Interference Channels with Very Weak Interference

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***Abstract*—We derive a genie-based outer bound for the sum rate of discrete memoryless interference channels. We define a class of very weak interference channels and study a sub-class called the binary skewed-Z interference channel. We use the genie-based outer bound to deduce the sum-capacity in a non-trivial regime of parameters for this sub-class.**

I. Introduction

The interference channel is a model for communication of two (or more) pairs of transmitters and receivers over a common medium. Each sender wants to send a private message to its intended receiver and one is interested in characterizing the region of rate-pairs that are simultaneously achievable, *i.e.* the capacity region. The characterization of the capacity region is a classical and fundamental open problem in multi-terminal information theory. For some background on this problem and problem definition, please refer to Chapter 6 in [1].

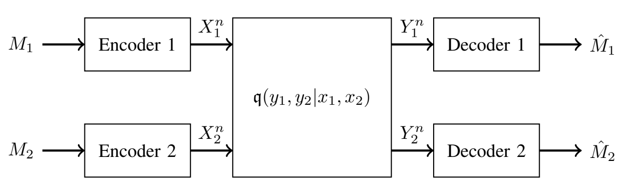

Fig. 1. Discrete memoryless interference channel

A rate pair (R_1, R_2) is said to be achievable if there is a sequence of encoding schemes such that $P_e := Pr\{(M_1, M_2) \neq (\hat{M}_1, \hat{M}_2)\} \to 0$ as $n \to \infty$, when the messages (M_1, M_2) are distributed uniformly over $[1 : \lfloor 2^{nR_1} \rfloor] \times [1 : \lfloor 2^{nR_2} \rfloor]$. The *capacity region* is the closure of the set of achievable rate pairs (R_1, R_2).

In this paper, we restrict ourselves to maximizing the sum-rate (R_1+R_2). Some of the technical arguments employed are novel (example bounding cardinalities of genies, see complete version [2]) and would be of independent interest.

II. Inner and outer bounds for the sum-rate

A. Inner bound

The sum-capacity of interference channel is not known in general. The best known achievable region is the Han-Kobayashi inner bound [3], [4], which subsumes all other known inner bounds. Performing Fourier-Motzkin elimination on this region will allow us to obtain the corresponding sum-rate inner bound.

Theorem 1 (Han-Kobayashi sum-rate inner bound)**.** *Any non-negative value $R_1 + R_2$ satisfying the constraints*

$$R_1 + R_2 \leq I(X_1; Y_1|U_2, Q) + I(X_2; Y_2|U_1, Q) \tag{1a}$$
$$R_1 + R_2 \leq I(U_2, X_1; Y_1|Q) + I(X_2; Y_2|U_2, U_1, Q) \tag{1b}$$
$$R_1 + R_2 \leq I(U_1, X_2; Y_2|Q) + I(X_1; Y_1|U_2, U_1, Q) \tag{1c}$$
$$R_1 + R_2 \leq I(U_2, X_1; Y_1|U_1, Q) + I(U_1, X_2; Y_2|U_2, Q) \tag{1d}$$

for some $p(q)p(u_1, x_1|q)p(u_2, x_2|q)$ is achievable.

While there are known outer bounds for the discrete memoryless interference channel, we will focus on our new outer bound.

B. Genie-based outer bound

In the scalar Gaussian interference channel it was shown that treating interference as noise is optimal, for sum-capacity, under a certain weak interference condition (see Chapter 6 in [1]). The optimality (or converse) was shown using "genie-aided" receivers. Using standard techniques we extend[1] the arguments to develop a genie-based outer bound and show that this new outer bound helps us determine the sum-capacity for certain new classes of discrete memoryless interference channels.

Theorem 2. *Let T_1, T_2 be any pair of random variables such that: $p(y_1, t_1|x_1, x_2) = p(t_1|x_1)p(y_1|t_1, x_1, x_2)$, $p(y_2, t_2|x_1, x_2) = p(t_2|x_2)p(y_2|t_2, x_1, x_2)$, and the marginals are consistent with the given channel transition probabilities, i.e. $p(y_1|x_1, x_2) = \mathfrak{q}(y_1|x_1, x_2)$ and $p(y_2|x_1, x_2) = \mathfrak{q}(y_2|x_1, x_2)$. The achievable sum-rate of the discrete memoryless interference channel characterized by $\mathfrak{q}(y_1, y_2|x_1, x_2)$ can be upper bounded as follows:*

$$\begin{aligned} R_1 + R_2 \leq \max_{p_1(x_1)p_2(x_2)} & I(X_1; T_1, Y_1) + I(X_2; T_2, Y_2) \\ & + \mathfrak{C}[I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)] \\ & - I(X_2; T_2|X_1, T_1) + I(X_2; Y_1|T_1, X_1) \\ & + \mathfrak{C}[I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)] \\ & - I(X_1; T_1|X_2, T_2) + I(X_1; Y_2|T_2, X_2), \end{aligned} \tag{2}$$

where $\mathfrak{C}[I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)]$ denotes the upper concave envelope of the function $I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)$ with respect to product distributions $p_a(x_1)p_b(x_2)$ evaluated at $p_1(x_1)p_2(x_2)$. Similarly the term

[1] A similar argument was independently developed by Khosravi-Farsani and used in [5, Theorem 1].

$\mathfrak{C}[I(X_1;T_1|X_2,T_2) - I(X_1;Y_2|T_2,X_2)]$ *denotes the upper concave envelope of the function* $I(X_1;T_1|X_2,T_2) - I(X_1;Y_2|T_2,X_2)$ *with respect to to product distributions* $p_a(x_1)p_b(x_2)$ *evaluated at* $p_1(x_1)p_2(x_2)$.

Due to lack of space, please check [2] for details.

Remark 1. *The following observations are worth noting.*

(a) Since every valid pair T_1, T_2 *(or genies) yields an outer bound, one minimizes the above expression over the choice of valid genies to obtain the best genie-based outer bound for the sum-rate. However, since every pair of valid genies yields an outer bound, it is not necessary to provide a cardinality bound on the size of the genie that one needs to consider to make the above region computable.*

(b) The above genie-based outer bound recovers the known result in the scalar Gaussian weak interference regime. Useful genies [6], [7], [8] turn out to be choices of T_1, T_2 *so that the functions* $I(X_2;T_2|X_1,T_1) - I(X_2;Y_1|T_1,X_1)$ *and* $I(X_1;T_1|X_2,T_2) - I(X_1;Y_2|T_2,X_2)$ *become concave in* $p_2(x_2)$ *and* $p_1(x_1)$ *respectively. For such genies observe that the outer bound reduces to*

$$R_1 + R_2 \leq \max_{p_1(x_1)p_2(x_2)} I(X_1;T_1,Y_1) + I(X_2;T_2,Y_2),$$

since the concave envelope of a concave function is itself. The maximizing distributions (X_{1*}, X_{2*}) *can be shown to be Gaussian by an application of EPI.*
Within this class of genies where Gaussian signaling is optimal, smart genies [6] ensure that $X_{i*} \rightarrow Y_i \rightarrow T_i, i = 1,2$ *becomes Markov. Therefore the presence of useful and smart genies reduces the upper bound to*

$$R_1 + R_2 \leq I(X_{1*};Y_{1*}) + I(X_{2*};Y_{2*}),$$

which is achievable by treating interference as noise.

(c) Concave envelopes are just a compact way of representing maximizations over auxiliary random variables.

III. Very weak interference class of interference channels

In this section we define the very weak interference class, bearing in mind that our interest is in computing the sum-capacity. Our definition (and nomenclature) is motivated in part by the definition of very strong interference channel [9] presented below.

Definition 1. *A DM-IC is said to have very strong interference if*

$$I(X_1;Y_1|X_2) \leq I(X_1;Y_2),$$
$$I(X_2;Y_2|X_1) \leq I(X_2;Y_1)$$

for all $p_1(x_1)p_2(x_2)$.

Remark 2. *In layman's terms a phrasing of the definition is the following: If the interference at the unintended receiver is so strong that one can decode the interfering signal treating ones own signal as noise at a higher rate than the rate at which the true receiver can decode its intended signal even if some genie provides the interfering signal, then the interference is said to be very strong. The optimal strategy indeed turns out to be to decode the interfering signal first and then decodes ones intended signal.*

In a very weak interference setting one expects the intended receiver to treat the interference signal as noise. Additionally, the true receiver should not even try to decode any part of the interfering signal. Motivated by this intuition, we make the following definition.

Definition 2. *A discrete memoryless interference channel characterized by the transition matrix* $\mathfrak{q}(y_1,y_2|x_1,x_2)$ *is called a very weak interference channel if for every pair of auxiliaries* (U_1,U_2) *such that the joint probability distribution takes the form* $p_1(u_1,x_1)p_2(u_2,x_2)\mathfrak{q}(y_1,y_2|x_1,x_2)$ *the following inequalities hold:*

$$\begin{aligned} I(U_1;Y_1) &\geq I(U_1;Y_2|X_2) \\ I(U_2;Y_2) &\geq I(U_2;Y_1|X_1). \end{aligned} \tag{3}$$

Remark 3. *The following remarks capture some of the intuition as well as limitations of the above definition of very weak interference channels. It would be nice to formally prove this in the sense of [10]. Such a formal proof seems currently out of reach.*

1) *The term* $I(U_1;Y_1)$ *captures the rate of information from* U_1 *(a part of* X_1 *or a cloud centre among* X_1^n *sequences) to* Y_1 *when* Y_1 *tries to decode* U_1 *while treating the rest as noise. However, the receiver* Y_1 *could do some interference cancellation of part of* X_2 *before decoding* U_1*; hence this is an underestimate of the information rate from* U_1 *to* Y_1.
The term $I(U_1;Y_2|X_2)$ *captures the rate of information from* U_1 *to receiver* Y_2*, after* Y_2 *has (magically) cleaned any effect from* X_2*. This is the maximum rate from* U_1 *that receiver* Y_2 *can hope to decode.*
Thus the direction of the inequality states that if U_1 *(part of* X_1*) is to be decoded at* Y_2 *then this imposes a penalty on the rate from* U_1 *to* Y_1 *even under the most favorable (unfavorable) decoding scenario at* Y_2 *(*Y_1*). Thus if one is interested in maximizing* $R_1 + R_2$ *then one would expect that* Y_2 *should not attempt to decode any part of* X_1.
2) *Note that if one is interested in optimizing* $\lambda R_1 + R_2$, $\lambda \neq 1$*, then one must use a different criterion than the one given above to expect treating interference as noise to remain optimal.*

Proposition 1. *The conditions given in* (3) *are equivalent to the following conditions: for a fixed* $p_2(x_2)$ *the function* $I(X_1;Y_1) - I(X_1;Y_2|X_2)$ *is concave in* $p_1(x_1)$ *and for a fixed* $p_1(x_1)$ *the function* $I(X_2;Y_2) - I(X_2;Y_1|X_1)$ *is concave in* $p_2(x_2)$.

Proof. Since $U_1 \rightarrow X_1 \rightarrow (X_2,Y_1,Y_2)$ is Markov observe that

$$\begin{aligned} &I(U_1;Y_1) \geq I(U_1;Y_2|X_2) \iff \\ &I(X_1;Y_1) - I(X_1;Y_2|X_2) \geq I(X_1;Y_1|U_1) - I(X_1;Y_2|U_1X_2). \end{aligned}$$

The right hand side is clearly equivalent to concavity w.r.t. $p_1(x_1)$. ☐

Proposition 2. *Let $SR_{HK}(\mathfrak{q})$ denote the maximum sum-rate achievable using the Han-Kobayashi encoding strategy. Under the very weak interference channel definition in* (3)*, the Han-Kobayashi sum-rate reduces to*

$$SR_{HK}(\mathfrak{q}) = \max_{p_1(x_1)p_2(x_2)} I(X_1;Y_1) + I(X_2;Y_2).$$

Proof. Clearly by setting $Q = U_1 = U_2 = 0$ the trivial random variable (i.e. by treating interference as noise) one can indeed achieve the above sum-rate using the Han-Kobayashi scheme.

To observe the reverse direction consider equation (1d) and note the following

$$\begin{aligned}
&I(U_2, X_1; Y_1|U_1, Q) + I(U_1, X_2; Y_2|U_2, Q)\\
&\overset{(a)}{=} I(U_2, X_1; Y_1|Q) - I(U_1; Y_1|Q)\\
&\quad + I(U_1, X_2; Y_2|Q) - I(U_2; Y_2|Q)\\
&= I(X_1; Y_1|Q) + I(U_2; Y_1|X_1 Q) - I(U_2; Y_2|Q)\\
&\quad + I(X_2; Y_2|Q) - I(U_1; Y_1|Q) + I(U_1; Y_2|X_2, Q)\\
&\overset{(b)}{\leq} I(X_1; Y_1|Q) + I(X_2; Y_2|Q).
\end{aligned}$$

Here (a) is a consequence of the Markov chains $U_1 \to X_1 \to (U_2, X_2, Y_1, Y_2)$ and $U_2 \to X_2 \to (U_1, X_1, Y_1, Y_2)$ which hold conditioned on $Q = q$. Inequality (b) is an immediate consequence of our definition of very weak interference channel. Since the average over Q is dominated by the maximum value the lemma is established. ☐

IV. Binary skewed-Z interference channel

In this section we introduce a class of channels that satisfy the very weak interference condition for a certain set of parameters. We focus on the sum-rate capacity of this class of channels under very weak interference for the rest of the article.

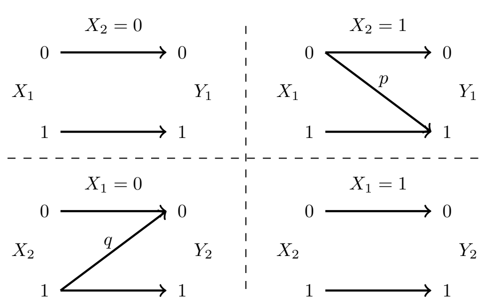

Fig. 2. Binary skewed-Z interference channel (BSZIC)

Figure 2 depicts the transition probabilities of the direct channels for different values of interfering signal. We call such a channel to be *binary skewed-Z interference channel (BSZIC)*.

Proposition 3. *The binary skewed-Z interference channel shown in Figure 2 is a very weak interference channel*[2] *if and only if* $0 \leq p + q \leq 1$.

[2]For the symmetric case, i.e. $p = q$, the condition reduces to $p \leq \frac{1}{2}$ and was derived by Yin Zi as part of a class project in 2011.

Proof. From Proposition 1, it suffices to determine the conditions under which $I(X_1;Y_1) - I(X_1;Y_2|X_2)$ is concave in $p_1(x_1)$ for all fixed $p_2(x_2)$. Let $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ denote the binary entropy function. Let $\mathrm{P}(X_2 = 0) = a$ and $\mathrm{P}(X_1 = 0) = x$. We need to determine the values of $p, q \in [0,1]$ with which $I(X_1;Y_1) - I(X_1;Y_2|X_2)$ is concave in x for all $a \in [0,1]$.

$$\begin{aligned}
&I(X_1;Y_1) - I(X_1;Y_2|X_2)\\
&\quad = H(x(1-\bar{a}p)) - xH(1-\bar{a}p) - \bar{a}H(xq) + \bar{a}xH(q),
\end{aligned}$$

where $\bar{a} = 1 - a$. Note that the second and the last terms are linear in x. After taking second derivative, one could see that the concavity of the above expression w.r.t x is equivalent to showing that

$$\frac{1-\bar{a}p}{1-x(1-\bar{a}p)} \geq \frac{\bar{a}q}{1-xq},$$
$$i.e. \quad (1-\bar{a}p)(1-xq) \geq \bar{a}q(1-x(1-\bar{a}p)).$$

The above condition must hold for every $x \in [0,1]$. Since both sides of the inequality are linear in x, it suffices to verify only at $x = 0$ and $x = 1$. Substituting, we obtain the following two conditions, respectively.

$$\begin{cases} 1-\bar{a}p \geq \bar{a}q, \\ (1-\bar{a}p)(1-q) \geq pq\bar{a}^2. \end{cases}$$

Both conditions have to be satisfied simultaneously for all $a \in [0,1]$. It is easy to check that this is equivalent to $p+q \leq 1$. ☐

Remark 4. *Scalar Gaussian Z-interference channels with crossover gains $a \leq 1, b = 0$ satisfy the very weak interference condition and here it is known [11], [12] that the optimal sum-rate is attained by treating interference as noise. However, we are not able to isolate any non-trivial subset of parameters with crossover gains $a > 0, b > 0$ that satisfies the very weak interference condition.*

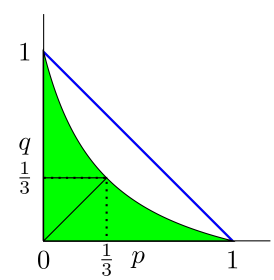

Fig. 3. Regime of parameters where the sum-capacity is established for the Skewed-Z interference channel

Theorem 3. *Treating interference as noise is sum-rate optimal for BSZIC with channel parameters (p, q) satisfying*

$$\begin{aligned} &0 \leq p \leq \frac{1}{3}, \\ &p \leq q \leq \frac{1-p}{1+3p} \end{aligned} \quad \textit{or} \quad \begin{aligned} &0 \leq q \leq \frac{1}{3}, \\ &q \leq p \leq \frac{1-q}{1+3q} \end{aligned}$$

The regime of parameters (as a subset of the weak-interference regime) is shown in Figure 3.

Proof. Let $x = Pr(X_1 = 0)$, $y = Pr(X_2 = 1)$. Consider binary genies T_1, T_2 with the following joint distribution:

X_1	X_2	Y_1	T_1	Probability
0	0	0	0	$x(1-y)(1-p)a$
0	1	0	0	$xy(1-p)a$
0	0	0	1	$x(1-y)((1-p)(1-a)+p)$
1	0	1	1	$(1-x)(1-y)$
0	1	0	1	$xy(1-p)(1-a)$
0	1	1	1	xyp
1	1	1	1	$(1-x)y.$

X_1	X_2	Y_2	T_2	Probability
1	1	1	0	$(1-x)y(1-q)c$
0	1	1	0	$xy(1-q)c$
1	1	1	1	$(1-x)y((1-q)(1-c)+q)$
0	1	0	1	xyq
1	0	0	1	$(1-x)(1-y)$
0	1	1	1	$xy(1-q)(1-c)$
0	0	0	1	$x(1-y).$

Set

$$a = \frac{1+q}{2(1-pq)}, \quad c = \frac{p^2 + p\bar{p}\bar{a}}{\bar{q}(p^2 + p\bar{p}\bar{a} + \bar{p}\bar{a})}.$$

One can verify (see [2] for details) that genies T_1, T_2 satisfy the conditions of Theorem 2; the functions

$$I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1),$$
$$I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)$$

are concave in $p_1(x_1)p_2(x_2)$; and further $X_1 \to Y_1 \to T_1$ and $X_2 \to Y_2 \to T_2$ are Markov. This would then reduce the sum rate outer bound provided by Theorem 2 to that achievable by treating interference as noise. (This proof structure is very similar to the identification of *smart* and *useful* genies in the Gaussian interference channel.) □

Remark 5. *In [2] we show that the above conditions on (p, q) are necessary for the existence of genies such that the difference of mutual information terms are concave and the Markov chain holds.*

A. More on the genie-based outer bound

In this section, we analyze the necessary conditions[3] when the genie-based outer bound for the skewed-Z interference channel reduces to the sum-rate yielded by treating interference as noise. Due to lack of space, some of the proofs are omitted here and the reader may refer [2] for the details. However since our setting is a discrete setting we are able to perform a much more exhaustive analysis of the bound than that possible in the Gaussian setting.

For a given (valid) pair of genies (T_1, T_2) consider the sum-rate outer bound given by Theorem 2. Further let $p_1^*(x_1)p_2^*(x_2)$ be a maximizing product distribution (i.e. the product distribution that yields the outer bound for this particular choice of genies). For the expression in (2) to reduce to

$$I(X_1; Y_1) + I(X_2; Y_2)$$

at $p_1^*(x_1)p_2^*(x_2)$, it is easy to see that the following equalities must hold:

$$\begin{aligned}
&I(X_1; T_1|Y_1) = 0,\\
&I(X_2; T_2|Y_2) = 0,\\
&\mathfrak{C}[I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)]\\
&\quad = I(X_2; T_2|X_1, T_1) + I(X_2; Y_1|T_1, X_1),\\
&\mathfrak{C}[I(X_1; T_1|X_2, T_2) - I(X_1; Y_2|T_2, X_2)]\\
&\quad = I(X_1; T_1|X_2, T_2) + I(X_1; Y_2|T_2, X_2).
\end{aligned}$$

However these inequalities need to hold only at the maximizing distribution $p_1^*(x_1)p_2^*(x_2)$. Further if such genies exist, by virtue of the fact that the expression $I(X_1; Y_1) + I(X_2; Y_2)$ at $p_1^*(x_1)p_2^*(x_2)$ yields an outer bound to the sum-rate, it must also hold that $p_1^*(x_1)p_2^*(x_2)$ is also a maximizer of the expression $I(X_1; Y_1)+I(X_2; Y_2)$ over all product distributions (since the maximum of $I(X_1; Y_1)+I(X_2; Y_2)$ is an achievable sum-rate).

We first restrict our attention to genies (taking values in some finite alphabet) such that the Markov chains $X_1 \to Y_1 \to T_1$ and $X_2 \to Y_2 \to T_2$ hold at some distribution $\mathrm{P}(X_1 = 0) = x_*$ and $\mathrm{P}(X_2 = 1) = y_*$. One can easily verify that for the Markov chains to hold, the probability distributions must take the form

X_1	X_2	Y_1	T_1	Probability
0	0	0	i	$x_*(1-y_*)((1-p)a_i + pb_i))$
1	0	1	i	$(1-x_*)(1-y)b_i$
0	1	0	i	$x_*y_*(1-p)a_i$
0	1	1	i	$x_*y_*pb_i$
1	1	1	i	$(1-x_*)y_*b_i$

for some $0 \le a_i, b_i \le 1$. A similar structure also holds for the distribution of (X_1, X_2, Y_2, T_2). An interesting observation is that if the Markov chain holds for some $x_*, y_* > 0$ then the Markov condition continues to hold for any product distribution. This is a chance observation (peculiar to the Binary skewed-Z interference channel) which greatly simplified our analysis.

Among the class of genies that satisfy the Markov chain, one is further interested in a subclass for which the upper concave envelopes of the differences of mutual information match the function value at $p_1^*(x_1)p_2^*(x_2)$. To this end, define $f(x, y)$ as

$$I(X_2; T_2|X_1, T_1) - I(X_2; Y_1|T_1, X_1)|_{\mathrm{P}(X_1=0)=x, \mathrm{P}(X_2=1)=y}.$$

Expanding the terms and noting the linearity in x can express $f(x, y) = (1-x)g_0(y) + xg_1(y)$, where $g_0(y) = f(0, y)$ is a concave function of y and $g_1(y) = f(1, y)$ is in general neither convex nor concave in the entire interval $y \in [0, 1]$.

The following proposition aids in our computation of the upper concave envelope of $f(x, y)$.

[3] Note that the previous result only dealt with the sufficient conditions.

Proposition 4. *Let* $\mathfrak{C}[f](x,y)$ *denote the upper concave envelope of* $f(x,y)$ *over the space of product distributions notated by* $\mathrm{P}(X_1=0)=x, \mathrm{P}(X_2=1)=y$. *Then*

$$\mathfrak{C}[f](x,y) = (1-x)\mathfrak{C}[g_0](y) + x\mathfrak{C}[g_1(y)],$$

where $\mathfrak{C}[g_0](y), \mathfrak{C}[g_1](y)$ *denotes the upper concave envelope ofs* $g_0(y), g_1(y)$ *respectively over* $y \in [0,1]$.

Proof. Consider a maximizing convex combination: i.e. a probability vector $\{\alpha_i\}$ and points $(x_i, y_i) \in [0,1]\times[0,1]$ such that $\sum_i \alpha_i f(x_i,y_i) = \mathfrak{C}[f](x,y)$. We know that

$$\sum_i \alpha_i x_i y_i = xy, \sum_i \alpha_i x_i = x, \sum_i \alpha_i y_i = y.$$

Obtain a new convex combination as follows: with probability $\alpha_i(1-x_i)$ choose $(0,y_i)$ and with probability $\alpha_i x_i$ choose $(1,y_i)$. Observe that

$$\begin{aligned}
&\sum_i \alpha_i(1-x_i)f(0,y_i) + \alpha_i x_i f(1,y_i)\\
&\quad= \sum_i \alpha_i((1-x_i)f(0,y_i) + x_i f(1,y_i))\\
&\quad= \sum_i \alpha_i f(x_i,y_i) = \mathfrak{C}[f](x,y).
\end{aligned}$$

Since $\sum_i \frac{\alpha_i(1-x_i)}{(1-x)} = 1$ and $\sum_i \frac{\alpha_i(1-x_i)}{(1-x)} y_i = y$ we have $\sum_i \alpha_i(1-x_i)f(0,y_i) \leq (1-x)\mathfrak{C}[g_0](y)$. Similarly we have $\alpha_i x_i f(1,y_i) \leq x\mathfrak{C}[g_1(y)]$. Thus $\mathfrak{C}[f](x,y) \leq (1-x)\mathfrak{C}[g_0](y) + x\mathfrak{C}[g_1(y)]$.

The other direction is immediate as one can always take the convex combination that achieves $\mathfrak{C}[g_0](y)$ and the convex combination that achieves $\mathfrak{C}[g_1](y)$ to obtain a value $(1-x)\mathfrak{C}[g_0](y) + x\mathfrak{C}[g_1(y)]$. □

For the binary skewed-Z interference channel, $g_0(y)$ is concave and hence $\mathfrak{C}[g_0](y) = g_0(y)$. We will seek to answer the following question: In the class of genies such that the Markov chain holds, are there genies such that $\mathfrak{C}[g_1(y)] = g_1(y)$ at y^*, the maximizing distribution? If the answer is affirmative whenever $p+q \leq 1$, then the genie-based outer bound will yield the sum-capacity in the entire weak interference regime of parameters. However, we shall see that this is not the case.

1) Genie approach in an intermediate regime: We restrict our attention to the symmetric case where $p=q$. When $p=q\leq\frac{1}{3}$ we observe that there are genies for which $g_1(y)$ is concave when $y\in[0,1]$.

Now we consider the range $\frac{1}{3}\leq p=q\leq\frac{1}{2}$. Suppose we restrict ourselves to genies with binary alphabets, then $g_1(y)$ displays an interesting behavior. The function is concave in some interval $[0,\hat{y}]$ and convex in the remainder. Hence the concave envelope of $g_1(y)$ matches the function in the interval $[0,y^\dagger]$ ($y^\dagger \leq \hat{y}$) and follows the tangent to the curve $g_1(y)$ (at $y^\dagger$) in the interval $[y^\dagger,1]$. Here $y^\dagger$ is the unique point in $[0,1]$ such the tangent to the curve $g_1(y)$ at $y^\dagger$ passes through $g_1(1)$ when $y=1$.

Numerical simulations indicate that there are such genies when $0\leq p=q\leq 0.39$. Since we have very explicit expressions, it is not difficult to convert the simulations to a complete argument, but we refrain from doing so because of the following negative result.

Proposition 5. *For the binary skewed-Z interference channel when* $p=q=\frac{1}{2}$, *the genie-based outer bound is strictly greater than treating interference as noise inner bound.*

Proof. The proof essentially follows by showing that for any set of genies (with finite but arbitrary cardinalities) such that the Markov condition holds, the function $g_1(y)$ is convex. This precludes the concave envelope to match the function st $p_1^*(x_1)p_2^*(x_2)$. See [2] for details. □

CONCLUSION

We defined a very weak interference condition and showed that a subset of parameters of a binary skewed-Z interference channel belongs to this class. We developed a genie-based outer bound for the sum-rate of discrete memoryless interference channels and showed that treating interference as noise is optimal for a subset of parameters of the binary skewed-Z interference channel in the very weak interference regime. We also showed that the genie-based outer bound will not reduce to the sum-rate yielded by interference as noise in the entire very weak interference regime. This work shows that employing genies as a mathematical gadget for proving converses remains largely an unexplored area.

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