List of Formulae (7 Pages)

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USEFUL DEFINITIONS OF FORWARDS AND INVERSE TRANSFORMS

D-T. Convolution	$y[n]=\sum_{m=-\infty}^{\infty}x[m]h[n-m]$	C-T. Convolution	$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$
Periodic D-T Conv.	$y[n]=\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}x[m]h[n-m]$	Periodic C-T. Conv	$y(t)=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(\tau)h(t-\tau)d\tau$
Fourier Transform	$X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$	Inverse F. T.	$x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df$
Fourier Series	$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk2\pi t/T_0}$	Inversion of F.S.	$a_k=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}x(t)e^{-jk2\pi t/T_0}dt$
D-T FT	$\underline{X}(f)=\sum_{n=-\infty}^{\infty}x[n]e^{-j2\pi fn}$	Inverse: D-T FT	$x[n]=\int_{-\frac{1}{2}}^{\frac{1}{2}}\underline{X}(f)e^{j2\pi fn}df$
DFT	$X[k]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x[n]e^{-j2\pi nk/N}$	IDFT	$x[n]=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}X[k]e^{j2\pi nk/N}$
Laplace Transform	$\hat{X}(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt$	Inverse LT	$x(t)=\frac{1}{j2\pi}\int_{Re(s)=\sigma,\uparrow}\hat{X}(s)e^{st}ds$
Z Transform	$\hat{X}(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$	Inverse ZT	$x[n]=\frac{1}{j2\pi}\oint_{\lvert Z\rvert=r}\hat{X}(z)z^{n-1}dz$

Elementary Formulae

- $e^{ja}=\cos(a)+j\sin(a)$
- $\cos(a)=\frac{e^{ja}+e^{-ja}}{2}$
- $\sin(a)=\frac{e^{ja}-e^{-ja}}{2j}$
- $\text{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$
- $\text{rect}(x)=\begin{cases}1 & x\in(-\frac{1}{2},\frac{1}{2}]\\ 0 & \text{otherwise}\end{cases}$
- $\text{Tri}(x)=\begin{cases}1-\lvert x\rvert & x\in(-1,1]\\ 0 & \text{otherwise}\end{cases}$
- $\text{Tri}(x)=\text{rect}(x)*\text{rect}(x)$.
- $(a+b)^N=\sum_{n=0}^{N}\binom{N}{n}a^n b^{N-n}$.
- $\sum_{n=0}^{N-1}r^n=\frac{1-r^N}{1-r}$.
- $\int_{-\infty}^{\infty}x(t)\delta(t)dt=x(0)$.
- $\frac{du(t)}{dt}=\delta(t)$.
- $x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0)$.
- If $x(t)*h(t)=y(t)$, then $x'(t)*h(t)=x(t)*h'(t)=y'(t)$.
- $\delta(ax)=\frac{1}{\lvert a\rvert}\delta(x)$

Properties and Formulae: Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T}}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-\frac{j2\pi kt}{T}}dt$$

Properties

- *Multiplication*: If $x(t) \xrightarrow{FS} a_k$, $y(t) \xrightarrow{FS} b_k$ then $x(t)y(t) \xrightarrow{FS} \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$
- *Differentiation*: $\frac{d}{dt}x(t) \xrightarrow{FS} \frac{j2\pi k}{T}a_k$
- *Shifting*: $x(t-t_0) \xrightarrow{FS} e^{\frac{-j2\pi k}{T}t_0}a_k$
- *Frequency Shifting*: $e^{\frac{j2\pi Mt}{T}}x(t) \xrightarrow{FS} a_{k-M}$
- *Complex conjugation*: $x^*(t) \xrightarrow{FS} a^*_{-k}$.
- *Time Reversal*: $x(-t) \xrightarrow{FS} a_{-k}$.
- *Periodic Convolution*: If $x(t) \xrightarrow{FS} a_k$ and $y(t) \xrightarrow{FS} b_k$ then $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)y(t-\tau)d\tau \xrightarrow{FS} a_k b_k$
- *Parseval's theorem*: If $x(t) \xrightarrow{FS} a_k$ then $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$.

Some common Fourier Series

- $\sum_{k=-\infty}^{+\infty} a_k e^{j2\pi f_0 kt} \xrightarrow{FS} a_k$
- $e^{j2\pi f_0 t} \xrightarrow{FS} a_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{cases}$
- $\sum_{k=-\infty}^{+\infty} \delta(t-kT) \xrightarrow{FS} \frac{1}{T}$

Properties and Formulae: Fourier Transform

$$\hat{X}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \int_{-\infty}^{\infty} \hat{X}(f)e^{j2\pi ft}df$$

Properties

- *Linearity* If $x_1(t) \xrightarrow{FT} \hat{X}_1(f)$, $x_2(t) \xrightarrow{FT} \hat{X}_2(f)$ then $ax_1(t) + bx_2(t) \xrightarrow{FT} a\hat{X}_1(f) + b\hat{X}_2(f)$
- *Scaling*: $x(at) \xleftrightarrow{F.T.} \frac{1}{|a|}\hat{X}(\frac{f}{a})$
- *Multiplication*: If $x_1(t) \xrightarrow{FT} \hat{X}_1(f)$, $x_2(t) \xrightarrow{FT} \hat{X}_2(f)$ then $x_1(t) * x_2(t) \xrightarrow{FT} \hat{X}_1(f)\hat{X}_2(f)$
- *Differentiation (in time)*: $\frac{d}{dt}x(t) \xleftrightarrow{F.T.} j2\pi f\hat{X}(f)$

- *Integration (in time)*: $\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \overset{F.T.}{\longleftrightarrow} \frac{\hat{X}(f)}{j2\pi f} + \frac{1}{2}\hat{X}(0)\delta(f)$
- *Differentiation (in frequency)*: $-j2\pi t x(t) \overset{F.T.}{\longleftrightarrow} \frac{d}{df}\hat{X}(f)$
- *Shifting in time*: $x(t-t_0) \xrightarrow{F.T.} e^{-j2\pi f t_0}\hat{X}(f)$
- *Shifting in frequency*: $e^{j2\pi f_0 t}x(t) \xrightarrow{F.T.} \hat{X}(f-f_0)$
- *Complex conjugation*: $x^*(t) \xrightarrow{F.T.} \hat{X}^*(-f)$.
- *Parseval's theorem*: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{X}(f)|^2 df$.
- *Duality*: $F.T.(F.T.(f(t))) = f(-t)$.
- *Uncertainty Principle*: $\left(\int_{-\infty}^{\infty} t^2|x(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} f^2|\hat{X}(f)|^2 df\right) \geq \frac{1}{16\pi^2}$.
- *Poisson summation formula*: $\sum_{f\in\mathcal{Z}} \hat{X}(f) = \sum_{k\in\mathcal{Z}} x(k)$

Some common transforms

- $\text{rect}(t) \xrightarrow{FT} \text{sinc}(f)$
- $e^{-at}u(t) \overset{F.T.}{\longleftrightarrow} \frac{1}{a+j2\pi f}$ when $a > 0$.
- $e^{at}u(-t) \overset{F.T.}{\longleftrightarrow} \frac{1}{a-j2\pi f}$ when $a > 0$.
- $\sum_n \delta(t-nT) \overset{F.T.}{\longleftrightarrow} \frac{1}{T}\sum_k \delta\left(f - \frac{k}{T}\right)$.
- $u(t) \overset{F.T.}{\longleftrightarrow} \frac{1}{\mathrm{j}2\pi f} + \frac{1}{2}\delta(f)$.
- $e^{j2\pi f_0 t} \overset{F.T.}{\longleftrightarrow} \delta(f-f_0)$
- $\sum_{k=-\infty}^{+\infty} a_k e^{j2\pi f_0 k t} \overset{F.T.}{\longleftrightarrow} \sum_{k=-\infty}^{+\infty} a_k \delta(f - kf_0)$

Properties and Formulae: Discrete Time Fourier Transform

$$\hat{X}(f) = \sum_n x[n]e^{-j2\pi f n}$$

$$x(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)e^{j2\pi f n} df$$

Properties

- *Multiplication*: If $x_1(n) \xrightarrow{DTFT} \hat{X}_1(f)$, $x_2(t) \xrightarrow{DTFT} \hat{X}_2(f)$ then $x_1[n] * x_2[n] \xrightarrow{DTFT} \hat{X}_1(f)\hat{X}_2(f)$
- *Differentiation (in frequency)*: $-j2\pi n x[n] \xrightarrow{DTFT} \frac{d}{df}\hat{X}(f)$
- *Shifting in time*: $x[n-n_0] \xrightarrow{DTFT} e^{-j2\pi f n_0}\hat{X}(f)$

- *Shifting in frequency*: $e^{j2\pi f_0 n}x(t) \xrightarrow{DTFT} \hat{X}(f-f_0)$
- *Complex conjugation*: $x^*[n] \xrightarrow{DTFT} \hat{X}^*(-f)$.
- *Parseval's theorem*: $\sum_{-\infty}^{\infty} |x[n]|^2 dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{X}(f)|^2 df$.
- *Zero-padding*:
$$x_{(k)}[n] = \begin{cases} x[r] & n = kr \\ 0 & o.w. \end{cases} \quad \overset{DTFT}{\longleftrightarrow} \hat{X}(kf)$$
- *Compression*:
$$x_{(k)}[n] = x[nk] \overset{DTFT}{\longleftrightarrow} \frac{1}{k}\sum_{r=0}^{k-1} \hat{X}\left(\frac{f-r}{k}\right).$$

Some common transforms

- constant function $1 \xrightarrow{DTFT} \sum_k \delta(f-k)$
- $\delta[n] \xrightarrow{DTFT} 1$
- $a^n u[n] \xrightarrow{DTFT} \frac{1}{1-ae^{-j2\pi f}}, \quad |a| < 1.$
- $\frac{(n+r-1)!}{n!(r-1)!}a^n u[n] \xrightarrow{DTFT} \frac{1}{(1-ae^{-j2\pi f})^r}, \quad |a| < 1.$
- $e^{j2\pi f_0 n} \xrightarrow{DTFT} \sum_{k=-\infty}^{\infty} \delta(f-f_0-k)$
- $\sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{N}} \xrightarrow{DTFT} \sum_{k=-\infty}^{\infty} \hat{a}_k \delta(f-\frac{k}{N})$ Here $\hat{a}_k = a_{k \mod N}$

Properties and Formulae: Discrete Fourier Transform

$$\hat{X}[k] = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$$
$$x[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \hat{X}[k]e^{j2\pi nk/N}$$

Properties

- *Linearity*: If $x_1[n] \xrightarrow{DFT} \hat{X}_1[k]$, $x_2[n] \xrightarrow{DFT} \hat{X}_2[k]$ then $ax_1[n] + bx_2[n] \xrightarrow{DFT} a\hat{X}_1[k] + b\hat{X}_2[k]$
- *Time shift*: $x[n-n_0] \xrightarrow{DFT} \hat{X}[k]e^{-\frac{j2\pi n_0}{N}k}$
- *Frequency shift*: $e^{\frac{j2\pi M}{N}n}x[n] \xrightarrow{DFT} \hat{X}[k-M]$
- *Periodic Convolution*: If $x_1[n] \xrightarrow{DFT} \hat{X}_1[k]$, $x_2[n] \xrightarrow{DFT} \hat{X}_2[k]$ then $\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1} x_1[r]x_2[n-r] \xrightarrow{DFT} \hat{X}_1[k]\hat{X}_2[k]$ Here the convolution is periodic convolution.

- *Multiplication*: If $x_1[n] \xrightarrow{DFT} \hat{X}_1[k]$, $x_2[n] \xrightarrow{DFT} \hat{X}_2[k]$ then $x_1[n]x_2[n] \xrightarrow{DFT} \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \hat{X}_1[\ell]\hat{X}_2[n-\ell]$ Here the convolution is periodic convolution.

- *Unitarity*: Let $A_N = (e^{-j2\pi nk/N})_{0\leq n,k\leq N-1}$ be the DFT matrix. Then $A_N A_N^* = A_N^* A_N = I$

- Let $A_N = (e^{-j2\pi nk/N})_{0\leq n,k\geq N-1}$ be the DFT matrix. Then $A_N^4 = I$

- *Fast Fourier Transform*: Suppose $N = 2^K$ for some $K \in \mathbb{N}$. Consider a discrete-time signal $x[n]$ of length N. Let $x_e[n] = x[2n], n = 0, 1, \ldots, 2^{K-1} - 1$ and $x_o[n] = x[2n+1], n = 0, 1, \ldots, 2^{K-1} - 1$. Then
  $X[k] = \sqrt{\frac{1}{2}}(X_e[k] + X_o[k]e^{-2\pi j \frac{k}{N}})$ for $k = 0, \ldots, \frac{N}{2} - 1$,
  $X[k] = \sqrt{\frac{1}{2}}\left(X_e[k - \frac{N}{2}] + X_o[k - \frac{N}{2}]e^{-2\pi j \frac{k}{N}}\right)$ for $k = \frac{N}{2}, \ldots, N-1$

Properties and Formulae: Laplace Transform

$$\hat{X}(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$x(t) = \frac{1}{j2\pi}\int_{Re(s)=\sigma,\uparrow} \hat{X}(s)e^{st}ds$$

Properties

- *Linearity*: $ax_1(t) + bx_2(t) \xrightarrow{LT} a\hat{X}_1(s) + b\hat{X}_2(s)$, ROC $\supset R_1 \cap R_2$

- *Time Shifting*: $x(t - t_0) \xrightarrow{LT} e^{-st_0}\hat{X}(s)$, ROC $= R$

- *Shifting in s-Domain*: $e^{s_0 t}x(t) \xrightarrow{LT} \hat{X}(s - s_0)$, ROC $= \{s + s_0 : s \in R\}$

- *Time scaling*: $x(at) \xrightarrow{LT} \frac{1}{|a|}\hat{X}(\frac{s}{a})$, ROC $= \{sa : s \in R\}$

- *Conjugation*: $x^*(t) \xrightarrow{LT} \hat{X}^*(s^*)$, ROC $= R$

- *Convolution*: $x_1(t) * x_2(t) \xrightarrow{LT} \hat{X}_1(s)\hat{X}_2(s)$, ROC $\supset R_1 \cap R_2$

- *Differentiation in Time Domain*: $\frac{d}{dt}x(t) \xrightarrow{LT} s\hat{X}(s)$, ROC $\supset R$

- *Differentiation in s-Domain*: $-tx(t) \xrightarrow{LT} \frac{d}{ds}\hat{X}(s)$, ROC $= R$

- *Integration in Time Domain*: $\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{LT} \frac{1}{s}\hat{X}(s)$, ROC $\supset R \cap \{s : \text{Re}(s) > 0\}$

- *Initial Value Theorem*: If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$ then $\lim_{t\to 0+} x(t) = \lim_{s\to+\infty} s\hat{X}(s)$

- *Final Value Theorem*: If $x(t) = 0$ for $t < 0$ and $\lim_{t\to+\infty} x(t)$ exists and is finite then $\lim_{t\to+\infty} x(t) = \lim_{s\to 0} s\hat{X}(s)$

Some common transforms

- $\delta(t) \xrightarrow{LT} 1,$ ROC: All s
- $u(t) \xrightarrow{LT} \frac{1}{s},$ ROC: $\text{Re}(s) > 0$
- $-u(-t) \xrightarrow{LT} \frac{1}{s},$ ROC: $\text{Re}(s) < 0$
- $\frac{t^{n-1}}{(n-1)!}u(t) \xrightarrow{LT} \frac{1}{s^n},$ ROC: $\text{Re}(s) > 0$
- $-\frac{t^{n-1}}{(n-1)!}u(-t) \xrightarrow{LT} \frac{1}{s^n},$ ROC: $\text{Re}(s) < 0$
- $e^{-\alpha t}u(t) \xrightarrow{LT} \frac{1}{s+\alpha},$ ROC: $\text{Re}(s) > -\text{Re}(\alpha)$
- $-e^{-\alpha t}u(-t) \xrightarrow{LT} \frac{1}{s+\alpha},$ ROC: $\text{Re}(s) < -\text{Re}(\alpha)$
- $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t) \xrightarrow{LT} \frac{1}{(s+\alpha)^n},$ ROC: $\text{Re}(s) > -\text{Re}(\alpha)$
- $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t) \xrightarrow{LT} \frac{1}{(s+\alpha)^n},$ ROC: $\text{Re}(s) < -\text{Re}(\alpha)$
- $\delta(t-T) \xrightarrow{LT} e^{-sT},$ ROC: All s
- $u(t)\cos 2\pi f_0 t \xrightarrow{LT} \frac{s}{s^2+(2\pi f_0)^2},$ ROC: $\text{Re}(s) > 0$
- $u(t)\sin 2\pi f_0 t \xrightarrow{LT} \frac{2\pi f_0}{s^2+(2\pi f_0)^2},$ ROC: $\text{Re}(s) > 0$
- $u(t)e^{-\alpha t}\cos 2\pi f_0 t \xrightarrow{LT} \frac{s+\alpha}{(s+\alpha)^2+(2\pi f_0)^2},$ ROC: $\text{Re}(s) > -\text{Re}(\alpha)$
- $u(t)e^{-\alpha t}\sin 2\pi f_0 t \xrightarrow{LT} \frac{2\pi f_0}{(s+\alpha)^2+(2\pi f_0)^2},$ ROC: $\text{Re}(s) > -\text{Re}(\alpha)$
- $\frac{d^n}{dt^n}\delta(t) \xrightarrow{LT} s^n,$ ROC: All s
- $\underbrace{u(t) * \cdots * u(t)}_{n \text{ times}} \xrightarrow{LT} \frac{1}{s^n},$ ROC: $\text{Re}(s) > 0$

Properties and Formulae: Z Transform

$$\hat{X}(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{j2\pi}\oint_{|Z|=r} \hat{X}(z)z^{n-1}dz$$

Properties

- *Linearity*: $ax_1[n] + bx_2[n] \xrightarrow{ZT} a\hat{X}_1(z) + b\hat{X}_2(z),$ ROC $\supset R_1 \cap R_2$
- *Time Shifting*: $x[n-n_0] \xrightarrow{ZT} z^{-n_0}\hat{X}(z),$ ROC $= R$ or $R \cup \{0\}$ or $R \setminus \{0\}$

- *Scaling in z-Domain*: $a^n x[n] \xrightarrow{ZT} \hat{X}(a^{-1}z)$,   ROC $= \{za : z \in R\}$
- *Time Reversal*: $x[-n] \xrightarrow{ZT} \hat{X}(z^{-1})$,   ROC $= \left\{z^{-1} : z \in R\right\}$
- *Time expansion*: $x_{(k)}[n] := \begin{cases} x[r], & \text{if } n = rk \text{ for some integer } r \\ 0, & \text{otherwise} \end{cases} \xrightarrow{ZT} \hat{X}(z^k)$,   ROC $= \left\{z^{1/k} : z \in R\right\}$
- *Conjugation*: $x^*[n] \xrightarrow{ZT} \hat{X}^*(z^*)$,   ROC $= R$
- *Convolution*: $x_1[n] * x_2[n] \xrightarrow{ZT} \hat{X}_1(z)\hat{X}_2(z)$,   ROC $\supset R_1 \cap R_2$
- *Accumulation*: $\sum_{k=-\infty}^{n} x[k] \xrightarrow{ZT} \frac{1}{1-z^{-1}}\hat{X}(z)$,   ROC $\supset R \cap \{z : |z| > 1\}$
- *Differentiation in z-Domain*: $nx[n] \xrightarrow{ZT} -z\frac{d}{dz}\hat{X}(z)$,   ROC $= R$
- *Initial Value theorem*: If $x[n] = 0$ for $n < 0$ then $x[0] = \lim_{z\to\infty} \hat{X}(z)$

Some common transforms

- $\delta[n] \xrightarrow{ZT} 1$,   ROC: All z
- $u[n] \xrightarrow{ZT} \frac{1}{1-z^{-1}}$,   ROC: $|z| > 1$
- $-u[-n-1] \xrightarrow{ZT} \frac{1}{1-z^{-1}}$,   ROC: $|z| < 1$
- $\delta[n-m] \xrightarrow{ZT} z^{-m}$,   ROC: All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
- $\alpha^n u[n] \xrightarrow{ZT} \frac{1}{1-\alpha z^{-1}}$,   ROC: $|z| > |\alpha|$
- $-\alpha^n u[-n-1] \xrightarrow{ZT} \frac{1}{1-\alpha z^{-1}}$,   ROC: $|z| < |\alpha|$
- $n\alpha^n u[n] \xrightarrow{ZT} \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$,   ROC: $|z| > |\alpha|$
- $-n\alpha^n u[-n-1] \xrightarrow{ZT} \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$,   ROC: $|z| < |\alpha|$
- $n^2\alpha^n u[n] \xrightarrow{ZT} \frac{\alpha z^{-1}(1+\alpha z^{-1})}{(1-\alpha z^{-1})^3}$,   ROC: $|z| > |\alpha|$
- $-n^2\alpha^n u[-n-1] \xrightarrow{ZT} \frac{\alpha z^{-1}(1+\alpha z^{-1})}{(1-\alpha z^{-1})^3}$,   ROC: $|z| < |\alpha|$
- $\binom{n+m-1}{m-1} a^n u[n] \xrightarrow{ZT} \frac{1}{(1-az^{-1})^m}$, for positive integer m,   ROC: $|z| > |\alpha|$
- $(-1)^m \binom{-n-1}{m-1} a^n u[-n-m] \xrightarrow{ZT} \frac{1}{(1-az^{-1})^m}$, for positive integer m,   ROC: $|z| < |\alpha|$