The Capacity Region for Classes of Sum-Broadcast Channels

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Abstract

We define a class of *primary*-broadcast channels, which includes some of the previously studied classes such as less-noisy and semi-deterministic broadcast channels. We establish the capacity region for the "sum" of primary-broadcast channels. This is achieved by showing that an auxiliary receiver outer bound, previously introduced by some of the authors, matches Marton's inner bound.

Index Terms

Shannon theory, outer bound, broadcast channel, auxiliary receiver

I. Introduction

As depicted in Figure 1, a two-receiver discrete-memoryless broadcast channel $(T_{YZ|X}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a mathematical model in which the sender aims to communicate a private message M_1 to receiver Y, a private message M_2 to receiver Z, and a common message M_0 to both receivers (see [8, Chapters 5, 8] for detailed definitions). This model was first proposed by Cover [4]. However, a computable characterization of the capacity region remains a fundamental open problem in the field of network information theory. Let $\mathcal{C}(T)$ denote the capacity region for the broadcast channel $T_{YZ|X}$.

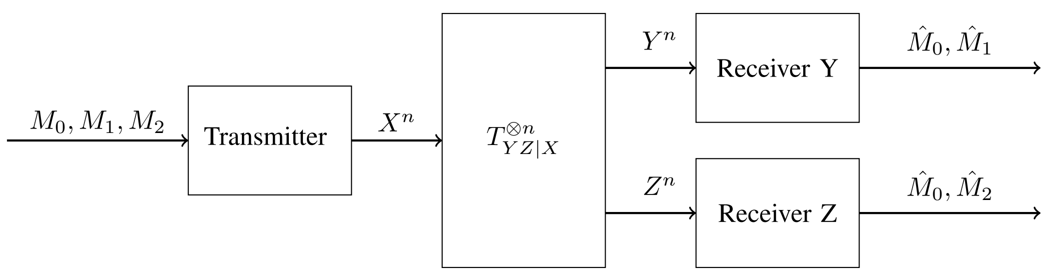

Fig. 1. Two-receiver broadcast communication system.

Theorem 1. *(Marton's inner bound), [15] The union of non-negative rate triples* (R_0, R_1, R_2) *satisfying the constraints*

$$\begin{aligned} R_0 &\leq \min\{I(W;Y), I(W;Z)\}, \\ R_0 + R_1 &\leq I(U,W;Y), \\ R_0 + R_2 &\leq I(V,W;Z), \\ R_0 + R_1 + R_2 &\leq \min\{I(W;Y), I(W;Z)\} + I(U;Y|W) + I(V;Z|W) - I(U;V|W), \end{aligned}$$

for any triples of random variables (U,V,W) *such that* $(U,V,W) \multimap X \multimap (Y,Z)$ *is achievable for a broadcast channel* $T(y,z|x)$*. We denote this region as* $\mathcal{M}(T)$*.*

Marton's inner bound [15] is the best-known achievable region for two-receiver broadcast channels. Numerous studies have demonstrated that Marton's inner bound matches the capacity region for certain classes of broadcast channels. Notable examples include results in [1], [2], [5], [9]–[11], [17]–[20]. In several cases where Marton's inner bound matches the capacity region, it also coincides with the following outer bound:

Theorem 2. *(UVW-Outer Bound), [16] If a rate triple* (R_0, R_1, R_2) *is achievable for a broadcast channel* $T(y,z|x)$*, then it must satisfy the inequalities*

$$R_0 \leq \min\{I(W;Y), I(W;Z)\}, \tag{1a}$$

$$R_0 + R_1 \leq \min\{I(W;Y), I(W;Z)\} + I(U;Y|W), \tag{1b}$$

$$R_0+R_2 \leq \min\{I(W;Y), I(W;Z)\} + I(V;Z|W), \tag{1c}$$
$$R_0+R_1+R_2 \leq \min\{I(W;Y), I(W;Z)\} + I(U;Y|W) + I(X;Z|U,W), \tag{1d}$$
$$R_0+R_1+R_2 \leq \min\{I(W;Y), I(W;Z)\} + I(V;Z|W) + I(X;Y|V,W), \tag{1e}$$

for some pmf $p(u)p(v)p(w|u,v)p(x|w,u,v)$. *Further, it suffices to consider* (U,V,W) *satisfying* $|\mathcal{W}| \leq |\mathcal{X}|+5, |\mathcal{U}| \leq |\mathcal{X}|+1, |\mathcal{V}| \leq |\mathcal{X}|+1$. *We denote this region as* $\mathcal{O}(T)$.

Remark 1. The UVW outer bound, like Marton's inner bound, matches the capacity region for certain classes of broadcast channels, [1], [2], [5], [9], [10], [17]–[20]. In all these cases, $\mathcal{O}(T) = \mathcal{C}(T) = \mathcal{M}(T)$.

The capacity region of a product and sum broadcast channels was studied by El Gamal in [6]. In the same paper, the capacity region of a sum (and product) of reversely degraded broadcast channels was established. In [11], a class of product broadcast channels was identified for which Marton's inner bound (Theorem 1) matches the capacity region, while the UVW outer bound evaluates to a strictly larger region, specifically $\mathcal{O}(T) \supsetneq \mathcal{C}(T) = \mathcal{M}(T)$. The authors demonstrated that the two-letter Marton's inner bound is the capacity region for both reversely more-capable product broadcast channels and reversely semi-deterministic product broadcast channels. There, the authors established an outer bound that hinged on the product nature of the underlying broadcast channel. Consequently, progress on the capacity region of classes of sum-broadcast channels had eluded the previous approach.

In [13], the following outer bound, using auxiliary receivers, was presented to the capacity region of the broadcast channel. This outer bound is at least as good as the UVW outer bound and the tailored outer bound for the product broadcast channels developed in [11].

Theorem 3 (Theorem 8, [13])**.** *Given a broadcast channel* $T_{YZ|X}$, *any achievable non-negative rate triple* (R_0,R_1,R_2) *must satisfy the following constraints*

$$R_0 \leq \min\{I(W^\dagger;G) + I(W;Y|G), I(W^\dagger;Z|K) + I(W;K)\}, \tag{2a}$$
$$R_0+R_1 \leq I(U^\dagger,W^\dagger;G) + I(U,W;Y|G), \tag{2b}$$
$$R_0+R_1 \leq I(W^\dagger;Z|K) + I(W,G;K) + I(U^\dagger;G|W^\dagger,K) + I(U;Y|W,G), \tag{2c}$$
$$R_0+R_2 \leq I(W^\dagger,K;G) + I(W;Y|G) + I(V^\dagger;Z|W^\dagger,K) + I(V;K|W,G), \tag{2d}$$
$$R_0+R_2 \leq I(V,W;K) + I(V^\dagger,W^\dagger;Z|K), \tag{2e}$$
$$\begin{aligned} R_0+R_1+R_2 \leq{}& \min\{I(W^\dagger,K;G)+I(W;Y|G),\ I(W^\dagger;Z|K)+I(W,G;K)\} \\ &+ I(U;Y|W,G) + I(X;K|U,W,G) \\ &+ \min\big\{I(U^\dagger;G|W^\dagger,K) + I(X;Z|U^\dagger,W^\dagger,K),\ I(V^\dagger;Z|W^\dagger,K) + I(X;G|V^\dagger,W^\dagger,K)\big\}, \end{aligned} \tag{2f}$$
$$\begin{aligned} R_0+R_1+R_2 \leq{}& \min\{I(W^\dagger,K;G)+I(W;Y|G),\ I(W^\dagger;Z|K)+I(W,G;K)\} \\ &+ I(V^\dagger;Z|W^\dagger,K) + I(X;G|V^\dagger,W^\dagger,K) \\ &+ \min\big\{I(U;Y|W,G) + I(X;K|U,W,G),\ I(V;K|W,G) + I(X;Y|V,W,G)\big\}, \end{aligned} \tag{2g}$$

for any $T_{G,K|X,Y,Z}$ *and some* $p(w,v,u|x)p(w^\dagger,v^\dagger,u^\dagger|x)p(x)$ *satisfying* $|\mathcal{W}^\dagger|, |\mathcal{W}| \leq |\mathcal{X}|+7$, $|\mathcal{U}^\dagger|, |\mathcal{V}| \leq |\mathcal{X}|+2$, $|\mathcal{V}^\dagger|, |\mathcal{U}| \leq |\mathcal{X}|+1$. *Let us call this region* $\mathcal{O}_{aux}(T)$.

Remark 2. In Theorem 3, let us set $G=Y$ and let K be a constant random variable. The constraints then become:

$$\begin{aligned} R_0 &\leq \min\{I(W^\dagger;Y), I(W^\dagger;Z)\}, \\ R_0+R_1 &\leq I(U^\dagger,W^\dagger;Y), \\ R_0+R_1 &\leq I(W^\dagger;Z) + I(U^\dagger;Y|W^\dagger), \\ R_0+R_2 &\leq I(W^\dagger;Y) + I(V^\dagger;Z|W^\dagger), \\ R_0+R_2 &\leq I(V^\dagger,W^\dagger;Z), \\ R_0+R_1+R_2 &\leq \min\{I(W^\dagger;Y),\ I(W^\dagger;Z)\} + \min\big\{I(U^\dagger;Y|W^\dagger) + I(X;Z|U^\dagger,W^\dagger),\ I(V^\dagger;Z|W^\dagger) + I(X;Y|V^\dagger,W^\dagger)\big\}, \\ R_0+R_1+R_2 &\leq \min\{I(W^\dagger;Y),\ I(W^\dagger;Z)\} + I(V^\dagger;Z|W^\dagger) + I(X;Y|V^\dagger,W^\dagger). \end{aligned}$$

A moment's reflection reveals that these constraints are equivalent to those in the UVW outer bound. Given that $\mathcal{O}_{aux}(T)$ is the intersection of the region over all possible choices of $T_{G,K|X,Y,Z}$, it is immediate that $\mathcal{O}_{aux}(T) \subseteq \mathcal{O}(T)$.

In this paper, we consider the sum of two broadcast channels. The communication model for sum broadcast channels was first introduced in [6]. In that work, the capacity region for the sum of reversely degraded broadcast channels was established. However, the capacity regions for sums of other classes of broadcast channels remain unknown. In this work, we characterize the sum of a broader family of broadcast channels. Specifically, we demonstrate that Marton's inner bound is tight for sum broadcast channels when the component channels are less noisy or semi-deterministic. The proof utilizes the auxiliary receiver outer bound [13], highlighting the efficacy of this recent converse technique in yielding tight characterizations of channel capacity. The auxiliary receiver approach has also been successfully applied to other problems in network information theory, such as the relay channel and the interference channel (see, e.g., [3], [7], [12], [13], [21]).

A. *Preliminaries*

Throughout this paper, $\bar{\alpha}$ denotes $1-\alpha$. All logarithms in this paper are in base two. The binary entropy function is defined as

$$H_2(q) = -q\log(q) - (1-q)\log(1-q), \qquad \forall q \in [0,1].$$

The following equation is useful in our arguments: for any $\lambda, \alpha, \beta \geq 0$

$$\max_{q\in[0,1]} \lambda H_2(q) + q\alpha + \bar{q}\beta = \lambda \log_2\left(2^{\alpha/\lambda} + 2^{\beta/\lambda}\right). \tag{3}$$

Let $\sqcup$ denote the disjoint union of two sets. Given two broadcast channels T_a and T_b, whose input and output sets satisfy $\mathcal{X}_a \cap \mathcal{X}_b = \emptyset$, $\mathcal{Y}_a \cap \mathcal{Y}_b = \emptyset$, $\mathcal{Z}_a \cap \mathcal{Z}_b = \emptyset$, we make the following definition.

Definition 1. A broadcast channel T is termed a sum of two broadcast channels T_a and T_b, denoted by $T = T_a \oplus T_b$, if its input and output sets are defined as:

$$\mathcal{X} = \mathcal{X}_a \sqcup \mathcal{X}_b, \quad \mathcal{Y} = \mathcal{Y}_a \sqcup \mathcal{Y}_b, \quad \mathcal{Z} = \mathcal{Z}_a \sqcup \mathcal{Z}_b,$$

and the channel transition probabilities are given by:

$$T(y,z|x) = \begin{cases} T_a(y,z|x), & (x,y,z) \in \mathcal{X}_a \times \mathcal{Y}_a \times \mathcal{Z}_a, \\ T_b(y,z|x), & (x,y,z) \in \mathcal{X}_b \times \mathcal{Y}_b \times \mathcal{Z}_b, \\ 0, & o.w.. \end{cases}$$

The (convex) capacity region of a broadcast channel $T_{YZ|X}$ is completely characterized by supporting hyperplanes of the form $\lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2$ for $\lambda_0 \geq \lambda_1 \geq \lambda_2$, and $\lambda_0 R_0 + \lambda_1 R_2 + \lambda_2 R_1$ for $\lambda_0 \geq \lambda_1 \geq \lambda_2$. This is because if (R_0, R_1, R_2) is achievable, so is $(0, R_0 + R_1, R_2)$ and $(0, R_1, R_0 + R_2)$. Assume $\lambda_0 < \max(\lambda_1, \lambda_2)$. This means that all the common rate R_0 can be moved to either R_1 or R_2. Then, increasing λ_0 to $\max(\lambda_1, \lambda_2)$ will not change the maximum weighted sum rate.

II. Primary Broadcast Channels

Definition 2. We say that a broadcast channel $T_{YZ|X}$ belongs to the *primary class* $\mathcal{P}$ if for any $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq 0$, as well as $\alpha \in [0,1]$, the following two inequalities hold:

$$SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) \geq \min\{SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T), SR_{OB,1n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T)\}, \tag{4a}$$

$$SR_{IB,2}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) \geq \min\{SR_{OB,2m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T), SR_{OB,2n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T)\}, \tag{4b}$$

where

$$SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) := \max_{p(w,u,v,x)} \lambda_0\alpha I(W;Y) + \lambda_0\bar{\alpha} I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(V;Z|W) - \lambda_2 I(U;V|W),$$

$$SR_{IB,2}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) := \max_{p(w,u,v,x)} \lambda_0\alpha I(W;Y) + \lambda_0\bar{\alpha} I(W;Z) + \lambda_1 I(V;Z|W) + \lambda_2 I(U;Y|W) - \lambda_2 I(U;V|W),$$

$$SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) := \max_{p(w,u,x)} \lambda_0\alpha I(W;Y) + \lambda_0\bar{\alpha} I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(X;Z|U,W),$$

$$SR_{OB,1n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) := \max_{p(w,u,v,x)} \lambda_0\alpha I(W;Y) + \lambda_0\bar{\alpha} I(W;Z) + (\lambda_1-\lambda_2) I(U;Y|W) + \lambda_2\big(I(V;Z|W) + I(X;Y|V,W)\big),$$

$$SR_{OB,2m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) := \max_{p(w,v,x)} \lambda_0\alpha I(W;Y) + \lambda_0\bar{\alpha} I(W;Z) + \lambda_1 I(V;Z|W) + \lambda_2 I(X;Y|V,W),$$

$$SR_{OB,2n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) := \max_{p(w,u,v,x)} \lambda_0\alpha I(W;Y) + \lambda_0\bar{\alpha} I(W;Z) + (\lambda_1-\lambda_2) I(V;Z|W) + \lambda_2\big(I(U;Y|W) + I(X;Z|U,W)\big).$$

Remark 3. The definition of a *primary* class $\mathcal{P}$ is inspired by the comparison between weighted sum-rates of Marton's inner bound and UVW outer bound (as is evidenced by Lemma 1.) It is also evident that it is optimal to set $U = X$ (and, $V = X$) in the optimization for $SR_{OB,1n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T)$ (and, respectively, $SR_{OB,2n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T)$).

Lemma 1. *If a broadcast channel $T \in \mathcal{P}$, then Marton's inner bound $\mathcal{M}(T)$ (Theorem 1) and UVW outer bound $\mathcal{O}(T)$(Theorem 2) coincide, yielding the capacity region.*

Proof. It suffices to show that the weighted sum-rate achieved by Marton's inner bound is at least as large as the one yielded by the UVW outer bound.

Let $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq 0$ be given. Consider the weighted sum-rate $\lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2$; the other case $\lambda_0 R_0 + \lambda_1 R_2 + \lambda_2 R_1$ follows similarly. Observe that,

$$\begin{aligned}
&\max_{(R_0,R_1,R_2)\in\mathcal{M}(T)} \lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2 \\
&\quad \geq \max_{p(w,u,v,x)} \lambda_0 \min\{I(W;Y), I(W;Z)\} + \lambda_1 I(U;Y|W) + \lambda_2 I(V;Z|W) - \lambda_2 I(U;V|W) \\
&= \max_{p(w,u,v,x)} \min_{\alpha\in[0,1]} \lambda_0 \alpha I(W;Y) + \lambda_0 \bar{\alpha} I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(V;Z|W) - \lambda_2 I(U;V|W) \\
&\quad \overset{(a)}{=} \min_{\alpha\in[0,1]} \max_{p(w,u,v,x)} \lambda_0 \alpha I(W;Y) + \lambda_0 \bar{\alpha} I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(V;Z|W) - \lambda_2 I(U;V|W) \\
&\quad \overset{(b)}{\geq} \min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T), SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T)\} \\
&\quad \overset{(c)}{\geq} \min_{\alpha\in[0,1]} \max_{p(w,u,v,x)} \lambda_0 \alpha I(W;Y) + \lambda_0 \bar{\alpha} I(W;Z) + (\lambda_1 - \lambda_2) I(U;Y|W) \\
&\qquad + \lambda_2 \min\left\{I(U;Y|W) + I(X;Z|U,W), I(V;Z|W) + I(X;Y|V,W)\right\} \\
&\quad \geq \max_{p(w,u,v,x)} \lambda_0 \min\{I(W;Y), I(W;Z)\} + (\lambda_1 - \lambda_2) I(U;Y|W) \\
&\qquad + \lambda_2 \min\left\{I(U;Y|W) + I(X;Z|U,W), I(V;Z|W) + I(X;Y|V,W)\right\} \\
&\quad \geq \max_{(R_0,R_1,R_2)\in\mathcal{O}(T)} \lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2.
\end{aligned}$$

In the above, the max-min interchange in (a) is established in [11] and follows from a theorem of Telkersen. (b) follows from (4a), and (c) follows by noting that

$$\begin{aligned}
&\min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T), SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T)\} \\
&\quad \geq \max_{p(w,u,v,x)} \lambda_0 \alpha I(W;Y) + \lambda_0 \bar{\alpha} I(W;Z) + (\lambda_1 - \lambda_2) I(U;Y|W) \\
&\qquad + \lambda_2 \min\left\{I(U;Y|W) + I(X;Z|U,W), I(V;Z|W) + I(X;Y|V,W)\right\},
\end{aligned} \tag{5}$$

as $\min\max$ is always at least as large as $\max\min$. □

The next lemma shows that the class of primary broadcast channels contains some of the previously studied ones.

Lemma 2. *It $T_{YZ|X}$ is a broadcast channel where one of the receivers, Y, is less-noisy [14] than Z, then $T_{YZ|X} \in \mathcal{P}$. Similarly, if Y is a deterministic function of X, then again $T_{YZ|X} \in \mathcal{P}$ (this class is called semi-deterministic).*

The proof details found in Appendix A.

Remark 4. Indeed, the class of primary broadcast channels contains larger classes such as essentially less-noisy [17], or even the vector Gaussian broadcast channel. In this work, we go beyond non-primary and non-product broadcast channels and establish their capacity region.

The following is the main theorem in this paper.

Theorem 4. *If a broadcast channel T is a sum of two primary broadcast channels T_a, T_b, such that $T = T_a \oplus T_b$, then $\mathcal{M}(T) = \mathcal{C}(T) = \mathcal{O}_{aux}(T)$.*

Proof. Let $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq 0$. It suffices to show that

$$\max_{(R_0,R_1,R_2)\in\mathcal{M}(T)} \lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2 \geq \max_{(R_0,R_1,R_2)\in\mathcal{O}_{aux}(T)} \lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2. \tag{6}$$

The other case follows similarly (by interchanging the roles of receivers and the corresponding auxiliary variables). Since these two collections of hyperplanes characterize the convex capacity region $\mathcal{C}(T)$, we are done.

Observe that

$$\begin{aligned}
&\max_{(R_0,R_1,R_2)\in\mathcal{M}(T)} \lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2 \\
&\quad \overset{(a)}{=} \min_{\alpha\in[0,1]} \max_{p(w,u,v,x)} \lambda_0 \alpha I(W;Y) + \lambda_0 \bar{\alpha} I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(V;Z|W) - \lambda_2 I(U;V|W) \\
&\quad \overset{(b)}{\geq} \min_{\alpha\in[0,1]} \max_{q} \lambda_0 H_2(q) + q\Bigg(\max_{p(u_a,v_a,w_a,x_a)} \lambda_0 \alpha I(W_a;Y_a) + \lambda_0 \bar{\alpha} I(W_a;Z_a) + \lambda_1 I(U_a;Y_a|W_a) + \lambda_2 I(V_a;Z_a|W_a)
\end{aligned}$$

$$
\begin{aligned}
&- \lambda_2 I(U_a;V_a|W_a)\Big) + \bar{q}\Big(\max_{p(w_b,u_b,v_b,x_b)} \lambda_0\alpha I(W_b;Y_b) + \lambda_0\bar{\alpha} I(W_b;Z_b) + \lambda_1 I(U_b;Y_b|W_b) \\
&+ \lambda_2 I(V_b;Z_b|W_b) - \lambda_2 I(U_b;V_b|W_b)\Big) \qquad (7)\\
&= \min_{\alpha\in[0,1]} \max_q \lambda_0 H_2(q) + q SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a) + \bar{q} SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b) \qquad (8)\\
&\overset{(c)}{=} \min_{\alpha\in[0,1]} \lambda_0 \log\left(2^{\frac{1}{\lambda_0}SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a)} + 2^{\frac{1}{\lambda_0}SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b)}\right). \qquad (9)
\end{aligned}
$$

As before, here (a) follows from the min-max theorem established in [11]. To obtain (b), consider $p(u,v,w,x)$. where $W = (Q,\tilde{W})$ and

$$
\begin{cases}
p(\tilde{W}=w_a, U=u_a, V=v_a, X=x_a|Q=a) = p(w_a,u_a,v_a,x_a),\\
p(\tilde{W}=w_b, U=u_b, V=v_b, X=x_b|Q=b) = p(w_b,u_b,v_b,x_b).
\end{cases}
$$

Since this is a particular choice of a generic $p(w,u,v,x)$, the inequality (b) is immediate. Finally, (c) follows from (3).

We now turn to the outer bound in Theorem 3. The following constraints on the rates correspond to (2a), (2b), (2c), (2f), and (2g).

$$
\begin{aligned}
R_0 &\le \min\{I(W^\dagger;G) + I(W;Y|G), I(W^\dagger;Z|K) + I(W;K)\}, \qquad (10a)\\
R_0 + R_1 &\le I(U^\dagger,W^\dagger;G) + I(U,W;Y|G), \qquad (10b)\\
R_0 + R_1 &\le I(W^\dagger;Z|K) + I(W,G;K) + I(U^\dagger;G|W^\dagger,K) + I(U;Y|W,G), \qquad (10c)\\
R_0 + R_1 + R_2 &\le \min\{I(W^\dagger,K;G) + I(W;Y|G),\ I(W^\dagger;Z|K) + I(W,G;K)\}\\
&\quad + I(U;Y|W,G) + I(X;K|U,W,G)\\
&\quad + \min\left\{I(U^\dagger;G|W^\dagger,K) + I(X;Z|U^\dagger,W^\dagger,K),\ I(V^\dagger;Z|W^\dagger,K) + I(X;G|V^\dagger,W^\dagger,K)\right\}, \qquad (10d)\\
R_0 + R_1 + R_2 &\le \min\{I(W^\dagger,K;G) + I(W;Y|G),\ I(W^\dagger;Z|K) + I(W,G;K)\}\\
&\quad + I(V^\dagger;Z|W^\dagger,K) + I(X;G|V^\dagger,W^\dagger,K)\\
&\quad + \min\left\{I(U;Y|W,G) + I(X;K|U,W,G),\ I(V;K|W,G) + I(X;Y|V,W,G)\right\}, \qquad (10e)
\end{aligned}
$$

Let $Q \in \{a,b\}$ denote the component of the sum channel as before. Define the auxiliary receiver $T_{G,K|X,Y,Z}$ as follows: we define $(\bar{G},\bar{K})$ such that $\bar{G} = \emptyset$ and $\bar{K} = Z_a$ when $Q = a$, while $\bar{G} = Y_b$ and $\bar{K} = \emptyset$ when $Q = b$. Using an abuse of notation, let $\mathbb{P}(Q=a) = q$ (Note: This need not be the same q used in the inner bound above.). Setting $G = (Q,\bar{G}), K = (Q,\bar{K})$, the constraints (10a)-(10e) becomes, respectively,

$$
\begin{aligned}
R_0 &\le \min\big\{I(W^\dagger;Q) + \bar{q}I(W^\dagger;Y_b|Q=b) + qI(W;Y_a|Q=a),\\
&\qquad \bar{q}I(W^\dagger;Z_b|Q=b) + I(W;Q) + qI(W;Z_a|Q=a)\big\}, \qquad (11a)\\
R_0 + R_1 &\le I(U^\dagger,W^\dagger;Q) + \bar{q}I(U^\dagger,W^\dagger;Y_b|Q=b) + qI(U,W;Y_a|Q=a), \qquad (11b)\\
R_0 + R_1 &\le \bar{q}I(W^\dagger;Z_b|Q=b) + H(Q) + qI(W;Z_a|Q=a) + \bar{q}I(U^\dagger;Y_b|W^\dagger,Q=b) + qI(U;Y_a|W,Q=a), \qquad (11c)\\
R_0 + R_1 + R_2 &\le H(Q) + \min\{\bar{q}I(W^\dagger;Y_b|Q=b) + qI(W;Y_a|Q=a),\ \bar{q}I(W^\dagger;Z_b|Q=b) + qI(W;Z_a|Q=a)\}\\
&\quad + qI(U;Y_a|W,Q=a) + qI(X_a;Z_a|U,W,Q=a)\\
&\quad + \min\big\{\bar{q}I(U^\dagger;Y_b|W^\dagger,Q=b) + \bar{q}I(X_b;Z_b|U^\dagger,W^\dagger,Q=b),\\
&\qquad \bar{q}I(V^\dagger;Z_b|W^\dagger,Q=b) + \bar{q}I(X_b;Y_b|V^\dagger,W^\dagger,Q=b)\big\}, \qquad (11d)\\
R_0 + R_1 + R_2 &\le H(Q) + \min\{\bar{q}I(W^\dagger;Y_b|Q=b) + qI(W;Y_a|Q=a),\ \bar{q}I(W^\dagger;Z_b|Q=b) + qI(W;Z_a|Q=a)\}\\
&\quad + \bar{q}I(V^\dagger;Z_b|W^\dagger,Q=b) + \bar{q}I(X_b;Y_b|V^\dagger,W^\dagger,Q=b)\\
&\quad + \min\big\{qI(U;Y_a|W,Q=a) + qI(X_a;Z_a|U,W,Q=a),\\
&\qquad qI(V;Z_a|W,Q=a) + qI(X_a;Y_a|V,W,Q=a)\big\}, \qquad (11e)
\end{aligned}
$$

We can further bound the terms $I(W;Q)$ and $I(W^\dagger;Q)$ in (11a) from above by $H(Q)$. Thus, for any $\alpha \in [0,1]$ we can write:

$$
\begin{aligned}
R_0 &\le H(Q) + \min\big\{\bar{q}I(W^\dagger;Y_b|Q=b) + qI(W;Y_a|Q=a),\quad \bar{q}I(W^\dagger;Z_b|Q=b) + qI(W;Z_a|Q=a)\big\}\\
&\le H(Q) + \alpha\big(\bar{q}I(W^\dagger;Y_b|Q=b) + qI(W;Y_a|Q=a)\big) + \bar{\alpha}\big(\bar{q}I(W^\dagger;Z_b|Q=b) + qI(W;Z_a|Q=a)\big), \qquad (12)
\end{aligned}
$$

Similarly, we can weaken (11b) as follows:

$$
R_0 + R_1 \le H(Q) + \bar{q}I(U^\dagger,W^\dagger;Y_b|Q=b) + qI(U,W;Y_a|Q=a), \qquad (13)
$$

Finally, let us weaken (11d) as follows:

$$\begin{aligned}R_0+R_1+R_2 \le H(Q) &+ \alpha\big(\bar{q}I(W^\dagger;Y_b|Q=b)+qI(W;Y_a|Q=a)\big)+\bar{\alpha}\big(\bar{q}I(W^\dagger;Z_b|Q=b)+qI(W;Z_a|Q=a)\big)\\ &+qI(U;Y_a|W,Q=a)+qI(X_a;Z_a|U,W,Q=a)\\ &+\bar{q}\min\big\{I(U^\dagger;Y_b|W^\dagger,Q=b)+I(X_b;Z_b|U^\dagger,W^\dagger,Q=b),\\ &\qquad I(V^\dagger;Z_b|W^\dagger,Q=b)+I(X_b;Y_b|V^\dagger,W^\dagger,Q=b)\big\}.\end{aligned} \tag{14}$$

Similarly, we can weaken (11e) as

$$\begin{aligned}R_0+R_1+R_2 \le H(Q) &+ \alpha\big(\bar{q}I(W^\dagger;Y_b|Q=b)+qI(W;Y_a|Q=a)\big)+\bar{\alpha}\big(\bar{q}I(W^\dagger;Z_b|Q=b)+qI(W;Z_a|Q=a)\big)\\ &+\bar{q}I(V^\dagger;Z_b|W^\dagger,Q=b)+\bar{q}I(X_b;Y_b|V^\dagger,W^\dagger,Q=b)\\ &+q\min\big\{I(U;Y_a|W,Q=a)+I(X_a;Z_a|U,W,Q=a),\\ &\qquad I(V;Z_a|W,Q=a)+I(X_a;Y_a|V,W,Q=a)\big\}.\end{aligned} \tag{15}$$

Multiplying (12) by $(\lambda_0-\lambda_1)$, (13) by $(\lambda_1-\lambda_2)\alpha$, (11c) by $(\lambda_1-\lambda_2)\bar{\alpha}$, and (14) by λ_2 and summing them up (along with some elementary bounding yields)

$$\begin{aligned}&\lambda_0R_0+\lambda_1R_1+\lambda_2R_2\\ &\le \lambda_0H(Q)+q\Big(\lambda_0\alpha I(W;Y_a|Q=a)+\lambda_0\bar{\alpha}I(W;Z_a|Q=a)+\lambda_1I(U;Y_a|W,Q=a)+\lambda_2I(X_a;Z_a|U,W,Q=a)\Big)\\ &\quad+\bar{q}\Big(\lambda_0\alpha I(W^\dagger;Y_b|Q=b)+\lambda_0\bar{\alpha}I(W^\dagger;Z_b|Q=b)+(\lambda_1-\lambda_2)I(U^\dagger;Y_b|W^\dagger,Q=b)\Big)\\ &\quad+\bar{q}\lambda_2\min\big\{I(U^\dagger;Y_b|W^\dagger,Q=b)+I(X_b;Z_b|U^\dagger,W^\dagger,Q=b),\ I(V^\dagger;Z_b|W^\dagger,Q=b)+I(X_b;Y_b|V^\dagger,W^\dagger,Q=b)\big\},\end{aligned}$$

for some $p_Qp(u^\dagger,v^\dagger,w^\dagger,x_b|Q=b)p(u,w,x_a|Q=a)$, and any $\alpha\in[0,1]$. Therefore,

$$\max_{(R_0,R_1,R_2)\in\mathcal{O}_{aux}(T)}\lambda_0R_0+\lambda_1R_1+\lambda_2R_2 \tag{16}$$

$$\le\min_{\alpha\in[0,1]}\max_q\lambda_0H_2(q)+q\Big(\max_{p(u,w,x_a)}\lambda_0\alpha I(W;Y_a)+\lambda_0\bar{\alpha}I(W;Z_a)+\lambda_1I(U;Y_a|W)+\lambda_2I(X_a;Z_a|U,W)\Big) \tag{17}$$

$$\begin{aligned}&+\bar{q}\Big(\max_{p(u^\dagger,v^\dagger,w^\dagger,x_b)}\lambda_0\alpha I(W^\dagger;Y_b)+\lambda_0\bar{\alpha}I(W^\dagger;Z_b)+(\lambda_1-\lambda_2)I(U^\dagger;Y_b|W^\dagger)\\ &\qquad+\lambda_2\min\big\{I(U^\dagger;Y_b|W^\dagger)+I(X_b;Z_b|U^\dagger,W^\dagger),\ I(V^\dagger;Z_b|W^\dagger)+I(X_b;Y_b|V^\dagger,W^\dagger)\big\}\Big)\end{aligned} \tag{18}$$

$$\le\min_{\alpha\in[0,1]}\max_q\lambda_0H_2(q)+qSR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a)+\bar{q}\min\{SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b),SR_{OB,1n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b)\} \tag{19}$$

$$=\min_{\alpha\in[0,1]}\lambda_0\log\left(2^{\frac{1}{\lambda_0}SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a)}+2^{\frac{1}{\lambda_0}\min\{SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b),SR_{OB,1n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b)\}}\right), \tag{20}$$

where (19) follows similarly to (5), and (20) follows from (3).

Similarly, by multiplying (12) by $(\lambda_0-\lambda_1)$, (13) by $(\lambda_1-\lambda_2)\alpha$, (11c) by $(\lambda_1-\lambda_2)\bar{\alpha}$, and (15) by λ_2 and summing them up, and proceeding in exactly the same way as above yields:

$$\begin{aligned}&\max_{(R_0,R_1,R_2)\in\mathcal{O}_{aux}(T)}\lambda_0R_0+\lambda_1R_1+\lambda_2R_2\\ &\quad\le\min_{\alpha\in[0,1]}\lambda_0\log\left(2^{\frac{1}{\lambda_0}SR_{OB,1n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b)}+2^{\frac{1}{\lambda_0}\min\{SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a),SR_{OB,1n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a)\}}\right).\end{aligned} \tag{21}$$

Let $Q\in\{a,b\}$ denote the component of the sum channel as before. Define the auxiliary receiver $T_{G,K|X,Y,Z}$ as follows: define $(\bar{G},\bar{K})$ such that $\bar{G}=\emptyset$ and $\bar{K}=Z_b$ when $Q=b$, while $\bar{G}=Y_a$ and $\bar{K}=\emptyset$ when $Q=a$. Setting $G=(Q,\bar{G}),K=(Q,\bar{K})$, the constraints (10a)-(10e) becomes, respectively,

$$R_0\le\min\{I(W^\dagger;Q)+qI(W^\dagger;Y_a|Q=a)+\bar{q}I(W;Y_b|Q=b), \tag{22a}$$

$$qI(W^\dagger;Z_a|Q=a)+I(W;Q)+I(W;Z_b|Q=b)\}, \tag{22b}$$

$$R_0+R_1\le I(U^\dagger,W^\dagger;Q)+qI(U^\dagger,W^\dagger;Y_a|Q=a)+\bar{q}I(U,W;Y_b|Q=b), \tag{22c}$$

$$R_0+R_1\le H(Q)+qI(W^\dagger;Z_a|Q=a)+\bar{q}I(W;Z_b|Q=b)+qI(U^\dagger;Y_a|W^\dagger,Q=a)+\bar{q}I(U;Y_b|W,Q=b), \tag{22d}$$

$$R_0+R_1+R_2\le H(Q)+\min\{qI(W^\dagger;Y_a|Q=a)+\bar{q}I(W;Y_b|Q=b),\ qI(W^\dagger;Z_a|Q=a)+\bar{q}I(W;Z_b|Q=b)\}$$

$$
\begin{aligned}
&+ \bar{q}I(U;Y_b|W,Q=b) + \bar{q}I(X_b;Z_b|U,W,Q=b)\\
&+ q\min\big\{I(U^\dagger;Y_a|W^\dagger,Q=a) + I(X_a;Z_a|U^\dagger,W^\dagger,Q=a),\\
&\qquad I(V^\dagger;Z_a|W^\dagger,Q=a) + I(X_a;Y_a|V^\dagger,W^\dagger,Q=a)\big\}, \qquad (22e)
\end{aligned}
$$

$$
\begin{aligned}
R_0+R_1+R_2 \le\; & H(Q) + \min\{qI(W^\dagger;Y_a|Q=a) + \bar{q}I(W;Y_b|Q=b),\ qI(W^\dagger;Z_a|Q=a) + \bar{q}I(W;Z_b|Q=b)\}\\
&+ qI(V^\dagger;Z_a|W^\dagger,Q=a) + qI(X_a;Y_a|V^\dagger,W^\dagger,Q=a)\\
&+ \bar{q}\min\big\{I(U;Y_b|W,Q=b) + I(X_b;Z_b|U,W,Q=b),\ I(V;Z_b|W,Q=b) + I(X_b;Y_b|V,W,Q=b)\big\}, \qquad (22f)
\end{aligned}
$$

Take some $\alpha \in [0,1]$ and let us weaken (22b)-(22d) and (22f) as follows:

$$
R_0 \le H(Q) + \alpha\Big(qI(W^\dagger;Y_a|Q=a) + \bar{q}I(W;Y_b|Q=b)\Big) + \bar{\alpha}\Big(qI(W^\dagger;Z_a|Q=a) + I(W;Z_b|Q=b)\Big), \qquad (23a)
$$

$$
R_0+R_1 \le H(Q) + qI(U^\dagger,W^\dagger;Y_a|Q=a) + \bar{q}I(U,W;Y_b|Q=b), \qquad (23b)
$$

$$
R_0+R_1 \le H(Q) + qI(W^\dagger;Z_a|Q=a) + \bar{q}I(W;Z_b|Q=b) + qI(U^\dagger;Y_a|W^\dagger,Q=a) + \bar{q}I(U;Y_b|W,Q=b), \qquad (23c)
$$

$$
\begin{aligned}
R_0+R_1+R_2 \le\; & H(Q) + \alpha\Big(qI(W^\dagger;Y_a|Q=a) + \bar{q}I(W;Y_b|Q=b)\Big) + \bar{\alpha}\Big(qI(W^\dagger;Z_a|Q=a) + I(W;Z_b|Q=b)\Big)\\
&+ qI(V^\dagger;Z_a|W^\dagger,Q=a) + qI(X_a;Y_a|V^\dagger,W^\dagger,Q=a)\\
&+ \bar{q}\min\big\{I(U;Y_b|W,Q=b) + I(X_b;Z_b|U,W,Q=b),\ I(V;Z_b|W,Q=b) + I(X_b;Y_b|V,W,Q=b)\big\}, \qquad (23d)
\end{aligned}
$$

We have $q = \Pr(Q=a)$ and $\bar{q} = \Pr(Q=b)$. Multiplying (23a) by $(\lambda_0-\lambda_1)$, (23b) by $(\lambda_1-\lambda_2)\alpha$, (23c) by $(\lambda_1-\lambda_2)\bar{\alpha}$, and (23d) by λ_2 and summing them up

$$
\begin{aligned}
&\lambda_0R_0 + \lambda_1R_1 + \lambda_2R_2\\
&\le \lambda_0H(Q) + \lambda_0\alpha\Big(qI(W^\dagger;Y_a|Q=a) + \bar{q}I(W;Y_b|Q=b)\Big) + \lambda_0\bar{\alpha}\Big(qI(W^\dagger;Z_a|Q=a) + I(W;Z_b|Q=b)\Big)\\
&\quad + (\lambda_1-\lambda_2)\Big(qI(U^\dagger;Y_a|W^\dagger,Q=a) + \bar{q}I(U;Y_b|W,Q=b)\Big)\\
&\quad + \lambda_2\Big(qI(V^\dagger;Z_a|W^\dagger,Q=a) + qI(X_a;Y_a|V^\dagger,W^\dagger,Q=a)\\
&\qquad + \bar{q}\min\big\{I(U;Y_b|W,Q=b) + I(X_b;Z_b|U,W,Q=b),\ I(V;Z_b|W,Q=b) + I(X_b;Y_b|V,W,Q=b)\big\}\Big)
\end{aligned}
$$

for some $p_Q p(u,v,w,x_b|Q=b)p(u^\dagger,v^\dagger,w^\dagger,x_a|Q=a)$, and any $\alpha \in [0,1]$. Thus,

$$
\begin{aligned}
&\max_{(R_0,R_1,R_2)\in\mathcal{O}_{aux}(T)} \lambda_0R_0 + \lambda_1R_1 + \lambda_2R_2\\
&\le \min_{\alpha\in[0,1]} \max_q \lambda_0H_2(q) + q\Bigg(\max_{p(u^\dagger,v^\dagger,w^\dagger,x_a)} \lambda_0\alpha I(W^\dagger;Y_a) + \lambda_0\bar{\alpha}I(W^\dagger;Z_a) + (\lambda_1-\lambda_2)I(U^\dagger;Y_a|W^\dagger)\\
&\qquad\quad + \lambda_2(I(V^\dagger;Z_a|W^\dagger) + I(X_a;Y_a|V^\dagger,W^\dagger))\Bigg)\\
&\qquad + \bar{q}\Bigg(\max_{p(u,v,w,x_b)} \lambda_0\alpha I(W;Y_b) + \lambda_0\bar{\alpha}I(W;Z_b) + (\lambda_1-\lambda_2)I(U;Y_b|W)\\
&\qquad\quad + \lambda_2\min\big\{I(U;Y_b|W) + I(X_b;Z_b|U,W),\ I(V;Z_b|W) + I(X_b;Y_b|V,W)\big\}\Bigg)\\
&\le \min_{\alpha\in[0,1]} \max_q \lambda_0H_2(q) + qSR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_a) + \bar{q}\min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T_b), SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_b)\}\\
&= \min_{\alpha\in[0,1]} \lambda_0\log\left(2^{\frac{1}{\lambda_0}SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_a)} + 2^{\frac{1}{\lambda_0}\min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T_b),SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_b)\}}\right) \qquad (24)
\end{aligned}
$$

Combining (20), (21) and (24), we obtain

$$
\begin{aligned}
&\max_{(R_0,R_1,R_2)\in\mathcal{O}_{aux}(T)} \lambda_0R_0 + \lambda_1R_1 + \lambda_2R_2\\
&\le \min_{\alpha\in[0,1]} \lambda_0\log\left(2^{\frac{1}{\lambda_0}\min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T_b),SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_b)\}} + 2^{\frac{1}{\lambda_0}\min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T_a),SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_a)\}}\right)\\
&= \min_{\alpha\in[0,1]} \max_q \lambda_0H_2(q) + q\min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T_a), SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_a)\} + \bar{q}\min\{SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1m}(T_b), SR^{\lambda_0,\lambda_1,\lambda_2,\alpha}_{OB,1n}(T_b)\}
\end{aligned}
$$

$$\overset{(a)}{\leq} \min_{\alpha\in[0,1]} \max_{q} \lambda_0 H_2(q) + qSR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a) + \bar{q}SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b)$$
$$\overset{(b)}{\leq} \max_{(R_0,R_1,R_2)\in\mathcal{M}(T)} \lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2.$$

Here (a) holds as T_a and T_b are primary, and (b) holds from (8).

□

Remark 5. The above proof shows that for every $\alpha \in [0,1]$

$$SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a \oplus T_b) = \lambda_0 \log\left(2^{\frac{1}{\lambda_0}SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_a)} + 2^{\frac{1}{\lambda_0}SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T_b)}\right).$$

The following corollary is a direct consequence of Theorem 4 and Lemma 2.

Corollary 1. *For a sum of broadcast channels, if both components belong to one of the following categories: degraded broadcast channels, less-noisy broadcast channels, deterministic broadcast channels, or semi-deterministic broadcast channels, then Marton's inner bound, $\mathcal{M}(T)$, and the auxiliary receiver outer bound $\mathcal{O}_{aux}(T)$ coincide. This result generalizes the previously known result for the case of the sum of two reversely degraded broadcast channels [6].*

III. Summary

We investigate the sum channels of two-receiver broadcast channels. We defined a "primary" class of broadcast channels comprising previously studied classes, such as less-noisy and semi-deterministic. Then, we established the capacity region for a class of sum-broadcast channels whose both components are "primary". We utilized an outer bound using auxiliary receivers to fashion a converse.

The proof techniques employed here suggest that the auxiliary receiver outer bound is also tight for sum-broadcast channels where the components are more-capable comparable. However, this seems to involve a tailored manipulation of the outer bound. Here, our approach was to demonstrate a uniform manipulation for a class of channels.

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APPENDIX

A. The proof for two classes of broadcast channels being "primary"

1) Less Noisy broadcast channels are "primary": Without loss of generality, we assume $Y \overset{L.N.}{\succeq} Z$.

We have

$$\begin{aligned} SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha} &= \max_{p(u,v,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(V;Z|W) - \lambda_2 I(U;V|W) \\ &\overset{(a)}{\geq} \max_{p(w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(X;Y|W) \\ &\geq \max_{p(u,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(X;Y|U,W) \\ &\overset{(b)}{\geq} \max_{p(u,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(X;Z|U,W) \\ &= SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha} \end{aligned}$$

Here (a) follows as we maximize over a smaller set of distributions where $U = X$ and V is a constant random variable, and (b) uses the property of the less-noisy (indeed more-capable suffices) ordering.

Next, we have

$$\begin{aligned} SR_{IB,2}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) &= \max_{p(u,v,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(V;Z|W) + \lambda_2 I(U;Y|W) - \lambda_2 I(U;V|W) \\ &\overset{(a)}{=} \max_{p(v,w,x)} \alpha\lambda_0 I(V,W;Y) + \bar{\alpha}\lambda_0 I(V,W;Z) + \lambda_2 I(X;Y|V,W) \\ &= \max_{p(w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_2 I(X;Y|W) \end{aligned}$$

where (a) follows as moving from (U,V,W) to $\tilde{U} = X, \tilde{W} = (W,V)$, and $\tilde{V}$ constant does not decrease the expression; this follows from $\alpha\lambda_0 I(V,W;Y) \geq \alpha\lambda_0 I(W;Y) + \alpha\lambda_1 I(V;Z|W)$, and $\bar{\alpha}\lambda_0 I(V,W;Z) \geq \bar{\alpha}\lambda_0 I(W;Z) + \bar{\alpha}\lambda_1 I(V;Z|W)$.

Similarly,

$$\begin{aligned} SR_{OB,2m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) &= \max_{p(u,v,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(V;Z|W) + \lambda_2 I(X;Y|V,W) \\ &\overset{(a)}{=} \max_{p(v,w,x)} \alpha\lambda_0 I(V,W;Y) + \bar{\alpha}\lambda_0 I(V,W;Z) + \lambda_2 I(X;Y|V,W) \\ &= \max_{p(w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_2 I(X;Y|W) \end{aligned}$$

where (a) follows as moving (V,W) to where $\tilde{W} = (W,V)$, and $\tilde{V}$ constant does not decrease the expression by a similar argument as above. Therefore, $SR_{IB,2}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T) = SR_{OB,2m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}(T)$, and completes the proof.

2) Semi-deterministic broadcast channels are "primary": Assume, $Z = f(X)$.

$$\begin{aligned} SR_{ib,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha} &= \max_{p(u,v,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(V;Z|W) - \lambda_2 I(U;V|W) \\ &\overset{(a)}{=} \max_{p(u,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 H(Z|UW) \\ &= \max_{p(u,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(U;Y|W) + \lambda_2 I(X;Z|U,W). \end{aligned}$$

Here (a) follows as it is optimal to set $V = Z$. Hence, $SR_{IB,1}^{\lambda_0,\lambda_1,\lambda_2,\alpha} \geq SR_{OB,1m}^{\lambda_0,\lambda_1,\lambda_2,\alpha}$ is satisfied.

Similarly,

$$\begin{aligned} SR_{IB,2}^{\lambda_0,\lambda_1,\lambda_2,\alpha} &= \max_{p(u,v,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 I(V;Z|W) + \lambda_2 I(U;Y|W) - \lambda_2 I(U;V|W) \\ &\overset{(a)}{=} \max_{p(u,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + \lambda_1 H(Z|W) + \lambda_2 I(U;Y|W) - \lambda_2 I(U;Z|W) \\ &= \max_{p(u,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + (\lambda_1 - \lambda_2) H(Z|W) + \lambda_2 I(U;Y|W) + \lambda_2 H(Z|U,W) \\ &\geq \max_{p(u,v,w,x)} \alpha\lambda_0 I(W;Y) + \bar{\alpha}\lambda_0 I(W;Z) + (\lambda_1 - \lambda_2) I(V;Z|W) + \lambda_2 I(U;Y|W) + \lambda_2 I(X;Z|U,W). \end{aligned}$$

Here (a) follows as it is optimal to set $V = Z$. Hence, $SR_{IB,2}^{\lambda_0,\lambda_1,\lambda_2,\alpha} \geq SR_{OB,2n}^{\lambda_0,\lambda_1,\lambda_2,\alpha}$ is satisfied.

The above argument proves that semi-deterministic broadcast channels are "primary".