

INEQUALITIES IN ADDITIVE COMBINATORICS AND INFORMATION THEORY: A SHORT SURVEY OF THE TRANSLATION PROGRAMME

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ABSTRACT. Additive combinatorics and information theory each study the effect of additive operations on volumes: in the former, through cardinalities of sumsets such as $A + B$, and in the latter, through (differential) entropy and mutual information of random variables under addition. Ruzsa observed that many inequalities for sumsets in abelian groups admit entropic counterparts, and proposed a “theory of translation” between the two languages. Over the last two decades, this viewpoint has been developed in several directions, including exact equivalence theorems between cardinality and entropy inequalities; transfers of combinatorial proof ideas to information-theoretic settings; and conjectural entropic analogues of deep sumset results. In this survey we give a concise account of this translation programme, emphasizing Ruzsa-type equivalence theorems, entropic Ruzsa distances and triangle/sum-triangle inequalities, magnification ratios and their channel formulations, and the role of information-theoretic tools in small-doubling problems and the recently resolved Polynomial Freiman–Ruzsa/Marton conjecture. We also highlight discrete entropy power and Brascamp–Lieb inequalities on finite abelian groups, where uniform distributions on cosets play the role of Gaussians, and we point to a central open problem on the Gaussian Z-interference channel as an additive Gaussian analogue.

1. INTRODUCTION

Additive combinatorics is concerned with how sumsets

$$A + B := \{a + b : a \in A, b \in B\}$$

and related constructions behave as functions of the underlying sets A, B . Many central questions ask for bounds on $|A + B|$ in terms of $|A|$ and $|B|$, or for structural descriptions of sets with unusually small doubling $|A + A|/|A|$. In parallel, information theory studies Shannon entropy and mutual information of discrete or continuous random variables, and fundamental inequalities relate these functionals under addition and other linear operations.

Ruzsa observed a surprisingly rich analogy between cardinality inequalities and entropy inequalities, and initiated a programme to systematically translate results between the two settings. Broadly speaking, one sees three recurring themes.

- *Exact equivalence.* Certain linear cardinality inequalities for sumsets are equivalent to entropic inequalities, once one passes from uniform distributions on finite sets to general discrete random variables via typical sets.
- *Parallel proofs.* Even when a formal equivalence is not known, combinatorial proofs often admit information-theoretic analogues, typically via submodularity of entropy, data-processing inequalities, and coupling constructions.
- *Conjectural analogues.* Deep combinatorial results suggest natural entropic conjectures (for example, entropic forms of inverse theorems for small doubling and of reverse entropy power inequalities), whose resolution may require new ideas.

A number of works develop these ideas far beyond what can be covered in this short survey. On the combinatorial side, Tao’s paper on sumset and inverse sumset theory for Shannon entropy [24] systematically develops entropic analogues of Plünnecke–Ruzsa theory and inverse sumset results, including entropy versions of the Balog–Szemerédi–Gowers lemma and Freiman-type inverse

theorems. More recently, Green, Manners, and Tao revisited the relationship between sumsets and entropy [9], proving, among other results, a “99% inverse theorem” for the entropic Ruzsa distance and showing that many sumset inequalities admit entropic analogues that are better behaved under homomorphisms and do not require independence assumptions. On the additive-combinatorics side, the resolution of Marton’s conjecture in $\mathbf{F}_2^n$ and its extension to abelian groups with bounded torsion [7, 6] is a major achievement that relies in an essential way on an entropy-like functional whose minimizers are uniform on cosets of subgroups.

From the information-theoretic perspective, a substantial body of work by Madiman, Kontoyiannis, Marcus, Tetali, and collaborators has developed entropy-based analogues of sumset inequalities in both discrete and continuous settings. For example, Madiman and Kontoyiannis [11] established differential-entropy versions of Ruzsa’s sum–difference, triangle, and Plünnecke inequalities, as well as a differential-entropy Freiman–Green–Ruzsa inverse theorem that can be viewed as a quantitative converse to the entropy power inequality. The work of Madiman, Marcus, and Tetali [17] introduced the notion of partition-determined functions and proved general submodularity inequalities for the entropy of such functions, yielding a unified framework that simultaneously implies cardinality and entropy inequalities for a wide range of sumset-like constructions. More recent developments (for instance [16, 15]) refine these ideas and connect entropic additive energy and sum–product phenomena to information-theoretic inequalities. The present article does not attempt to survey all these directions; instead, it highlights a few threads that are particularly close to the author’s interests and to potential applications in Gaussian interference channels.

Disclaimer. The author is not an expert in additive combinatorics or in the modern theory of small-doubling sets; the material surveyed here was encountered while exploring possible approaches to determining the capacity region of Gaussian interference channels, in particular the Gaussian Z-interference channel discussed below. The real experts in the additive-combinatorial and entropy–sumset aspects of this subject are Ruzsa, Tao, Green, Madiman, Kontoyiannis, and their collaborators, and the reader is encouraged to consult their original works for a deeper and more authoritative treatment.

Notation. We work primarily with abelian groups $\mathcal{G}$ written additively. For finite subsets A, B of an abelian group $\mathcal{G}$, we write

$$A + B := \{a + b : a \in A, b \in B\}, \quad A - B := \{a - b : a \in A, b \in B\}.$$

We write $[1 : n]$ for $\{1, 2, \dots, n\}$. Finitely generated torsion-free abelian groups are denoted by $\mathcal{T}$; such a group is isomorphic to $\mathbb{Z}^d$ for some d , and we freely identify $\mathcal{T}$ with $\mathbb{Z}^d$ when convenient.

For a discrete random variable X with finite support, $H(X)$ denotes its Shannon entropy (with respect to a fixed base, say 2), and $H(X, Y)$ the joint entropy. Mutual information is

$$I(X; Y) := H(X) + H(Y) - H(X, Y).$$

We write $H(Y | X) := H(X, Y) - H(X)$ for conditional entropy, and

$$I(X; Y | Z) := H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z)$$

for conditional mutual information.

1.1. Additive Gaussian channels and interference. While much of this survey focuses on discrete groups and combinatorial structure, the translation programme has close ties to network information theory in continuous settings, where additive Gaussian channels play a central role. A particularly important example is the scalar Gaussian one-sided interference (or Z-interference) channel, given by

$$Y_1 = X_1 + Z_1, \quad Y_2 = X_2 + aX_1 + Z_2,$$

where X_1, X_2 are independent input signals with average power constraints P_1, P_2 , $a \in \mathbb{R}$ is an interference gain, and Z_1, Z_2 are independent standard Gaussians.

The Gaussian Z-interference channel can be viewed as a two-dimensional additive system with structured interference, and its capacity region $\mathcal{C}$ exhibits behaviour that is naturally compared with continuous sumset problems phrased in terms of (differential) entropy. Readers may refer to El Gamal and Kim [4] for precise definitions of capacity regions and achievable rate pairs in multiuser channels.

The best known inner bound for general interference channels is the Han–Kobayashi (HK) achievable region. For the Gaussian Z-interference channel with $|a| \in (0, 1)$ it remains open whether Gaussian HK with power control achieves capacity. This conjecture goes back to the early work of Sato and of Han and Kobayashi in the 1980s, and a more precise formulation and discussion can be found in [19].

Conjecture 1.1 (Han–Kobayashi for Gaussian Z-interference). *For the Gaussian Z-interference channel with parameters (a, P_1, P_2) , the capacity region coincides with the Han–Kobayashi region with Gaussian signalling, i.e.*

$$\mathcal{C} = \mathcal{R}_{\text{HK-GS}}.$$

Remark 1.2 (Further developments on the Gaussian Z-interference channel). The Han–Kobayashi region is not equal to the capacity region for general interference channels; in particular, there are non-Gaussian Z-interference examples where the Han–Kobayashi achievable region is known to be strictly suboptimal [20]. For Gaussian interference channels, however, the underlying additive structure suggests that Han–Kobayashi with Gaussian signalling may still be optimal, which motivates Conjecture 1.1.

One way to view Conjecture 1.1 is as an extremal problem for a certain information-theoretic functional that is built from linear combinations of differential entropies corresponding to the Han–Kobayashi inequalities. This functional is structurally similar to the PFR-type entropy functionals appearing in small-doubling problems, where one seeks to identify extremizers (or near-extremizers) under additive constraints. Given the close analogy with PFR-type functionals, it is natural to hope that ideas and insights from additive combinatorics—such as entropic Ruzsa calculus, discrete BL–EPI inequalities, and extremizer-structure results—may play a useful role in understanding the Gaussian Z-interference problem.

There is also a broader line of work developing entropy-based analogues of sumset inequalities in both discrete and continuous settings. In particular, the differential-entropy sumset inequalities and inverse results of Madiman and Kontoyiannis [11], the partition-determined framework of Madiman, Marcus, and Tetali [17], and more recent work on entropic additive energy and sum–product inequalities [15] further support the analogy between additive structure and entropy. Taken together, these contributions suggest that tools developed for discrete and continuous sumset problems may eventually be relevant to extremal questions such as the Gaussian Z-interference conjecture discussed above.

2. EQUIVALENCE THEOREMS IN THE LINEAR SETTING

We begin with Ruzsa’s equivalence theorem, which gives an exact correspondence between a large class of linear sumset inequalities and entropic inequalities, and then discuss a generalized Ruzsa-type equivalence that incorporates optimization over couplings.

2.1. Graph-restricted sumsets and Ruzsa’s theorem. Many of the earliest equivalences are most naturally stated for graph-restricted sumsets.

Definition 2.1 (Graph-restricted sumset [22]). Let A, B be finite subsets of an abelian group $\mathcal{G}$, and let $G \subseteq A \times B$ be a bipartite graph on A and B . The G -restricted sumset and difference set are

$$A +_G B := \{a + b : (a, b) \in G\}, \quad A -_G B := \{a - b : (a, b) \in G\}.$$

Ruzsa's equivalence theorem relates linear inequalities in the cardinalities of such sets to linear entropy inequalities.

Theorem 2.2 (Ruzsa equivalence theorem, Equivalence Theorem 2 in [22]). *Let $f, g_1, \dots, g_k$ be linear functions in two variables with integer coefficients, and let $\alpha_1, \dots, \alpha_k > 0$. The following are equivalent.*

(i) *For every finite subset $A \subseteq \mathcal{T} \times \mathcal{T}$,*

$$|f(A)| \leq \prod_{i=1}^k |g_i(A)|^{\alpha_i}.$$

(ii) *For every pair (X, Y) of (not necessarily independent) random variables with values in $\mathcal{T}$ such that $H(g_i(X, Y)) < \infty$ for all i , we have $H(f(X, Y)) < \infty$ and*

$$H(f(X, Y)) \leq \sum_{i=1}^k \alpha_i H(g_i(X, Y)).$$

Remark 2.3 (Scope of Ruzsa's equivalence theorem). The original formulation of Theorem 2.2 applies to graph-restricted sumsets on finitely generated torsion-free abelian groups $\mathcal{T}$. In particular, both the use of a bipartite graph $G \subseteq A \times B$ and the torsion-free assumption on the ambient group are built into Ruzsa's proof via typical-set arguments and linear-forms manipulations on $\mathcal{T} \times \mathcal{T}$. Extending such an exact equivalence to general (not necessarily graph-restricted) sumsets or to ambient groups with torsion requires additional ideas beyond the original theorem.

As a concrete example, Theorem 2.2 yields an entropic version of the Katz–Tao sum–difference inequality.

Theorem 2.4 (Katz–Tao sum–difference inequality [10]). *Let $A, B \subseteq \mathcal{T}$ be finite and let $G \subseteq A \times B$. Then*

$$|A -_G B| \leq |A|^{2/3} |B|^{2/3} |A +_G B|^{1/2}.$$

Applying Theorem 2.2 with suitable linear forms, one obtains:

Theorem 2.5 (Entropic Katz–Tao inequality [22]). *Let X, Y be discrete random variables with finite support on $\mathcal{T}$. Then*

$$(2.1) \quad H(X - Y) \leq \frac{2}{3} H(X) + \frac{2}{3} H(Y) + \frac{1}{2} H(X + Y).$$

Theorem 2.6 (Entropic Katz–Tao inequality in mutual-information form [12, Thm. 5]). *Let X, Y be random variables with finite support on an abelian group $\mathcal{G}$. Then*

$$\frac{1}{2} I(X; X - Y) + \frac{1}{2} I(Y; X - Y) \leq \frac{3}{2} I(X; X + Y) + \frac{3}{2} I(Y; X + Y) + 3I(X; Y).$$

Remark 2.7. An entropy-only version of Theorem 2.6 (stated in terms of $H(X \pm Y)$ rather than mutual informations) appears as Proposition 3.6 in Tao's "Deleted scenes: entropy sumset estimates" [25]. The mutual-information formulation above was subsequently recorded and used in the entropic Ruzsa calculus developed in [12].

2.2. A generalized Ruzsa-type equivalence. Let $\mathcal{T}$ be a finitely generated torsion-free abelian group. Consider linear maps

$$f_1, \dots, f_k : \mathcal{T}^n \rightarrow \mathcal{T}, \quad g_1, \dots, g_\ell : \mathcal{T}^n \rightarrow \mathcal{T}$$

with integer coefficients, and positive weights $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell$. For each f_i , let $S_i \subseteq [1 : n]$ denote the set of coordinates on which f_i depends, and similarly for each g_j with index sets $T_j \subseteq$

$[1 : n]$. Assume that the sets $(S_i)_{i=1}^k$ are pairwise disjoint. For a family $(A_i)_{i=1}^n$ of finite subsets of $\mathcal{T}$, write

$$A_S := \bigotimes_{i \in S} A_i \subseteq \mathcal{T}^S.$$

For a family of random variables $(X_i)_{i=1}^n$ taking values in $\mathcal{T}$, and $S \subseteq [1 : n]$, write $X_S := (X_i)_{i \in S}$, and let $\Pi(X_S)$ denote the set of all joint laws on X_S with prescribed one-dimensional marginals $(p_{X_i})_{i \in S}$.

Theorem 2.8 (Generalized Ruzsa-type equivalence [12]). *Under the above assumptions, the following are equivalent.*

(a) *For all finite sets $A_1, \dots, A_n \subseteq \mathcal{T}$,*

$$\sum_{i=1}^k \alpha_i \log |f_i(A_{S_i})| \leq \sum_{j=1}^{\ell} \beta_j \log |g_j(A_{T_j})|.$$

(b) *For any discrete random variables $(X_1, \dots, X_n)$ with values in $\mathcal{T}$ and prescribed marginals p_{X_i} ,*

$$\sum_{i=1}^k \alpha_i \max_{\Pi(X_{S_i})} H(f_i(X_{S_i})) \leq \sum_{j=1}^{\ell} \beta_j \max_{\Pi(X_{T_j})} H(g_j(X_{T_j})).$$

Remark 2.9 (Torsion-free ambient groups). Theorem 2.8 removes the graph-restricted assumption: it applies to “regular” sumset inequalities without an explicit bipartite graph, and captures optimization over couplings on the entropic side. However, the ambient group is still required to be a finitely generated torsion-free abelian group $\mathcal{T}$, and the proof again exploits this structure (e.g., via identification with $\mathbb{Z}^d$ and linear typical-set constructions). Whether one can obtain a fully general equivalence theorem of this kind on arbitrary abelian groups, without any torsion-free hypothesis, remains open. Specific instances of such torsion-inclusive translations have been established in special cases (for example, certain sum–difference and Ruzsa-distance inequalities admit direct entropic proofs that work on all finite abelian groups), but a general theorem matching the scope of Theorem 2.8 is not yet available.

3. RUZSA DISTANCE AND ENTROPIC ANALOGUES

A central tool in additive combinatorics is the Ruzsa distance between finite subsets of an abelian group $\mathcal{G}$.

Definition 3.1 (Ruzsa distance). Let $A, B \subseteq \mathcal{G}$ be finite subsets. The Ruzsa distance between A and B is

$$d_R(A, B) := \log \frac{|A - B|}{|A|^{1/2} |B|^{1/2}}.$$

On the entropic side, given discrete random variables X, Y supported on finite subsets of $\mathcal{G}$, we define:

Definition 3.2 (Entropic Ruzsa distances). If $(X, Y) \sim p_X p_Y$ are independent, the *independent entropic Ruzsa distance* is

$$d_{\text{ER}}(X, Y) := H(X - Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y).$$

Given fixed marginals p_X, p_Y , the *coupling entropic Ruzsa distance* is

$$d_{\text{CR}}(X, Y) := \max_{(X, Y) \in \Pi(p_X, p_Y)} \left(H(X - Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) \right).$$

3.1. Triangle and sum–triangle inequalities.

Theorem 3.3 (Ruzsa triangle inequality). *Let A, B, C be finite subsets of an abelian group $\mathcal{G}$. Then*

$$d_R(A, B) \leq d_R(A, C) + d_R(C, B),$$

equivalently,

$$|A - B| |C| \leq |A - C| |C - B|.$$

Theorem 3.4 (Entropic Ruzsa triangle inequalities [23]). *Let X, Y, Z be discrete random variables with finite supports on an abelian group $\mathcal{G}$.*

(i) *If X, Y, Z are pairwise independent, then*

$$d_{ER}(Y, Z) \leq d_{ER}(Y, X) + d_{ER}(X, Z),$$

equivalently,

$$H(X) + H(Y - Z) \leq H(X - Y) + H(X - Z).$$

(ii) *For arbitrary marginals p_X, p_Y, p_Z and the coupling distance d_{CR} ,*

$$d_{CR}(Y, Z) \leq d_{CR}(Y, X) + d_{CR}(X, Z),$$

equivalently,

$$H(X) + \max_{\Pi(Y, Z)} H(Y - Z) \leq \max_{\Pi(X, Y)} H(X - Y) + \max_{\Pi(X, Z)} H(X - Z).$$

A closely related sum–triangle inequality states that, for finite $A, B, C \subseteq \mathcal{G}$,

$$|B + C| |A| \leq |A + B| |A + C|,$$

which, via Theorem 2.8, yields an entropic sum–triangle inequality on torsion-free groups $\mathcal{T}$. A purely entropic proof, avoiding the equivalence theorem, is not known at present.

Beyond the results stated here, Tao’s paper on sumset and inverse sumset theory for Shannon entropy [24] develops a systematic entropic analogue of much of the classical Plünnecke–Ruzsa theory. In particular, it establishes entropy versions of the Plünnecke inequalities, of the Balog–Szemerédi–Gowers lemma, and of Freiman-type inverse sumset theorems, and shows how these can be combined with combinatorial arguments to obtain structural information about approximate groups from entropy bounds.

The entropic approach has been sharpened further in the work of Green, Manners, and Tao on sumsets and entropy [9]. Among other results, they show that the independence hypothesis in many entropic Ruzsa inequalities can be dropped at the cost of a universal constant factor change in the entropic Ruzsa distance, and they prove a “99% inverse theorem” asserting that if the entropic Ruzsa distance is sufficiently small, then the underlying distribution is close (in an appropriate metric) to one uniform on a coset progression. Their arguments illustrate how entropic methods can sometimes yield cleaner or more flexible formulations than the underlying combinatorial statements, particularly with respect to homomorphisms and tensorization.

3.2. Magnification ratios and channels. For a bipartite graph $G \subseteq A \times B$, the magnification ratio measures expansion from A to B .

Definition 3.5 (Magnification ratio). Let A, B be finite subsets of an abelian group $\mathcal{G}$, and $G \subseteq A \times B$ a bipartite graph. The magnification ratio from A to B is

$$\mu_{A \rightarrow B}(G) := \min_{\emptyset \neq S \subseteq A} \frac{|N_G(S)|}{|S|},$$

where $N_G(S) := \{b \in B : \exists a \in S \text{ with } (a, b) \in G\}$.

There is also an information-theoretic characterization in terms of channels consistent with G . Let $\mathcal{W}$ be the set of all stochastic matrices $W_{Y|X}$ from A to B , and define

$$\mathcal{W}(G) := \{W_{Y|X} \in \mathcal{W} : W(Y = b \mid X = a) = 0 \text{ whenever } (a, b) \notin G\}.$$

For a fixed input distribution p_X on A , set

$$\lambda_{A \rightarrow B}(G; p_X) := \max_{W \in \mathcal{W}(G)} (H(Y) - H(X)),$$

and then

$$\lambda_{A \rightarrow B}(G) := \min_{p_X} \lambda_{A \rightarrow B}(G; p_X).$$

Theorem 3.6 (Entropic characterization of magnification ratio [12]). *Let A, B be finite and $G \subseteq A \times B$. Then*

$$\log \mu_{A \rightarrow B}(G) = \lambda_{A \rightarrow B}(G).$$

Thus magnification ratios can be seen as worst-case entropy gains of channels with support constrained by G , providing a direct dictionary between combinatorial expansion and information-theoretic contraction/expansion.

4. SUM-DIFFERENCE INEQUALITIES AND COVERING LEMMAS

We now discuss Ruzsa's sum-difference inequality and some of its entropic generalizations, including formulations that parallel the Green-Ruzsa covering lemma and related sumset inequalities.

4.1. Ruzsa sum-difference inequality and entropic forms. Ruzsa's sum-difference inequality can be stated in terms of d_R as follows.

Theorem 4.1 (Ruzsa sum-difference inequality [21, Thm. 5.3]). *Let A, B be finite subsets of an abelian group $\mathcal{G}$. Then*

$$d_R(A, -B) \leq 3 d_R(A, B),$$

equivalently,

$$|A + B| |A| |B| \leq |A - B|^3.$$

An entropic analogue for d_{ER} follows from Tao's mutual-information form of the Katz-Tao inequality.

Corollary 4.2 (Entropic Ruzsa sum-difference inequalities). *Let X, Y be discrete random variables with finite support on an abelian group $\mathcal{G}$.*

(i) *If X, Y are independent, then*

$$d_{ER}(X, -Y) \leq 3 d_{ER}(X, Y),$$

equivalently,

$$H(X - Y) \leq 3H(X + Y) - H(X) - H(Y).$$

(ii) *For arbitrary marginals p_X, p_Y , the coupling entropic Ruzsa distance also satisfies*

$$d_{CR}(X, -Y) \leq 3 d_{CR}(X, Y),$$

equivalently,

$$\max_{\Pi(X, Y)} \left[H(X - Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) \right] \leq 3 \max_{\Pi(X, Y)} \left[H(X + Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y) \right].$$

In particular, the coupling form (ii) provides a direct entropic analogue of Ruzsa's sum-difference inequality phrased in terms of the coupling distance d_{CR} .

Remark 4.3. Part (i) of Corollary 4.2 is due to Tao in the independent setting, while part (ii) follows from a coupling-based entropic Ruzsa calculus developed in [12] and expanded in [14]. This mirrors the passage from independent Ruzsa distance to the coupling distance d_{CR} in Theorem 3.4.

4.2. Green–Ruzsa covering lemma and an entropic counterpart. The Green–Ruzsa covering lemma provides a powerful tool for controlling difference sets.

Lemma 4.4 (Green–Ruzsa covering lemma [8]). *Let A, B be finite subsets of an abelian group $\mathcal{G}$. Then there exists a set $X \subseteq B$ with*

$$|X| \leq 2 \frac{|A+B|}{|A|} - 1$$

such that for every $y \in B$ there are at least $|A|/2$ triples $(x, a, a') \in X \times A \times A$ with

$$x + a - a' = y.$$

In particular, $B - B \subseteq A - A + X - X$, and the same conclusions hold with $|A+B|/|A|$ replaced by $|A-B|/|A|$.

As an application, Ruzsa proved a nontrivial inequality for iterated sum–difference sets.

Theorem 4.5 (Ruzsa sumset inequality [21]). *Let A, B be finite subsets of an abelian group $\mathcal{G}$. Then*

$$|(B+B) - (B+B)| \leq \frac{|A+B|^4 |A-A|}{|A|^4}.$$

On the entropic side, Tao derived an inequality for $Y_1 - Y_2 - Y_3 + Y_4$ (a signed sum of four copies of Y) in terms of entropies involving $X + Y$ and $X_5 - X_6$ when X and Y are independent; in one convenient form, this can be written as

$$H(Y_1 - Y_2 - Y_3 + Y_4) \leq 4H(X + Y) + H(X_5 - X_6) - 4H(X),$$

for independent copies $Y_1, \dots, Y_4$ of Y and X_5, X_6 of X . This gives an independent entropic analogue of Theorem 4.5.

In Lau’s thesis, this is extended to a coupling version on finite abelian groups [14, Thm. 4.2.33]. Roughly speaking, if X, Y are discrete random variables on a finite abelian group with prescribed marginals, then for any choice of couplings realizing these marginals and for suitable copies $Y_1, \dots, Y_4, X_5, X_6$ (not necessarily independent), one has

$$H(Y_1 - Y_2 - Y_3 + Y_4) \leq 4 \max_{\Pi(X,Y)} H(X + Y) + \max_{\Pi(X_5,X_6)} H(X_5 - X_6) - 4H(X),$$

where the maxima are taken over all couplings with the given one-dimensional marginals. In particular, the right-hand side depends only on the marginals of X and Y , not on a specific independent product structure. The proof in [14] proceeds by constructing appropriate copies with linear relations reflecting those in Theorem 4.5, then applying submodularity and chain rules for entropy, and finally optimizing over couplings.

5. SMALL DOUBLING, PFR/MARTON, AND ENTROPIC FORMULATIONS

We now turn to structural questions for sets with small doubling and their information-theoretic counterparts, focusing on the Polynomial Freiman–Ruzsa (PFR) / Marton conjecture in finite abelian groups and its entropic avatars.

5.1. The PFR/Marton conjecture in finite fields. Let A be a finite subset of an abelian group $\mathcal{G}$, and define its doubling constant

$$K := \frac{|A+A|}{|A|}.$$

Inverse theorems for small doubling ask for a structural description of A when K is bounded. In the setting of $\mathbf{F}_2^n$, this is known as Marton’s conjecture.

Conjecture 5.1 (Polynomial Freiman–Ruzsa / Marton [7, Conj. 1.1]). *Let $A \subset \mathbf{F}_2^n$ be nonempty with $|A + A| \leq K|A|$. Then there exists a subgroup $H \leq \mathbf{F}_2^n$ and elements $x_1, \dots, x_m \in \mathbf{F}_2^n$ with*

$$m \leq 2K^C$$

for some absolute constant C , such that

$$A \subseteq \bigcup_{i=1}^m (x_i + H).$$

Theorem 5.2 (Marton’s conjecture in $\mathbf{F}_2^n$ [7]). *Let $A \subset \mathbf{F}_2^n$ be nonempty and suppose $|A + A| \leq K|A|$. Then there exists a subgroup $H \leq \mathbf{F}_2^n$ with $|H| \leq |A|$ such that A is covered by at most $2K^C$ translates of H , for some absolute constant C .*

Gowers, Green, Manners, and Tao have established polynomial bounds of this form in $\mathbf{F}_2^n$ [7]. In subsequent work they extend these methods to prove Marton-type results in abelian groups of bounded torsion, again with polynomial control on the number of cosets in terms of the doubling constant and the torsion parameter [6].

5.2. An entropic viewpoint. A natural way to recast small-doubling phenomena is via Shannon entropy and entropic Ruzsa distances. Let X be a $\mathbf{F}_2^n$ -valued random variable, and let X' be an independent copy. If X is uniform on a finite set A , then

$$H(X) = \log |A|, \quad H(X + X') = \log |A + A|,$$

so

$$H(X + X') - H(X) = \log \frac{|A + A|}{|A|} = \log K.$$

Equivalently, for the independent entropic Ruzsa distance,

$$d_{\text{ER}}(X, X') = H(X - X') - H(X) = \log \frac{|A - A|}{|A|}$$

in the uniform case. Thus small doubling corresponds to small entropic growth under addition.

This suggests an *entropic Marton conjecture*, asserting roughly that if $d_{\text{ER}}(X, X')$ is small (or, symmetrically, if $H(X + X') - H(X)$ is small), then X must be close in distribution to a mixture of uniform distributions on cosets of a subgroup. Specializing to the case where X is uniform on A recovers Theorem 5.2 via an entropy-counting argument.

A key ingredient in the resolution of PFR/Marton is an entropy-like functional whose minimizers are uniform on cosets. In information-theoretic language, one works with independent random variables X, Y, X_0, Y_0 on a finite abelian group of characteristic 2 and considers functionals built from expressions such as

$$H(X + Y) - \frac{1}{2}H(X) - \frac{1}{2}H(Y),$$

together with suitable perturbations and tensorization arguments across product groups. In the formulation used in [7], this leads to an entropic functional τ on pairs of distributions that satisfies a small collection of axioms (monotonicity under addition, a conditional version, translation invariance, and continuity). They show that any minimizer of τ must be uniform on a coset of a subgroup, and then connect the value of τ for a given small-doubling set to its covering number. This entropic extremal characterization is the bridge between small doubling in $\mathbf{F}_2^n$ and the covering conclusion of Theorem 5.2.

6. INFORMATION-THEORETIC TECHNIQUES

6.1. Discrete BL–EPI inequalities. On the analytic side, continuous entropy power inequalities (EPIs) and Brascamp–Lieb inequalities capture the behaviour of entropy under linear combinations of independent random variables, with Gaussians as extremizers. Anantharam, Jog, and Nair introduced a unified BL–EPI family of entropic inequalities for linear images of Euclidean random vectors, and proved that Gaussians are extremal by a “doubling-followed-by-rotation” argument [1].

Building on this work, a discrete analogue of the unified BL–EPI inequality for finite abelian groups was developed in [13]. For a suitable BL–EPI datum on a finite abelian group $\mathcal{G}$, one defines discrete entropy functionals mirroring those of [1], proves a subadditivity property via a doubling-plus-rotation construction adapted to the group setting, and shows that uniform distributions on cosets of subgroups are extremizers. Thus, uniform laws on subgroups play the role of Gaussians in the discrete BL–EPI setting.

Very roughly, for independent random variables $X_1, \dots, X_n$ on a finite abelian group and integer linear forms $L_j = \sum_i c_{j,i} X_i$, one considers functionals of the form

$$\sum_j \beta_j H(L_j) - \sum_i \alpha_i H(X_i)$$

subject to a Brascamp–Lieb-type balance condition on the coefficients. Under appropriate admissibility conditions, such inequalities hold in the discrete setting, and their extremizers are uniform on cosets of subgroups. The proofs combine:

- a discrete analogue in the spirit of the Darmois–Skitovich theorem, characterizing independence of linear forms in terms of uniform-on-subgroup structure;
- a rotation trick and a superadditive lifted functional built from conditional entropies;
- tensorization and extremal arguments reminiscent of those used for continuous unified EPI–Brascamp–Lieb inequalities.

From the viewpoint of additive combinatorics and information theory, the key takeaway is that uniform distributions on cosets play the role of discrete Gaussians. Discrete EPI/Brascamp–Lieb inequalities thus provide a flexible tool for understanding small-doubling regimes, entropic Ruzsa distances, and the structure of extremizers in PFR-type problems.

In a related continuous-direction development, Klartag and Lehec have recently resolved Bourgain’s hyperplane (slicing) conjecture by proving that the isotropic constant of any convex body in $\mathbb{R}^n$ is bounded by a universal constant; their proof, via stochastic localization and sharp small-ball estimates for isotropic log-concave measures, is tightly connected to stability questions for both the Brunn–Minkowski inequality and the entropy power inequality (EPI). Earlier work of Bobkov and Madiman [2, 3] had already provided an entropic reformulation of Bourgain’s conjecture, showing that it is equivalent (up to universal constants) to a dimension-free bound on a suitable notion of “entropy distance to Gaussianity” for isotropic log-concave measures, and formulating an *entropic hyperplane conjecture* asserting uniform bounds on the per-coordinate entropy deficit relative to the Gaussian. Very roughly, the recent results of Klartag and Lehec may be interpreted as giving a sharp quantitative stability statement for Brunn–Minkowski and, in entropic form, for the EPI: if a pair of independent random vectors nearly attains equality in an appropriate EPI-type inequality, then their laws must be close, in natural metrics such as transportation distance or relative entropy, to Gaussian laws, with quantitative control on the deviation. This can be viewed as an “inverse sumset” phenomenon in the continuous setting, parallel in spirit to Freiman-type inverse theorems and to the entropic inverse theorems of Madiman and Kontoyiannis for discrete sumsets and for differential entropy.

6.2. Copy lemma techniques and non-Shannon phenomena. A recurring tool in the work of Green, Tao, and subsequent authors is the use of *copy-lemma* type constructions. Informally,

one enlarges the probability space by adding auxiliary random variables that are conditionally independent “copies” of existing variables given some sigma-algebra, arranged so that certain sums or differences are preserved. Entropy submodularity and data-processing inequalities applied to this enlarged system then yield new linear inequalities among the entropies of the original variables.

In additive combinatorics, early uses of such ideas appear in Green’s entropy-based arguments for sumset inequalities and in Tao’s entropy proofs of Ruzsa-type inequalities and inverse theorems, where carefully constructed independent copies are introduced to reflect combinatorial configurations such as coverings and multi-fold sumsets. More recent work develops this viewpoint systematically: in proving entropic analogues of covering-lemma and Plünnecke–Ruzsa-type bounds (including the entropy version of Theorem 4.5), one constructs joint laws of several copies of the basic variables X, Y together with additional “copies” that preserve specified sums or differences. Subadditivity and chain-rule manipulations on this extended system, combined with optimization over couplings as in Theorem 2.8, then produce the desired entropic inequalities.

From a broader information-theoretic perspective, copy lemmas are central in the construction of *non-Shannon-type* information inequalities, starting from the seminal work of Zhang and Yeung and in subsequent developments surveyed, for instance, by Csirmaz. In that setting, as in the additive-combinatorial applications considered here, one introduces suitable copies of variables, imposes conditional independence constraints, and then applies submodularity of entropy and chain rules to derive new linear inequalities. Thus the sumset-related inequalities in this survey and the classical non-Shannon inequalities can be viewed as members of a larger family generated from common basic ingredients: copy constructions together with submodularity, rather than belonging to fundamentally different regimes.

7. CONCLUDING REMARKS AND OPEN DIRECTIONS

We close with a few indicative directions where the translation programme between additive combinatorics and information theory needs further evolution.

- *Beyond linear forms.* Most equivalence results discussed here concern inequalities for linear forms in random variables or sumsets. Extending the translation programme to nonlinear operations, higher-degree additive energies, and incidence-type inequalities remains largely unexplored. There has also been recent interest in extending the entropic viewpoint from sums to combinations of sums and products: Goh [5] introduces an entropic analogue of additive energy and formulates sum–product-type conjectures suggesting an Erdős–Szemerédi sum–product phenomenon for entropy, Máthé and O’Regan [18] develop discretised sum–product theorems using Shannon-type inequalities, and Tao [26] studies entropies of “slopes” and associated sum–difference exponents. These developments lie somewhat beyond the scope of the present article, but they point to a broader emerging theory of entropy in sum–product phenomena and other nonlinear contexts.
- *Reverse inequalities and stability.* Classical reverse EPIs and stability results for sumset inequalities suggest parallel entropic reverse inequalities and stability principles. Quantitative versions of entropic triangle, sum–triangle, and sum–difference inequalities that characterize near-equality cases could lead to finer structural theorems for approximate groups and near-small-doubling sets.
- *Equivalence theorems beyond torsion-free groups.* Generalize Theorems 2.2 and 2.8 to arbitrary finitely generated abelian groups, removing the torsion-free assumption on the ambient group. In particular, given a linear cardinality inequality for sumsets (with or without graph restrictions) that holds uniformly over all finite subsets of an abelian group with torsion, determine whether there is an exact entropic counterpart of the same form for all discrete random variables on that group, and identify the additional ideas needed beyond typical-set and Sanov-type arguments.
- *Interference channels and additive Gaussians.* The conjecture that the Han–Kobayashi region with Gaussian signalling equals capacity for the Gaussian Z-interference channel (Conjecture 1.1)

is a central open problem in network information theory. It can be viewed as an additive Gaussian extremal problem in the spirit of EPI and Brascamp–Lieb, and it would be very interesting to understand whether the discrete translation techniques surveyed here (equivalence theorems, coupling optimizations, and extremizer analysis) shed light on this conjecture.

- *Copy lemmas and non-Shannon-type inequalities.* The copy-lemma-based constructions appearing in the work surveyed here parallel the techniques used to derive non-Shannon-type information inequalities. It is an intriguing open direction to investigate whether additive-combinatorial configurations give rise to new non-Shannon-type inequalities, or conversely whether known non-Shannon inequalities admit meaningful sumset interpretations.
- *Unified frameworks and new extremizers.* Discrete analogues of unified BL–EPI inequalities and their continuous counterparts suggest the existence of broader “master inequalities” from which many known combinatorial and information-theoretic inequalities can be derived. Identifying such frameworks in other settings—for example, nonabelian groups, approximate groups, or hypergraph sumsets—may reveal new extremal distributions or sets beyond cosets and progressions.

The interface between additive combinatorics and information theory has already produced conceptually new proofs and resolved longstanding conjectures. At the same time, the range of known equivalences and analogies is still modest compared to the breadth of each field, suggesting that many further translations—and perhaps fundamentally new inequalities—remain to be discovered.

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