

MIEG 2051/ESTR 2360
Signals and Systems
Problem Set 5

Due: Oct 24, 2025 (11:30 P.M.)

1. Calculate the inverse DTFT of $\hat{X}(f)$ defined as

$$\hat{X}(f) = \frac{1}{(1 - ae^{-j2\pi f})^2}$$

where $|a| < 1$.

2. Let

$$(x[0], x[1], x[2], \dots, x[2N-2], x[2N-1]) \xleftrightarrow{DFT} (X[0], X[1], \dots, X[2N-1]).$$

Let $y[n] = x[2n], n = 0, \dots, N-1$ be the even part of $x[n]$.

$$(y[0], y[1], \dots, y[N-2], y[N-1]) \xleftrightarrow{DFT} (Y[0], Y[1], \dots, Y[N-1]).$$

Let $z[n] = x[2n+1], n = 0, \dots, N-1$ be the odd part of $x[n]$.

$$(z[0], z[1], \dots, z[N-1], z[N-1]) \xleftrightarrow{DFT} (Z[0], Z[1], \dots, Z[N-1]).$$

Note: For $k \geq N$, $Y(k)$ is defined as $Y(k-N)$ (its periodic extension), and similarly for $Z(k)$.

Express $X(k)$ as a linear combination of $Y(k)$ and $Z(k)$, for $k = 0, \dots, 2N-1$.

3. Compute the DFT of the following vector

$$[1, -1, 2, -2, -1, 1, -2, 2].$$

Note that $e^{j\frac{2\pi}{8}} = \frac{1}{\sqrt{2}}(1 + j)$.

(Hint: Use the ideas of splitting into even and odd recursively, as in the previous question, to simplify your computations.)

4. The DTFT of a sequence $x[n]$ is $\hat{X}(f)$. Since $\hat{X}(f)$ is periodic we can compute a Fourier series representation for it as $X(f) = \sum_k a_k e^{j2\pi f k}$. Then compute the relation between the sequences a_k and $x[n]$.
5. Let $\mathbf{x} = [x(0) \cdots x(N-1)]^T$ be a discrete sequence. Let $\mathbf{X} = [X(0) \cdots X(N-1)]^T$ be its discrete Fourier transform.
 - (a) If $\mathbf{X} = A\mathbf{x}$, where A is a $N \times N$ matrix; express $A \equiv [a_{kn}]$ in terms of $\omega_N = e^{-\frac{j2\pi}{N}}$.
 - (b) Prove that $AA^\dagger = I_N$, where $A^\dagger$ - the conjugate transpose; I_N - the identity matrix.
 - (c) Let $\mathbf{y} = AA\mathbf{x}$, then compute $y(i), 0 \leq i \leq N-1$.
 - (d) Prove that $A^4 = I_N$ where I_N is the identity matrix.
6. *Heisenberg's inequality - Compulsory for Elite students*

The support size of a discrete sequence is the number of nonzero elements of it. Let $\mathbf{x} = [x(0) \cdots x(N-1)]^T$ be a discrete sequence, which has support size N_T . Let $\mathbf{X} = [X(0) \cdots X(N-1)]^T$ be its discrete Fourier Transform with support size N_F . Heisenberg's inequality assures that

$$N_T \cdot N_F \geq N$$

- (i) Let $\mathbf{x}' = [x(i_1) \cdots x(i_{N_T})]$ be the nonzero part of $\mathbf{x}$. Similar to question 5, consider the matrix A of size $N \times N_T$, where $\mathbf{X} = A\mathbf{x}'$ and express $A \equiv [a_{kn}]$ in terms of $\omega_N = e^{-\frac{j2\pi}{N}}$.
- (ii) Now consider any N_T consecutive elements of $\mathbf{X}$, e.g. $\mathbf{X}'$. Express $\mathbf{X}' = A'\mathbf{x}'$, where A' is a $N_T \times N_T$ matrix. Show that A' has a nonzero determinant.
Hint: determinant of A' equals to some non-zero constant times the determinant of some Vandermonde matrix V of the following form

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^{n-1} \end{bmatrix}, \quad \det(V) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i).$$

- (iii) Use a counting argument to show that $N_T \cdot N_F \geq N$.
Hint: Not only the previous part holds for any N_T consecutive elements of $\mathbf{X}$ but also it holds for N_T cyclic consecutive elements of $\mathbf{X}$, in other words, when $B > N - N_T$ the previous part holds for the elements

$$[X(B), X(B+1), \dots, X(N-1), X(0), \dots, X(N_T - N + B - 1)].$$