dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} -e^{-st} dt \\
&= \left. \frac{e^{-st}}{s} \right|_{-\infty}^{0} \\
&= \frac{1}{s}.
\end{aligned}
$$

∎

Example 4. $e^{-at}u(t) \xrightarrow{\text{LT}} \frac{1}{s+a}$ with ROC $\{s : \operatorname{Re}(s) > -\operatorname{Re}(a)\}$, and $-e^{-at}u(-t) \xrightarrow{\text{LT}} \frac{1}{s+a}$ with ROC $\{s : \operatorname{Re}(s) < -\operatorname{Re}(a)\}$, for any $a \in \mathbb{C}$.

Proof. We already know this when $a = 0$. Applying the frequency shifting property of the Laplace transform gives

$$
\begin{aligned}
\mathcal{L}\{e^{-at}u(t)\}(s) &= \mathcal{L}\{u(t)\}(s - (-a)) \\
&= \frac{1}{s+a},
\end{aligned}
$$

with ROC

$$
\begin{aligned}
\{s : \operatorname{Re}(s) > 0\} - a &= \{s - a : \operatorname{Re}(s) > 0\} \\
&= \{\tilde{s} : \operatorname{Re}(\tilde{s} + a) > 0\} \\
&= \{\tilde{s} : \operatorname{Re}(\tilde{s}) > -\operatorname{Re}(a)\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{L}\{-e^{-at}u(-t)\}(s) &= \mathcal{L}\{-u(-t)\}(s - (-a)) \\
&= \frac{1}{s+a},
\end{aligned}
$$

with ROC

$$
\begin{aligned}
\{s : \operatorname{Re}(s) < 0\} - a &= \{s - a : \operatorname{Re}(s) < 0\} \\
&= \{\tilde{s} : \operatorname{Re}(\tilde{s} + a) < 0\} \\
&= \{\tilde{s} : \operatorname{Re}(\tilde{s}) < -\operatorname{Re}(a)\}.
\end{aligned}
$$

∎

Example 5. $e^{-a|t|} \xrightarrow{\text{LT}} \frac{2a}{a^2-s^2}$ with ROC $\{s : -\operatorname{Re}(a) < \operatorname{Re}(s) < \operatorname{Re}(a)\}$, for any $a \in \mathbb{C}$.

Proof. The ROC of $\mathcal{L}\{e^{-a|t|}\}(s)$ is

$$
\begin{aligned}
\left\{s : \int_{-\infty}^{+\infty} |e^{-a|t|}e^{-st}| dt < \infty\right\} &= \left\{s : \int_{-\infty}^{+\infty} e^{-\operatorname{Re}(a)|t| - \operatorname{Re}(s)t} dt < \infty\right\} \\
&= \left\{s : \int_{-\infty}^{0} e^{\operatorname{Re}(a)t - \operatorname{Re}(s)t} dt + \int_{0}^{+\infty} e^{-\operatorname{Re}(a)t - \operatorname{Re}(s)t} dt < \infty\right\} \\
&= \left\{s : \int_{-\infty}^{0} e^{(\operatorname{Re}(a) - \operatorname{Re}(s))t} dt < \infty \text{ and } \int_{0}^{+\infty} e^{(-\operatorname{Re}(a) - \operatorname{Re}(s))t} dt < \infty\right\} \\
&= \{s : \operatorname{Re}(a) - \operatorname{Re}(s) > 0 \text{ and } -\operatorname{Re}(a) - \operatorname{Re}(s) < 0\} \\
&= \{s : -\operatorname{Re}(a) < \operatorname{Re}(s) < \operatorname{Re}(a)\}.
\end{aligned}
$$

Hence, for s in the ROC,

$$\begin{aligned}
\mathcal{L}\{e^{-a|t|}\}(s) &= \int_{-\infty}^{+\infty} e^{-a|t|}e^{-st}dt \\
&= \int_{-\infty}^{0} e^{at-st}dt + \int_{0}^{+\infty} e^{-at-st}dt \\
&= \left.\frac{e^{(a-s)t}}{a-s}\right|_{-\infty}^{0} + \left.\frac{e^{(-a-s)t}}{-a-s}\right|_{0}^{+\infty} \\
&= \frac{1}{a-s} + \frac{1}{a+s} \\
&= \frac{2a}{a^2-s^2}.
\end{aligned}$$

■

Example 6. $u(t)\sin\omega t \xrightarrow{\text{LT}} \frac{\omega}{s^2+\omega^2}$ and $u(t)\cos\omega t \xrightarrow{\text{LT}} \frac{s}{s^2+\omega^2}$, both with ROC $\{s : \operatorname{Re}(s) > 0\}$, for any $\omega \in \mathbb{R}\setminus\{0\}$.

Proof. The ROC of $\mathcal{L}\{u(t)\sin\omega t\}(s)$ is

$$\begin{aligned}
\left\{s : \int_{-\infty}^{+\infty} |(u(t)\sin\omega t)e^{-st}|dt < \infty\right\} &= \left\{s : \int_{0}^{+\infty} e^{-\operatorname{Re}(s)t}|\sin\omega t|dt < \infty\right\} \\
&= \{s : \operatorname{Re}(s) > 0\}.
\end{aligned}$$

Hence, for s in the ROC,

$$\begin{aligned}
\mathcal{L}\{u(t)\sin\omega t\}(s) &= \int_{-\infty}^{+\infty} (u(t)\sin\omega t)e^{-st}dt \\
&= \int_{0}^{+\infty} \frac{e^{j\omega t}-e^{-j\omega t}}{2j}e^{-st}dt \\
&= \frac{1}{2j}\left.\left(\frac{e^{(j\omega-s)t}}{j\omega-s} - \frac{e^{(-j\omega-s)t}}{-j\omega-s}\right)\right|_{0}^{+\infty} \\
&= -\frac{1}{2j}\left(\frac{1}{j\omega-s} - \frac{1}{-j\omega-s}\right) \\
&= \frac{\omega}{s^2+\omega^2}.
\end{aligned}$$

The ROC of $\mathcal{L}\{u(t)\cos\omega t\}(s)$ is

$$\begin{aligned}
\left\{s : \int_{-\infty}^{+\infty} |(u(t)\cos\omega t)e^{-st}|dt < \infty\right\} &= \left\{s : \int_{0}^{+\infty} e^{-\operatorname{Re}(s)t}|\cos\omega t|dt < \infty\right\} \\
&= \{s : \operatorname{Re}(s) > 0\}.
\end{aligned}$$

Hence, for s in the ROC,

$$\begin{aligned}
\mathcal{L}\{u(t)\cos\omega t\}(s) &= \int_{-\infty}^{+\infty} (u(t)\cos\omega t)e^{-st}dt \\
&= \int_{0}^{+\infty} \frac{e^{j\omega t}+e^{-j\omega t}}{2}e^{-st}dt \\
&= \frac{1}{2}\left.\left(\frac{e^{(j\omega-s)t}}{j\omega-s} + \frac{e^{(-j\omega-s)t}}{-j\omega-s}\right)\right|_{0}^{+\infty}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left(\frac{1}{j\omega - s} + \frac{1}{-j\omega - s}\right) \\
&= \frac{s}{s^2 + \omega^2}.
\end{aligned}
$$

■

Example 7. $t^n u(t) \xrightarrow{\text{LT}} \frac{n!}{s^{n+1}}$ with ROC $\{s : \text{Re}(s) > 0\}$, for any non-negative integer n.

Proof. The ROC of $\mathcal{L}\{t^n u(t)\}(s)$ is

$$
\begin{aligned}
\left\{s : \int_{-\infty}^{+\infty} |t^n u(t) e^{-st}| dt < \infty\right\} &= \left\{s : \int_{0}^{+\infty} t^n e^{-\text{Re}(s)t} dt < \infty\right\} \\
&= \{s : \text{Re}(s) < 0\}.
\end{aligned}
$$

Hence, for s in the ROC,

$$
\begin{aligned}
\mathcal{L}\{t^n u(t)\}(s) &= \int_{-\infty}^{+\infty} t^n u(t) e^{-st} dt \\
&= \int_{0}^{+\infty} t^n e^{-st} dt \\
&= -\frac{1}{s} \int_{0}^{+\infty} t^n \left(\frac{d}{dt} e^{-st}\right) dt \\
&= -\frac{1}{s} \left(t^n e^{-st}\Big|_0^{+\infty} - \int_0^{+\infty} \left(\frac{d}{dt} t^n\right) e^{-st} dt \right) \qquad \text{(integration by parts)} \\
&= \frac{n}{s} \int_0^{+\infty} t^{n-1} e^{-st} dt \\
&= \frac{n}{s} \cdot \mathcal{L}\{t^{n-1} u(t)\}(s) \\
&= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \mathcal{L}\{t^{n-2} u(t)\}(s) \\
&= \dots \\
&= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \dots \cdot \frac{1}{s} \cdot \mathcal{L}\{u(t)\}(s) \\
&= \frac{n!}{s^n} \cdot \mathcal{L}\{u(t)\}(s) \\
&= \frac{n!}{s^n} \cdot \frac{1}{s} \\
&= \frac{n!}{s^{n+1}}.
\end{aligned}
$$

■

7.5 Applications of Laplace transform

7.5.1 Linear constant-coefficient differential equations

Consider the differential equation

$$
\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = x(t).
$$

Taking Laplace transform gives

$$\sum_{n=0}^{N} a_n s^n \hat{Y}(s) = \hat{X}(s).$$

Then

$$\hat{Y}(s) = \frac{\hat{X}(s)}{\sum_{n=0}^{N} a_n s^n}.$$

7.5.2 RLC circuits

The voltage-current relationships across a resistor (with resistance R), an inductor (with inductance L), and a capacitor (with capacitance C) are respectively given by

$$\begin{aligned} V(t) &= RI(t), \\ V(t) &= L\frac{dI(t)}{dt}, \\ I(t) &= C\frac{dV(t)}{dt}, \end{aligned}$$

where $V(t), I(t)$ are the instantaneous voltage and instantaneous current at time t, respectively.

Taking Laplace transform gives the equations in frequency domain.

$$\begin{aligned} \hat{V}(s) &= R \cdot \hat{I}(s), \\ \hat{V}(s) &= sL \cdot \hat{I}(s), \\ \hat{V}(s) &= \frac{1}{sC} \cdot \hat{I}(s). \end{aligned}$$

Exercises

1. (23 M2)

   Consider a causal LTI system that satisfies the following differential equation:

$$5\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t) + 2x(t).$$

(a) Determine the Lapalace Transform, $\hat{H}(s)$, of the above system and its corresponding Region of Convergence, ROC.

(b) Invert the Laplace transform to determine the impulse response, $h(t)$.

(c) We know that for an LTI system, the input $e^{j2\pi ft}$ produces an output given by $\hat{H}(f)e^{j2\pi ft}$. Let $|\hat{H}(f)|$ be called the frequency amplification of the system at frequency f. For the system in question, determine

$$v = \max_{f\in\mathbb{R}} |\hat{H}(f)|,$$

the largest value of the frequency amplification.

(d) Compute the output of the LTI system when the input is $e^{-2t}u(t)$.

2. (22 Final) An LTI system satisfies a differential equation of the form

$$\frac{d^4y(t)}{dt^4} + 2\frac{d^3y(t)}{dt^3} - 2\frac{dy(t)}{dt} - y(t) = x(t).$$

For these parts assume that the system is a **causal** system.

(a) Compute the Laplace Transform $\hat{H}(s)$ and its ROC. (Hint: the poles are integers)

(b) Invert the Laplace transform to find $h(t)$
For these parts assume that the system is a **stable** system.

(c) Using the new ROC invert the Laplace transform to find $h(t)$

(d) Compute $\int_{\mathbb{R}} |h(t)|dt$. (Hint: there is a very quick way to do this)

3. A causal LTI system satisfies a differential equation of the form

$$\frac{d^4y(t)}{dt^4} + 324y(t) = x(t).$$

(a) Compute the Laplace Transform $\hat{H}(s)$, the locations of its poles, and its ROC. (Hint: $324 = 18^2$).

(b) Invert the Laplace transform to find $h(t)$.

(c) Is the system stable?

(d) For the same $\hat{H}(s)$, how many possible choices of ROCs exist?

Chapter 8

Z-transform

We have seen that $\{t \mapsto e^{st}\}_{s\in\mathbb{C}}$ is a set of eigenfunctions for all LTI systems. In the discrete-time setting, there is a similar set of eigenfunctions for all LSI systems, which is $\{n \mapsto z^n\}_{z\in\mathbb{C}\setminus\{0\}}$.

Consider the discrete-time signal $n \mapsto z^n$ where $z \in \mathbb{C} \setminus \{0\}$. With this being the input, the output of the LSI system with impulse response $h(t)$ is

$$\begin{aligned} z^n * h[n] &= \sum_{m=-\infty}^{+\infty} h[m]z^{n-m} \\ &= z^n \left(\sum_{m=-\infty}^{+\infty} h[m]z^{-m} \right) \\ &=: \hat{H}(z)z^n, \end{aligned}$$

where we have put $\hat{H}(z) := \sum_{m=-\infty}^{+\infty} h[m]z^{-m}$, assuming that this summation is well-defined. Thus, as long as z^n is a valid input to an LSI system, it is also an eigenfunction with eigenvalue $\hat{H}(z)$. The gives the definition of the *Z-transform*, which is the discrete-time analog of the Laplace transform.

Definition 8.1. The *Z-transform* of a discrete-time signal $x[n]$ is defined as

$$\hat{X}(z) := \sum_{n=-\infty}^{+\infty} x[n]z^{-n}.$$

○

8.1 Region of convergence

Like the Laplace transform, we also need to specify the region of convergence to fully characterize the signal. For the Z-transform, this is the region of $z \in \mathbb{C}$ in which the Z-transform is absolutely summable.

Definition 8.2. The *region of convergence* (ROC) of the Z-transform of a discrete-time signal $x[n]$ is the set of all $z \in \mathbb{C}$ such that

$$\sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| < \infty.$$

○

Remark 8.1. We adopt the convention that $0 \cdot \infty = 0$ and

$$0^n := \begin{cases} 0, & n > 0, \\ 1, & n = 0, \\ \infty, & n < 0. \end{cases}$$

Hence 0 belongs to the ROC of the Z-transform of $x[n]$ if and only if $x[n] = 0$ for all $n > 0$. ○

Recall that the ROC of any Laplace transform is a strip parallel to the imaginary axis. Similarly, the ROC of any Z-transform is an annulus, disk, or punctured disk centered at the origin.

Proposition 82. The ROC of any Z-transform is of the form $\{z \in \mathbb{C} : |z| \in I\}$ where I is some interval in $[0, +\infty)$. In other words, the ROC is always an annulus, disk, or punctured disk centered at the origin. ○

Proof. Suppose that the Z-transform of $x[n]$ has ROC $\mathcal{R}$. It is enough to show the following two, where the first claim means that $z \in \mathcal{R}$ if and only if $|z| \in \mathcal{R}$, and the second claim means that $\mathcal{R} \cap [0, +\infty)$ is an interval in $[0, +\infty)$.

1. If $z \in \mathcal{R}$, then $az \in \mathcal{R}$ for any $|a| = 1$.
2. If z_1, z_2 are two non-negative real numbers in $\mathcal{R}$, then any $z \in (z_1, z_2)$ is also in $\mathcal{R}$.

To show the first claim, if $z \in \mathcal{R}$ and $|a| = 1$ then

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} |x[n](az)^{-n}| &= \sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| \cdot |a|^{-n} \\ &= \sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| \\ &< \infty. \end{aligned}$$

Therefore $az \in \mathcal{R}$.

To show the second claim, if $z_1, z_2 \in \mathcal{R}$ are non-negative real numbers and $z \in (z_1, z_2)$ then

$$\begin{aligned} z^{-n} &\leq \max\left\{z_1^{-n}, z_2^{-n}\right\} \\ &\leq z_1^{-n} + z_2^{-n} \end{aligned}$$

for any $n \in \mathbb{Z}$. Hence,

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| &= \sum_{n=-\infty}^{+\infty} |x[n]|z^{-n} \\ &\leq \sum_{n=-\infty}^{+\infty} |x[n]|(z_1^{-n} + z_2^{-n}) \\ &= \sum_{n=-\infty}^{+\infty} |x[n]z_1^{-n}| + \sum_{n=-\infty}^{+\infty} |x[n]z_2^{-n}| \\ &< \infty. \end{aligned}$$

Therefore $z \in \mathcal{R}$. ■

Proposition 83. Let $\mathcal{R}$ be the ROC of the Z-transform of a discrete-time signal $x[n]$.

1. Suppose that $x[n]$ is *right-sided*, i.e. there exists $n_0 \in \mathbb{R}$ such that $x[n] = 0$ for any $n < n_0$. Then $\{z : |z| > |z_0|\} \subseteq \mathcal{R}$ for any $z_0 \in \mathcal{R}$.
2. Suppose that $x[n]$ is *left-sided*, i.e. there exists $n_0 \in \mathbb{R}$ such that $x[n] = 0$ for any $n > n_0$. Then $\{z : 0 < |z| < |z_0|\} \subseteq \mathcal{R}$ for any $z_0 \in \mathcal{R}$.

○

Proof. We only show the result for right-sided signals as the result for left-sided signals follows a similar proof.

If z_0 is in the ROC of the Z-transform of a right-sided signal $x[n]$ (with $x[n] = 0$ for any $n < n_0$), then for any $|z| > |z_0|$,

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| &= \sum_{n=n_0}^{+\infty} |x[n]z^{-n}| \\
&= \sum_{n=n_0}^{+\infty} |x[n]z_0^{-n}| \cdot \left|\frac{z^{-n}}{z_0^{-n}}\right| \\
&= \sum_{n=n_0}^{+\infty} |x[n]z_0^{-n}| \cdot \left(\frac{|z|}{|z_0|}\right)^{-n} \\
&\leq \sum_{n=n_0}^{+\infty} |x[n]z_0^{-n}| \cdot \left(\frac{|z|}{|z_0|}\right)^{-n_0} \\
&= \left(\frac{|z|}{|z_0|}\right)^{-n_0} \sum_{n=-\infty}^{+\infty} |x[n]z_0^{-n}| \\
&< \infty.
\end{aligned}$$

Hence z is in the ROC. ■

Proposition 84. An LSI system with impulse response $h[n]$ is stable if and only if 1 belongs to the ROC of the Z-transform of $h[n]$. ○

Proof. Recall that an LSI system is stable if and only if its impulse response $h[t]$ is absolutely summable, i.e. $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$, which is equivalent to that 1 belongs to the ROC of the Z-transform of $h[n]$. ■

8.2 Inverse Z-transform

The Z-transform $\hat{X}(z)$ of a signal $x[n]$ at $z = re^{j2\pi f}$ ($r > 0$, $f \in \mathbb{R}$), where r is in the ROC, is evaluated as

$$\begin{aligned}
\hat{X}(re^{j2\pi f}) &= \sum_{n=-\infty}^{+\infty} x[n](re^{j2\pi f})^{-n} \\
&= \sum_{n=-\infty}^{+\infty} (x[n]r^{-n})e^{-j2\pi f n}.
\end{aligned}$$

That is, $f \mapsto \hat{X}(re^{j2\pi f})$ is the DTFT of $n \mapsto x[n]r^{-n}$. Inverting the DTFT gives

$$x[n]r^{-n} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(re^{j2\pi f})e^{j2\pi f n} df.$$

Hence

$$\begin{aligned}
x[n] &= r^n \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(re^{j2\pi f}) e^{j2\pi f n} df \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(re^{j2\pi f})(re^{j2\pi f})^n df \\
&= \frac{1}{j2\pi} \oint_\Gamma \hat{X}(z) z^n \cdot \frac{1}{z} dz \qquad \text{(with } z := re^{j2\pi f}\text{)} \\
&= \frac{1}{j2\pi} \oint_\Gamma \hat{X}(z) z^{n-1} dz,
\end{aligned}$$

where Γ is the circle $\{z : |z| = r\}$, oriented counterclockwise. In fact, similar to the case for the Laplace transform, as a consequence of Cauchy's integral theorem and the holomorphicity of the Z-transform, one can choose the contour Γ to be an arbitrary counterclockwise contour encircling the origin in the ROC. This yields the formula for the inverse Z-transform.

Definition 8.3. The *inverse Z-transform* of a function $\hat{X}(z)$ on some subset of $\mathbb{C} \setminus \{0\}$ is defined as the contour integral

$$x[n] := \frac{1}{j2\pi} \oint_\Gamma \hat{X}(z) z^{n-1} dz,$$

where Γ is any counterclockwise contour encircling the origin in the domain of $\hat{X}(z)$. ○

8.3 Properties of Z-transform

8.3.1 Linearity

Proposition 85. Let the Z-transform of $x[n], y[n]$ be $\hat{X}(z), \hat{Y}(z)$ with ROC $\mathcal{R}_1, \mathcal{R}_2$, respectively. Then, for any $a, b \in \mathbb{C}$, the Z-transform of $ax[n]+by[n]$ is $a\hat{X}(z)+b\hat{Y}(z)$ with ROC $\mathcal{R} \supseteq \mathcal{R}_1 \cap \mathcal{R}_2$. ○

Proof. To show that $z \in \mathcal{R}_1 \cap \mathcal{R}_2$ belongs to the ROC of the Z-transform of $ax[n] + by[n]$, note that

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} |(ax[n] + by[n])z^{-n}| &\le |a| \sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| + |b| \sum_{n=-\infty}^{+\infty} |y[n]z^{-n}| \\
&< \infty,
\end{aligned}$$

since both summations are finite. Now, the Z-transform of $ax[n] + by[n]$ is

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} (ax[n] + by[n])z^{-n} &= a \sum_{n=-\infty}^{+\infty} x[n]z^{-n} + b \sum_{n=-\infty}^{+\infty} y[n]z^{-n} \\
&= a\hat{X}(z) + b\hat{Y}(z).
\end{aligned}$$

■

8.3.2 Time shifting

Proposition 86. Let the Z-transform of $x[n]$ be $\hat{X}(z)$ with ROC $\mathcal{R}$. Then the Z-transform of $x[n-n_0]$ is $z^{-n_0}\hat{X}(z)$ with ROC $\mathcal{R}$ or $\mathcal{R} \setminus \{0\}$. ○

Proof.

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} x[n-n_0]z^{-n} &= \sum_{m=-\infty}^{+\infty} x[m]z^{-(m+n_0)} && \text{(with } m := n-n_0\text{)} \\
&= z^{-n_0} \sum_{m=-\infty}^{+\infty} x[m]z^{-m} \\
&= z^{-n_0}\hat{X}(z),
\end{aligned}$$

with ROC \{0} being

$$\begin{aligned}
\left\{z \neq 0 : \sum_{n=-\infty}^{+\infty} |x[n-n_0]z^{-n}| < \infty\right\} &= \left\{z \neq 0 : \sum_{m=-\infty}^{+\infty} |x[m]z^{-(m+n_0)}| < \infty\right\} && \text{(with } m := n-n_0\text{)} \\
&= \left\{z \neq 0 : |z^{-n_0}| \sum_{m=-\infty}^{+\infty} |x[m]z^{-m}| < \infty\right\} \\
&= \left\{z \neq 0 : \sum_{m=-\infty}^{+\infty} |x[m]z^{-m}| < \infty\right\} \\
&= \mathcal{R} \setminus \{0\}.
\end{aligned}$$

■

8.3.3 Scaling in the z-domain

Proposition 87. Let the Z-transform of $x[n]$ be $\hat{X}(z)$ with ROC $\mathcal{R}$. Then, for any $a \in \mathbb{C} \setminus \{0\}$, the Z-transform of $a^{-n}x[n]$ is $\hat{X}(az)$ with ROC $\frac{1}{a}\mathcal{R} = \left\{\frac{z}{a} : z \in \mathcal{R}\right\}$. ○

Proof.

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} (a^{-n}x[n])z^{-n} &= \sum_{n=-\infty}^{+\infty} x[n](az)^{-n} \\
&= \hat{X}(az),
\end{aligned}$$

with ROC

$$\begin{aligned}
\left\{z : \sum_{n=-\infty}^{+\infty} |(a^{-n}x[n])z^{-n}| < \infty\right\} &= \left\{z : \sum_{n=-\infty}^{+\infty} |x[n](az)^{-n}| < \infty\right\} \\
&= \{z : az \in \mathcal{R}\} \\
&= \left\{\frac{\tilde{z}}{a} : \tilde{z} \in \mathcal{R}\right\} \\
&= \frac{1}{a}\mathcal{R}.
\end{aligned}$$

■

8.3.4 Zero-padding

Proposition 88. Let the Z-transform of $x[n]$ be $\hat{X}(z)$ with ROC $\mathcal{R}$. Let k be a positive integer. Then the Z-transform of the zero-padded signal

$$n \mapsto \begin{cases} x[m], & n = km \text{ for some } m \in \mathbb{Z},, \\ 0, & \text{otherwise}, \end{cases}$$

is $\hat{X}(z^k)$ with ROC $\left\{z : z^k \in \mathcal{R}\right\}$. ○

Proof. The Z-transform of the zero-padded signal is

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} x[m] z^{-km} &= \sum_{m=-\infty}^{+\infty} x[m] (z^k)^{-m} \\ &= \hat{X}(z^k), \end{aligned}$$

with ROC

$$\begin{aligned} \left\{ z : \sum_{m=-\infty}^{+\infty} |x[m] z^{-km}| < \infty \right\} &= \left\{ z : \sum_{m=-\infty}^{+\infty} |x[m] (z^k)^{-m}| < \infty \right\} \\ &= \left\{ z : z^k \in \mathcal{R} \right\}. \end{aligned}$$

■

8.3.5 Compression

Proposition 89. Let the Z-transform of $x[n]$ be $\hat{X}(z)$ with ROC $\mathcal{R}$. Let k be a positive integer. Then the Z-transform of $x[kn]$ is $\frac{1}{k} \sum_{\ell=0}^{k-1} \hat{X}\left(e^{-j2\pi \frac{\ell}{k}} z^{\frac{1}{k}}\right)$, where $z^{\frac{1}{k}}$ is chosen to be any of the branches, with ROC containing $\left\{z^k : z \in \mathcal{R}\right\}$. ○

Proof. Suppose that $\tilde{z} \in \mathcal{R}$ and $z = \tilde{z}^k$. To show that z belongs to the ROC of the Z-transform of $x[kn]$, note that

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} |x[kn] z^{-n}| &= \sum_{n=-\infty}^{+\infty} |x[kn] (\tilde{z}^k)^{-n}| \\ &= \sum_{n=-\infty}^{+\infty} |x[kn] \tilde{z}^{-kn}| \\ &\leq \sum_{n=-\infty}^{+\infty} |x[n] \tilde{z}^{-n}| \\ &< \infty. \end{aligned}$$

since $\tilde{z} \in \mathcal{R}$. Now, consider

$$\begin{aligned} \frac{1}{k} \sum_{\ell=0}^{k-1} \hat{X}\left(e^{-j2\pi \frac{\ell}{k}} \tilde{z}\right) &= \frac{1}{k} \sum_{\ell=0}^{k-1} \sum_{n=-\infty}^{+\infty} x[n] \left(e^{-j2\pi \frac{\ell}{k}} \tilde{z}\right)^{-n} \\ &= \sum_{n=-\infty}^{+\infty} x[n] \tilde{z}^{-n} \left(\frac{1}{k} \sum_{\ell=0}^{k-1} e^{j2\pi \frac{n}{k} \ell} \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{+\infty} x[n]\tilde{z}^{-n} \left(\begin{cases} 1, & n \equiv 0 \pmod{k}, \\ 0, & \text{otherwise} \end{cases} \right) \\
&= \sum_{m=-\infty}^{+\infty} x[km]\tilde{z}^{-km} && (\text{with } m := \frac{n}{k}) \\
&= \sum_{m=-\infty}^{+\infty} x[km](\tilde{z}^{k})^{-m} \\
&= \sum_{m=-\infty}^{+\infty} x[km]z^{-m},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
\sum_{\ell=0}^{k-1} e^{j2\pi \frac{n}{k}\ell} &= \begin{cases} \sum_{\ell=0}^{k-1} 1, & n \equiv 0 \pmod{k}, \\ \frac{e^{j2\pi \frac{n}{k}k}-1}{e^{j2\pi \frac{n}{k}}-1}, & \text{otherwise} \end{cases} \\
&= \begin{cases} k, & n \equiv 0 \pmod{k}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Note that the choice of $\tilde{z}$ is arbitrary as long as $\tilde{z}^k = z$. Hence the Z-transform of $x[kn]$ is $\frac{1}{k}\sum_{\ell=0}^{k-1} \hat{X}\left(e^{-j2\pi\frac{\ell}{k}} z^{\frac{1}{k}}\right)$, with $z^{\frac{1}{k}}$ chosen to be any of the branches. ■

8.3.6 Time reversal

Proposition 90. Let the Z-transform of $x[n]$ be $\hat{X}(z)$ with ROC $\mathcal{R}$. Then the Z-transform of $x[-n]$ is $\hat{X}(z^{-1})$ with ROC $\left\{z^{-1} : z \in \mathcal{R}\right\}$. ○

Proof.

$$
\begin{aligned}
\sum_{n=-\infty}^{+\infty} x[-n]z^{-n} &= \sum_{m=-\infty}^{+\infty} x[m]z^{-(-m)} && (\text{with } m := -n) \\
&= \sum_{m=-\infty}^{+\infty} x[m](z^{-1})^{-m} \\
&= \hat{X}(z^{-1}),
\end{aligned}
$$

with ROC

$$
\begin{aligned}
\left\{z : \sum_{n=-\infty}^{+\infty} |x[-n]z^{-n}| < \infty\right\} &= \left\{z : \sum_{m=-\infty}^{+\infty} |x[m]z^{-(-m)}| < \infty\right\} && (\text{with } m := -n) \\
&= \left\{z : \sum_{m=-\infty}^{+\infty} |x[m](z^{-1})^{-m}| < \infty\right\} \\
&= \left\{z : z^{-1} \in \mathcal{R}\right\} \\
&= \left\{\tilde{z}^{-1} : \tilde{z} \in \mathcal{R}\right\}.
\end{aligned}
$$

■

8.3.7 Complex conjugation

Proposition 91. Let the Z-transform of $x[n]$ be $\hat{X}(z)$ with ROC $\mathcal{R}$. Then the Z-transform of $\overline{x[n]}$ is $\overline{\hat{X}(\overline{z})}$ with ROC $\mathcal{R}$. ○

Proof.

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \overline{x[n]} z^{-n} &= \overline{\sum_{n=-\infty}^{+\infty} x[n] \overline{z^{-n}}} \\ &= \overline{\sum_{n=-\infty}^{+\infty} x[n] \overline{z}^{-n}} \\ &= \overline{\hat{X}(\overline{z})},\end{aligned}$$

with ROC

$$\begin{aligned}\left\{ z : \sum_{n=-\infty}^{+\infty} |\overline{x[n]} z^{-n}| < \infty \right\} &= \left\{ z : \sum_{n=-\infty}^{+\infty} |x[n] z^{-n}| < \infty \right\} \\ &= \mathcal{R}.\end{aligned}$$

■

8.3.8 Convolution

Proposition 92. Let the Z-transform of $x[n], y[n]$ be $\hat{X}(z), \hat{Y}(z)$ with ROC $\mathcal{R}_1, \mathcal{R}_2$, respectively. Then the Z-transform of $x[n] * y[n]$ is $\hat{X}(z)\hat{Y}(z)$ with ROC $\mathcal{R} \supseteq \mathcal{R}_1 \cap \mathcal{R}_2$. ○

Proof. To show that $z \in \mathcal{R}_1 \cap \mathcal{R}_2$ belongs to the ROC of the Z-transform of $x[n] * y[n]$, note that

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} |(x[n] * y[n]) z^{-n}| &= \sum_{n=-\infty}^{+\infty} \left| \sum_{m=-\infty}^{+\infty} x[m] y[n-m] \right| |z^{-n}| \\ &\leq \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |x[m]| \, |y[n-m]| \, |z^{-n}| \\ &= \sum_{\ell=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |x[m]| \, |y[\ell]| \left| z^{-(m+\ell)} \right| && \text{(with } \ell := n-m\text{)} \\ &= \left(\sum_{m=-\infty}^{+\infty} |x[m] z^{-m}| \right) \left(\sum_{\ell=-\infty}^{+\infty} \left| y[\ell] z^{-\ell} \right| \right) \\ &< \infty,\end{aligned}$$

since both summations are finite. Now, the Z-transform of $x[n] * y[n]$ is

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} (x[n] * y[n]) z^{-n} &= \sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{+\infty} x[m] y[n-m] \right) z^{-n} \\ &= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x[m] y[n-m] z^{-m} z^{-(n-m)} \\ &= \sum_{\ell=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x[m] y[\ell] z^{-m} z^{-\ell} && \text{(with } \ell := n-m\text{)}\end{aligned}$$

$$= \left(\sum_{m=-\infty}^{+\infty} x[m]z^{-m}\right)\left(\sum_{\ell=-\infty}^{+\infty} y[\ell]z^{-\ell}\right)$$
$$= \hat{X}(z)\hat{Y}(z).$$

■

8.3.9 Differentiation in the z-domain

Proposition 93. Let the Z-transform of $x[n]$ be $\hat{X}(z)$ with ROC $\mathcal{R}$. Then the Z-transform of $-(n-1)x[n-1]$ is $\frac{d}{dz}\hat{X}(z)$ with ROC containing the interior of $\mathcal{R}$. ○

Proof. Let $\epsilon \neq 0$. By the z-domain scaling property, the Z-transform of $(1-\epsilon)^{-n}x[n]$ is $\hat{X}((1-\epsilon)z)$ with ROC $\frac{1}{1-\epsilon}\mathcal{R} = \left\{\frac{z}{1-\epsilon} : z \in \mathcal{R}\right\}$. By linearity of the Z-transform, the Z-transform of $\frac{1-(1-\epsilon)^{-n}}{\epsilon}x[n]$ is $\frac{\hat{X}(z)-\hat{X}((1-\epsilon)z)}{\epsilon}$. with ROC at least as large as $\mathcal{R} \cap \frac{1}{1-\epsilon}\mathcal{R}$. Letting $\epsilon \to 0$ gives that the Z-transform of $-nx[n]$ is $z\frac{d}{dz}\hat{X}(z)$ with ROC containing the interior of $\mathcal{R}$. Applying the time shifting property of the Z-transform yields the desired result. ■

8.3.10 Initial value theorem

Proposition 94. Let the Z-transform of $x[n]$ be $\hat{X}(z)$. Suppose that $x[n] = 0$ for all $n < 0$. Then

$$x[0] = \lim_{|z|\to+\infty} \hat{X}(z).$$

○

Proof.

$$\lim_{|z|\to+\infty} \hat{X}(z) = \lim_{|z|\to+\infty} \sum_{n=0}^{+\infty} x[n]z^{-n}$$
$$= \sum_{n=0}^{+\infty} x[n]\left(\lim_{|z|\to+\infty} z^{-n}\right)$$
$$= \sum_{n=0}^{+\infty} x[n]\left(\begin{cases}1, & n = 0,\\ 0, & \text{otherwise}\end{cases}\right)$$
$$= x[0].$$

■

8.3.11 Final value theorem

Proposition 95. Let the Z-transform of $x[n]$ be $\hat{X}(z)$. Suppose that $x[n] = 0$ for all $n < 0$. Then

$$\lim_{n\to+\infty} x[n] = \lim_{z\to1}(z-1)\hat{X}(z).$$

○

Proof.

$$\begin{aligned}
\lim_{z\to 1}(z-1)\hat{X}(z) &= \lim_{z\to 1}(z-1)\sum_{n=0}^{+\infty} x[n]z^{-n} \\
&= \lim_{z\to 1}\left(\sum_{n=0}^{+\infty} x[n]z^{-(n-1)} - \sum_{n=0}^{+\infty} x[n]z^{-n}\right) \\
&= \lim_{z\to 1}\left(x[0]z^{-1} + \sum_{n=0}^{+\infty} x[n+1]z^{-n} - \sum_{n=0}^{+\infty} x[n]z^{-n}\right) \\
&= x[0] + \lim_{z\to 1}\sum_{n=0}^{+\infty}(x[n+1]-x[n])z^{-n} \\
&= x[0] + \sum_{n=0}^{+\infty}(x[n+1]-x[n])\lim_{z\to 1} z^{-n} \\
&= x[0] + \sum_{n=0}^{+\infty}(x[n+1]-x[n]) \\
&= \lim_{n\to+\infty} x[n].
\end{aligned}$$

■

8.4 Examples of Z-transform

Example 1. $\delta[n] \xrightarrow{\text{ZT}} 1$ with ROC $\mathbb{C}$.

Proof.

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \delta[n]z^{-n} &= z^{-0} \\
&= 1,
\end{aligned}$$

with ROC

$$\begin{aligned}
\left\{z : \sum_{n=-\infty}^{+\infty} |\delta[n]z^{-n}| < \infty\right\} &= \{z : 1 < \infty\} \\
&= \mathbb{C}.
\end{aligned}$$

■

Example 2. $\delta[n-n_0] \xrightarrow{\text{ZT}} z^{-n_0}$ with ROC $\begin{cases} \mathbb{C}\setminus\{0\}, & n_0 > 0, \\ \mathbb{C}, & n_0 \le 0. \end{cases}$

Proof. We already know this when $n_0 = 0$. Apply the time shifting property of the Z-transform. ■

Example 3. $u[n] \xrightarrow{\text{ZT}} \frac{1}{1-z^{-1}}$ with ROC $\{z : |z| > 1\}$, and $-u[-n-1] \xrightarrow{\text{ZT}} \frac{1}{1-z^{-1}}$ with ROC $\{z : |z| < 1\}$.

Proof. The ROC of the Z-transform of $u[n]$ is

$$\begin{aligned}\left\{z : \sum_{n=-\infty}^{+\infty} |u[n]z^{-n}| < \infty\right\} &= \left\{z : \sum_{n=0}^{+\infty} |z^{-n}| < \infty\right\} \\ &= \left\{z : \sum_{n=0}^{+\infty} |z^{-1}|^n < \infty\right\} \\ &= \left\{z : |z^{-1}| < 1\right\} \\ &= \{z : |z| > 1\}.\end{aligned}$$

Hence, for z in the ROC,

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} u[n]z^{-n} &= \sum_{n=0}^{+\infty} z^{-n} \\ &= \sum_{n=0}^{+\infty} (z^{-1})^n \\ &= \frac{1}{1-z^{-1}}.\end{aligned}$$

The ROC of the Z-transform of $-u[-n-1]$ is

$$\begin{aligned}\left\{z : \sum_{n=-\infty}^{+\infty} |-u[-n-1]z^{-n}| < \infty\right\} &= \left\{z : \sum_{n=-\infty}^{-1} |z^{-n}| < \infty\right\} \\ &= \left\{z : \sum_{m=1}^{+\infty} |z^{m}| < \infty\right\} \qquad \text{(with } m := -n\text{)} \\ &= \left\{z : \sum_{m=1}^{+\infty} |z|^{m} < \infty\right\} \\ &= \{z : |z| < 1\}.\end{aligned}$$

Hence, for z in the ROC,

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} -u[-n-1]z^{-n} &= -\sum_{n=-\infty}^{-1} z^{-n} \\ &= -\sum_{m=1}^{+\infty} z^{m} \qquad \text{(with } m := -n\text{)} \\ &= -\frac{z}{1-z} \\ &= \frac{1}{1-z^{-1}}.\end{aligned}$$

■

Example 4. $a^n u[n] \xrightarrow{\text{ZT}} \frac{1}{1-az^{-1}}$ with ROC $\{z : |z| > |a|\}$, and $-a^n u[-n-1] \xrightarrow{\text{ZT}} \frac{1}{1-az^{-1}}$ with ROC $\{z : |z| < |a|\}$, for any $a \in \mathbb{C} \setminus \{0\}$.

Proof. We already know this when $a = 1$. Apply the z-domain scaling property of the Z-transform. ■

Example 5. $\binom{n+k-1}{k-1} a^n u[n] \xrightarrow{\text{ZT}} \frac{1}{(1-az^{-1})^k}$ with ROC $\{z : |z| > |a|\}$, and $(-1)^k \binom{-n-1}{k-1} a^n u[-n-k] \xrightarrow{\text{ZT}} \frac{1}{(1-az^{-1})^k}$ with ROC $\{z : |z| < |a|\}$, for any $a \in \mathbb{C} \setminus \{0\}$ and positive integer k.

Proof. It suffices to show that for $|z| > |a|$ and positive integer k we have

$$\frac{1}{(1-az^{-1})^k} = \sum_{n=0}^{+\infty} \binom{n+k-1}{k-1} a^n z^{-n}.$$

Note that from this we can also get the Z-transform of $(-1)^k \binom{-n-1}{k-1} a^n u[-n-k]$ by using the time shifting and time reversal properties of the Z-transform and substituting a by a^{-1}.

We induct on k. We already know the base case $k = 1$. We will verify the induction part by showing the equality

$$\sum_{n=0}^{\ell} \binom{n+k-1}{k-1} = \binom{\ell+k}{k}.$$

There is an elegant combinatorial proof for this. Consider the set $S := \{0, 1, \ldots, \ell + k - 1\}$. There are $\binom{\ell+k}{k}$ subsets of S of cardinality k in total. These subsets can be partitioned according to their minima. Note that for any subset T of S of cardinality k, we have $\min T = n$ if and only if $T = \{n\} \cup \tilde{T}$ for some subset $\tilde{T}$ of $\{n + 1, n + 2, \ldots, \ell + k - 1\}$ of cardinality $k - 1$. Hence, there are $\binom{\ell-n+k-1}{k-1}$ subsets of S of cardinality k with $\min T = n$ for $n \leq \ell$, and there are no such subsets for $n > \ell$. Thus,

$$\begin{aligned} \sum_{n=0}^{\ell} \binom{n+k-1}{k-1} &= \sum_{n=0}^{\ell} \binom{\ell-n+k-1}{k-1} \\ &= \binom{\ell+k}{k}. \end{aligned}$$

Thus, by the induction hypothesis, we have

$$\begin{aligned} \frac{1}{(1-az^{-1})^{k+1}} &= \frac{1}{1-az^{-1}} \cdot \frac{1}{(1-az^{-1})^k} \\ &= \left(\sum_{m=0}^{+\infty} a^m z^{-m}\right)\left(\sum_{n=0}^{+\infty} \binom{n+k-1}{k-1} a^n z^{-n}\right) \\ &= \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \binom{n+k-1}{k-1} a^{m+n} z^{-(m+n)} \\ &= \sum_{\ell=0}^{+\infty}\left(\sum_{n=0}^{\ell} \binom{n+k-1}{k-1}\right) a^{\ell} z^{-\ell} \qquad \text{(with } \ell := m+n\text{)} \\ &= \sum_{\ell=0}^{+\infty} \binom{\ell+k}{k} a^{\ell} z^{-\ell}. \end{aligned}$$

■

Example 6. $u[n]\sin\omega n \xrightarrow{\text{ZT}} \frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega+z^{-2}}$ with ROC $\{z : |z| > 1\}$, and $u[n]\cos\omega n \xrightarrow{\text{ZT}} \frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}$ with ROC $\{z : |z| > 1\}$, for any $\omega \in \mathbb{R}$ with $\omega \not\equiv 0 \pmod{\pi}$.

Proof. Consider

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} (u[n]\sin\omega n) z^{-n} &= \sum_{n=0}^{+\infty} z^{-n}\sin\omega n \\ &= \sum_{n=0}^{+\infty} z^{-n} \frac{e^{j\omega n} - e^{-j\omega n}}{2j} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2j}\left(\sum_{n=0}^{+\infty}(z^{-1}e^{j\omega})^n - \sum_{n=0}^{+\infty}(z^{-1}e^{-j\omega})^n\right) \\
&= \frac{1}{2j}\left(\frac{1}{1-z^{-1}e^{j\omega}} - \frac{1}{1-z^{-1}e^{-j\omega}}\right) \\
&= \frac{1}{2j}\frac{z^{-1}(e^{j\omega}-e^{-j\omega})}{1-z^{-1}(e^{j\omega}+e^{-j\omega})+z^{-2}} \\
&= \frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega+z^{-2}}, \\
\sum_{n=-\infty}^{+\infty}(u[n]\cos\omega n)z^{-n} &= \sum_{n=0}^{+\infty} z^{-n}\cos\omega n \\
&= \sum_{n=0}^{+\infty} z^{-n}\frac{e^{j\omega n}+e^{-j\omega n}}{2} \\
&= \frac{1}{2}\left(\sum_{n=0}^{+\infty}(z^{-1}e^{j\omega})^n + \sum_{n=0}^{+\infty}(z^{-1}e^{-j\omega})^n\right) \\
&= \frac{1}{2}\left(\frac{1}{1-z^{-1}e^{j\omega}} + \frac{1}{1-z^{-1}e^{-j\omega}}\right) \\
&= \frac{1}{2}\frac{2-z^{-1}(e^{j\omega}+e^{-j\omega})}{1-z^{-1}(e^{j\omega}+e^{-j\omega})+z^{-2}} \\
&= \frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}},
\end{aligned}
$$

with all of the summations converge absolutely if and only if $|z| > 1$. ■

8.5 Applications of Z-transform

8.5.1 Linear recurrences

Example 1. The *Fibonacci sequence* is defined as

$$
x[n] = \begin{cases} 0, & n \le 0, \\ 1, & n = 1, \\ x[n-1] + x[n-2], & n \ge 2. \end{cases}
$$

We want to find the general term for $x[n]$. Let $\hat{X}(z)$ be the Z-transform of $x[n]$. Note that, for any $n \in \mathbb{Z}$,

$$
x[n] = x[n-1] + x[n-2] + \delta[n-1].
$$

Taking Z-transform gives

$$
\hat{X}(z) = z^{-1}\hat{X}(z) + z^{-2}\hat{X}(z) + z^{-1}.
$$

Hence

$$
\begin{aligned}
\hat{X}(z) &= \frac{z^{-1}}{1-z^{-1}-z^{-2}} \\
&= \frac{z^{-1}}{(1-\varphi z^{-1})(1+\varphi^{-1}z^{-1})}
\end{aligned}
$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{1-\varphi z^{-1}} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1+\varphi^{-1} z^{-1}},$$

where $\varphi := \frac{1+\sqrt{5}}{2}$ is also known as the *golden ratio*. Note that $\hat{X}(z)$ has two poles at $z = \varphi$ and $z = -\varphi^{-1}$. The possible ROC are $\left\{z : |z| < \varphi^{-1}\right\}$, $\left\{z : \varphi^{-1} < |z| < \varphi\right\}$ and $\{z : |z| > \varphi\}$. Since $x[n]$ is right-sided, the corresponding ROC is $\{z : |z| > \varphi\}$. Inverting the Z-transform gives

$$\begin{aligned} x[n] &= \frac{1}{\sqrt{5}} \cdot \varphi^n u[n] - \frac{1}{\sqrt{5}} \cdot (-\varphi^{-1})^n u[n] \\ &= \frac{1}{\sqrt{5}} \left(\varphi^n - (-\varphi^{-1})^n\right) u[n] \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right) u[n]. \end{aligned}$$

Example 2. The *Pell numbers* is defined as

$$x[n] = \begin{cases} 0, & n \le 0, \\ 1, & n = 1, \\ 2x[n-1] + x[n-2], & n \ge 2. \end{cases}$$

We want to find the general term for $x[n]$. Let $\hat{X}(z)$ be the Z-transform of $x[n]$. Note that, for any $n \in \mathbb{Z}$,

$$x[n] = 2x[n-1] + x[n-2] + \delta[n-1].$$

Taking Z-transform gives

$$\hat{X}(z) = 2z^{-1}\hat{X}(z) + z^{-2}\hat{X}(z) + z^{-1}.$$

Hence

$$\hat{X}(z) = \frac{z^{-1}}{1 - 2z^{-1} - z^{-2}}.$$

The roots of $1 - 2z^{-1} - z^{-2} = 0$ are $z^{-1} = -1 \pm \sqrt{2}$. We want to split $\hat{X}(z)$ into partial fractions, i.e. to find constants A, B such that

$$\frac{z^{-1}}{1 - 2z^{-1} - z^{-2}} = \frac{A}{1 - \frac{1}{-1-\sqrt{2}} z^{-1}} + \frac{B}{1 - \frac{1}{-1+\sqrt{2}} z^{-1}},$$

or equivalently

$$z^{-1} = A\left(1 - \frac{1}{-1+\sqrt{2}} z^{-1}\right) + B\left(1 - \frac{1}{-1-\sqrt{2}} z^{-1}\right).$$

Putting $z^{-1} = -1 - \sqrt{2}$ and $z^{-1} = -1 + \sqrt{2}$, and then solving for A, B respectively gives

$$\begin{aligned} A &= -\frac{1}{2\sqrt{2}}, \\ B &= \frac{1}{2\sqrt{2}}. \end{aligned}$$

Thus,

$$\begin{aligned}\hat{X}(z) &= -\frac{1}{2\sqrt{2}} \cdot \frac{1}{1 - \frac{1}{-1-\sqrt{2}} z^{-1}} + \frac{1}{2\sqrt{2}} \cdot \frac{1}{1 - \frac{1}{-1+\sqrt{2}} z^{-1}} \\ &= -\frac{1}{2\sqrt{2}} \cdot \frac{1}{1-(1-\sqrt{2})z^{-1}} + \frac{1}{2\sqrt{2}} \cdot \frac{1}{1-(1+\sqrt{2})z^{-1}}.\end{aligned}$$

Note that $\hat{X}(z)$ has two poles at $z = 1-\sqrt{2}$ and $z = 1+\sqrt{2}$. The possible ROC are $\left\{z : |z| < \sqrt{2}-1\right\}$, $\left\{z : \sqrt{2}-1 < |z| < \sqrt{2}+1\right\}$ and $\left\{z : |z| > \sqrt{2}+1\right\}$. Since $x[n]$ is right-sided, the corresponding ROC is $\left\{z : |z| > \sqrt{2}+1\right\}$. Inverting the Z-transform gives

$$\begin{aligned}x[n] &= -\frac{1}{2\sqrt{2}}(1-\sqrt{2})^n u[n] + \frac{1}{2\sqrt{2}}(1+\sqrt{2})^n u[n] \\ &= \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right) u[n].\end{aligned}$$

Exercises

1.